Definition. Let $X$ be a nonempty set and $\mathcal{M} \subset \mathcal{P}(X)$ a $\sigma$-algebra.

(a) A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \rightarrow [-\infty, \infty]$ such that

(i) $\nu(\emptyset) = 0$

(ii) $\nu$ assumes at most one of the values $\pm \infty$.

(iii) If $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ is disjoint, then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n)$$

with the sum converging absolutely if $\nu\left(\bigcup_{n=1}^{\infty} E_n\right)$ is finite.

(b) A set $E \in \mathcal{M}$ is said to be positive (negative, null) for the signed measure $\nu$ if

$$F \in \mathcal{M}, \; F \subset E \Rightarrow \nu(F) \geq 0 \; (\leq 0, = 0)$$

1. Let $\nu$ be a signed measure on $(X, \mathcal{M})$. Prove that

(a) If $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ with $E_1 \subset E_2 \subset E_3 \subset \cdots$, then

$$\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{n \to \infty} \nu(E_n)$$

(b) If $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ with $E_1 \supset E_2 \supset E_3 \supset \cdots$ and $\nu(E_1)$ finite, then

$$\nu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{n \to \infty} \nu(E_n)$$

2. Let $\nu$ be a signed measure on $(X, \mathcal{M})$.

(a) Prove that if $E$ is a positive set for $\nu$ and $F \in \mathcal{M}$ with $F \subset E$, then $F$ is a positive set for $\nu$.

(b) Prove that if, for each $n \in \mathbb{N}$, $E_n$ is a positive set for $\nu$, then $\bigcup_{n=1}^{\infty} E_n$ is a positive set for $\nu$.

Definition. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) If $1 \leq p < \infty$, $L^p$ is the set of all measurable functions $f : X \to \mathbb{C}$ such that $\int |f(x)|^p \, d\mu < \infty$. For $f \in L^p$, we define

$$\|f\|_p = \left[ \int |f(x)|^p \, d\mu(x) \right]^{1/p}$$

(b) $L^\infty$ is the set of all measurable functions $f : X \to \mathbb{C}$ for which there exists an $M \geq 0$ such that $|f(x)| \leq M$ a.e.. For $f \in L^\infty$, we define

$$\|f\|_\infty = \inf \{ M \geq 0 \mid |f(x)| \leq M \text{ a.e.} \}$$

see over
3. In this problem, we prove the triangle inequality (a.k.a. the Minkowski inequality) \( \|f + g\|_p \leq \|f\|_p + \|g\|_p \), for all \( 1 \leq p \leq \infty \).

(a) Prove that \( \|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty \).

From now on assume that \( 1 \leq p < \infty \). If either \( \|f\|_p = 0 \) or \( \|g\|_p = 0 \), then \( f = 0 \) a.e. or \( g = 0 \) a.e. and \( \|f + g\|_p = \|f\|_p + \|g\|_p \). So from now on assume that \( \|f\|_p, \|g\|_p > 0 \). By replacing \( f \) by \( \frac{f}{\|f\|_p + \|g\|_p} \) and \( g \) by \( \frac{g}{\|f\|_p + \|g\|_p} \), we may assume that \( \|f\|_p + \|g\|_p = 1 \). From now on, do so. So we must prove that \( \|f + g\|_p \leq 1 \).

(b) (Concavity) Define, for all \( y \geq 0 \), \( h(y) = y^p \). Prove that, for all \( u, v \geq 0 \) and \( 0 \leq \lambda \leq 1 \),

\[
h(\lambda u + (1 - \lambda)v) \leq \lambda h(u) + (1 - \lambda)h(v)
\]

*Hint:* This is trivial for \( p = 1 \) and for \( u = v \), so assume that \( p > 1 \) and \( u \neq v \). Set

\[
H(\lambda) = h(\lambda u + (1 - \lambda)v) - [\lambda h(u) + (1 - \lambda)h(v)]
\]

and prove that \( H''(\lambda) > 0 \), for all \( 0 < \lambda < 1 \) and that this implies that the maximum of \( H \) must be achieved at \( \lambda = 0 \) or \( \lambda = 1 \), where \( H \) takes the value 0.

(c) Prove that \( \|f + g\|_p \leq \|f\|_p + \|g\|_p \), by proving that \( \|f + g\|_p^p \leq 1 \).

4. In this problem, we prove the Hölder inequality that \( \int |fg| \, d\mu \leq \|f\|_p \|g\|_q \), for all \( 1 \leq p, q \leq \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Since \( |f(x)g(x)| \leq |g(x)| \|f\|_\infty \) and \( |f(x)g(x)| \leq |f(x)| \|g\|_\infty \) for almost all \( x \), the cases \( p = 1, q = \infty \) and \( p = \infty, q = 1 \) are obvious. So we may assume that \( 1 < p, q < \infty \). Also if \( \|f\|_p = 0 \) or \( \|g\|_q = 0 \), then \( f = 0 \) a.e. or \( g = 0 \) a.e. and \( \|fg\|_1 = 0 \). So we may assume that \( \|f\|_p, \|g\|_q > 0 \). By replacing \( f \) by \( \frac{f}{\|f\|_p} \) and \( g \) by \( \frac{g}{\|g\|_q} \), we may assume that \( \|f\|_p = \|g\|_q = 1 \).

(a) Prove that the minimum value of the function \( h(c) = \frac{c^p}{p} + \frac{1}{q} - c \), for \( c \geq 0 \), is 0.

(b) Use part (a) to prove that \( \frac{a^p}{p} + \frac{b^q}{q} \geq ab \) for all \( a, b \geq 0 \).

(c) Prove the Hölder inequality by proving that \( \int |f(x)| \, |g(x)| \, d\mu(x) \leq 1 \) when \( \|f\|_p = \|g\|_q = 1 \).