Monotone Classes

Definition 1 Let X be a nonempty set. A collection $\mathcal{C} \subset \mathcal{P}(X)$ of subsets of X is called a **monotone class** if it is closed under countable increasing unions (that is, if $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ and $E_1 \subset E_2 \subset E_3 \subset \cdots$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$) and it is closed under countable decreasing intersections (that is, if $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ and $E_1 \supset E_2 \supset E_3 \supset \cdots$, then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{C}$).

Remark 2

(a) Monotone classes are closely related to σ–algebras. In fact, for us, their only use will be to help verify that a certain collection of subsets is a σ–algebra.

(b) Every σ–algebra is a monotone class, because σ–algebras are closed under arbitrary countable unions and intersections.

(c) If, for every index i in some index set $\mathcal{I}$, $\mathcal{C}_i$ is a monotone class, then $\bigcap_{i\in\mathcal{I}} \mathcal{C}_i$ is also a monotone class. In particular, for any $\mathcal{E} \subset \mathcal{P}(X)$, the collection

$$\mathcal{C}(\mathcal{E}) = \bigcap_{\substack{\mathcal{C} \text{ monotone class} \\ \mathcal{E}\subset\mathcal{C}}} \mathcal{C}$$

is a monotone class, called the monotone class generated by $\mathcal{E}$. It is the smallest monotone class that contains $\mathcal{E}$. So if $\mathcal{C}$ is any monotone class that contains $\mathcal{E}$, then $\mathcal{C}(\mathcal{E}) \subset \mathcal{C}$.

Lemma 3 *Let X be a nonempty set. If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, then $\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$. That is, the monotone class generated by $\mathcal{A}$ is the same as the σ–algebra generated by $\mathcal{A}$.*

Proof:

$\mathcal{C}(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$: By Remark 2.b, $\mathcal{M}(\mathcal{A})$ is a monotone class that contains $\mathcal{A}$. So, this follows by Remark 2.c.

$\mathcal{M}(\mathcal{A}) \subset \mathcal{C}(\mathcal{A})$: It suffices to prove that $\mathcal{C}(\mathcal{A})$ is a σ–algebra, because then we will know that $\mathcal{C}(\mathcal{A})$ is a σ–algebra containing $\mathcal{A}$ and hence $\mathcal{M}(\mathcal{A})$, which is the smallest σ–algebra containing $\mathcal{A}$.

By question # 6 of Problem Set 1, any algebra that is closed under countable increasing unions is a σ–algebra. So it suffices to prove that $\mathcal{C}(\mathcal{A})$ is an algebra (i.e. that $\mathcal{C}(\mathcal{A})$ is nonempty and closed under complements and finite intersections). So it suffices to prove

$$E, F \in \mathcal{C}(\mathcal{A}) \implies E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(\mathcal{A}) \tag{1}$$

(since X is automatically in $\mathcal{A}$, which is an algebra, and hence is automatically in $\mathcal{C}(\mathcal{A})$ and is an allowed choice for E). Define, for each $E \in \mathcal{C}(\mathcal{A})$,

$$\mathcal{D}(E) = \left\{ \; F \in \mathcal{C}(\mathcal{A}) \;\middle|\; E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(\mathcal{A}) \; \right\}$$

We wish to show that $\mathcal{C}(\mathcal{A}) \subset \mathcal{D}(E)$. To do so, it suffices to show that $\mathcal{D}(E)$ is a monotone class that contains $\mathcal{A}$. We first prove some properties of $\mathcal{D}(E)$.

(a) $\emptyset, E \in \mathcal{D}(E)$.

(b) $F \in \mathcal{D}(E) \iff E \in \mathcal{D}(F)$.

(c) $\mathcal{D}(E)$ is closed under countable increasing unions. To see this, let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(E)$ obey $F_1 \subset F_2 \subset F_3 \subset \cdots$ and set $F = \bigcup_{n=1}^{\infty} F_n$. Then $\big\{E \setminus F_n = E \cap F_n^c\big\}_{n\in\mathbb{N}} \subset \mathcal{C}(\mathcal{A})$ is decreasing, $\big\{F_n \setminus E = F_n \cap E^c\big\}_{n\in\mathbb{N}} \subset \mathcal{C}(\mathcal{A})$ is increasing and $\big\{E \cap F_n\big\}_{n\in\mathbb{N}} \subset \mathcal{C}(\mathcal{A})$ is increasing, so that

$$\begin{aligned}
E \setminus F &= E \cap \Big(\bigcup_{n=1}^{\infty} F_n\Big)^c = E \cap \Big(\bigcap_{n=1}^{\infty} F_n^c\Big) = \bigcap_{n=1}^{\infty} (E \cap F_n^c) \in \mathcal{C}(\mathcal{A}) \\
F \setminus E &= \Big(\bigcup_{n=1}^{\infty} F_n\Big) \cap E^c = \bigcup_{n=1}^{\infty} (F_n \cap E^c) \ = \bigcup_{n=1}^{\infty} (F_n \setminus E) \ \in \mathcal{C}(\mathcal{A}) \\
E \cap F &= E \cap \Big(\bigcup_{n=1}^{\infty} F_n\Big) \qquad\qquad\qquad\qquad = \bigcup_{n=1}^{\infty} (E \cap F_n) \in \mathcal{C}(\mathcal{A})
\end{aligned}$$

since $\mathcal{C}(\mathcal{A})$ is closed under countable decreasing intersections and countable increasing unions.

(d) $\mathcal{D}(E)$ is closed under countable decreasing intersections. To see this, let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(E)$ obey $F_1 \supset F_2 \supset F_3 \supset \cdots$ and set $F = \bigcap_{n=1}^{\infty} F_n$. As in part (c)

$$\begin{aligned}
E \setminus F &= E \cap \Big(\bigcap_{n=1}^{\infty} F_n\Big)^c = E \cap \Big(\bigcup_{n=1}^{\infty} F_n^c\Big) = \bigcup_{n=1}^{\infty} (E \cap F_n^c) \in \mathcal{C}(\mathcal{A}) \\
F \setminus E &= \Big(\bigcap_{n=1}^{\infty} F_n\Big) \cap E^c = \bigcap_{n=1}^{\infty} (F_n \cap E^c) \ = \bigcap_{n=1}^{\infty} (F_n \setminus E) \ \in \mathcal{C}(\mathcal{A}) \\
E \cap F &= E \cap \Big(\bigcap_{n=1}^{\infty} F_n\Big) \qquad\qquad\qquad\qquad = \bigcap_{n=1}^{\infty} (E \cap F_n) \in \mathcal{C}(\mathcal{A})
\end{aligned}$$

We are now ready to prove (1), or equivalently, that $\mathcal{C}(\mathcal{A}) \subset \mathcal{D}(E)$ for all $E \in \mathcal{C}(\mathcal{A})$. So let $E \in \mathcal{C}(\mathcal{A})$. By properties (c) and (d), $\mathcal{D}(E)$ is a monotone class, so it suffices to prove that $\mathcal{A} \subset \mathcal{D}(E)$. But

$$\begin{aligned}
F \in \mathcal{A} &\implies \mathcal{A} \subset \mathcal{D}(F) && \text{by the definition of } \mathcal{D}(F)\text{, since } \mathcal{A} \text{ is an algebra} \\
&\implies \mathcal{C}(\mathcal{A}) \subset \mathcal{D}(F) && \text{since } \mathcal{D}(\mathcal{F}) \text{ is a monotone class} \\
&\implies E \in \mathcal{D}(F) && \text{since } E \in \mathcal{C}(\mathcal{A}) \\
&\implies F \in \mathcal{D}(E) && \text{by property (b)}
\end{aligned}$$

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