

# Monotone Classes

**Definition 1** Let  $X$  be a nonempty set. A collection  $\mathcal{C} \subset \mathcal{P}(X)$  of subsets of  $X$  is called a **monotone class** if it is closed under countable increasing unions (that is, if  $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{C}$  and  $E_1 \subset E_2 \subset E_3 \subset \dots$ , then  $\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$ ) and it is closed under countable decreasing intersections (that is, if  $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{C}$  and  $E_1 \supset E_2 \supset E_3 \supset \dots$ , then  $\bigcap_{n=1}^{\infty} E_n \in \mathcal{C}$ ).

**Remark 2**

(a) Monotone classes are closely related to  $\sigma$ -algebras. In fact, for us, their only use will be to help verify that a certain collection of subsets is a  $\sigma$ -algebra.

(b) Every  $\sigma$ -algebra is a monotone class, because  $\sigma$ -algebras are closed under arbitrary countable unions and intersections.

(c) If, for every index  $i$  in some index set  $\mathcal{I}$ ,  $\mathcal{C}_i$  is a monotone class, then  $\bigcap_{i \in \mathcal{I}} \mathcal{C}_i$  is also a monotone class. In particular, for any  $\mathcal{E} \subset \mathcal{P}(X)$ , the collection

$$\mathcal{C}(\mathcal{E}) = \bigcap_{\substack{\mathcal{C} \text{ monotone class} \\ \mathcal{E} \subset \mathcal{C}}} \mathcal{C}$$

is a monotone class, called the monotone class generated by  $\mathcal{E}$ . It is the smallest monotone class that contains  $\mathcal{E}$ . So if  $\mathcal{C}$  is any monotone class that contains  $\mathcal{E}$ , then  $\mathcal{C}(\mathcal{E}) \subset \mathcal{C}$ .

**Lemma 3** Let  $X$  be a nonempty set. If  $\mathcal{A} \subset \mathcal{P}(X)$  is an algebra, then  $\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$ . That is, the monotone class generated by  $\mathcal{A}$  is the same as the  $\sigma$ -algebra generated by  $\mathcal{A}$ .

**Proof:**

$\mathcal{C}(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$ : By Remark 2.b,  $\mathcal{M}(\mathcal{A})$  is a monotone class that contains  $\mathcal{A}$ . So, this follows by Remark 2.c.

$\mathcal{M}(\mathcal{A}) \subset \mathcal{C}(\mathcal{A})$ : It suffices to prove that  $\mathcal{C}(\mathcal{A})$  is a  $\sigma$ -algebra, because then we will know that  $\mathcal{C}(\mathcal{A})$  is a  $\sigma$ -algebra containing  $\mathcal{A}$  and hence  $\mathcal{M}(\mathcal{A})$ , which is the smallest  $\sigma$ -algebra containing  $\mathcal{A}$ .

By question # 6 of Problem Set 1, any algebra that is closed under countable increasing unions is a  $\sigma$ -algebra. So it suffices to prove that  $\mathcal{C}(\mathcal{A})$  is an algebra (i.e. that  $\mathcal{C}(\mathcal{A})$  is nonempty and closed under complements and finite intersections). So it suffices to prove

$$E, F \in \mathcal{C}(\mathcal{A}) \implies E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(\mathcal{A}) \tag{1}$$

(since  $X$  is automatically in  $\mathcal{A}$ , which is an algebra, and hence is automatically in  $\mathcal{C}(\mathcal{A})$  and is an allowed choice for  $E$ ). Define, for each  $E \in \mathcal{C}(\mathcal{A})$ ,

$$\mathcal{D}(E) = \{ F \in \mathcal{C}(\mathcal{A}) \mid E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(\mathcal{A}) \}$$

We wish to show that  $\mathcal{C}(\mathcal{A}) \subset \mathcal{D}(E)$ . To do so, it suffices to show that  $\mathcal{D}(E)$  is a monotone class that contains  $\mathcal{A}$ . We first prove some properties of  $\mathcal{D}(E)$ .

- (a)  $\emptyset, E \in \mathcal{D}(E)$ .  
 (b)  $F \in \mathcal{D}(E) \iff E \in \mathcal{D}(F)$ .  
 (c)  $\mathcal{D}(E)$  is closed under countable increasing unions. To see this, let  $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(E)$  obey  $F_1 \subset F_2 \subset F_3 \subset \dots$  and set  $F = \bigcup_{n=1}^{\infty} F_n$ . Then  $\{E \setminus F_n = E \cap F_n^c\}_{n \in \mathbb{N}} \subset \mathcal{C}(\mathcal{A})$  is decreasing,  $\{F_n \setminus E = F_n \cap E^c\}_{n \in \mathbb{N}} \subset \mathcal{C}(\mathcal{A})$  is increasing and  $\{E \cap F_n\}_{n \in \mathbb{N}} \subset \mathcal{C}(\mathcal{A})$  is increasing, so that

$$\begin{aligned} E \setminus F &= E \cap \left( \bigcup_{n=1}^{\infty} F_n \right)^c = E \cap \left( \bigcap_{n=1}^{\infty} F_n^c \right) = \bigcap_{n=1}^{\infty} (E \cap F_n^c) \in \mathcal{C}(\mathcal{A}) \\ F \setminus E &= \left( \bigcup_{n=1}^{\infty} F_n \right) \cap E^c = \bigcup_{n=1}^{\infty} (F_n \cap E^c) = \bigcup_{n=1}^{\infty} (F_n \setminus E) \in \mathcal{C}(\mathcal{A}) \\ E \cap F &= E \cap \left( \bigcup_{n=1}^{\infty} F_n \right) = \bigcup_{n=1}^{\infty} (E \cap F_n) \in \mathcal{C}(\mathcal{A}) \end{aligned}$$

since  $\mathcal{C}(\mathcal{A})$  is closed under countable decreasing intersections and countable increasing unions.

- (d)  $\mathcal{D}(E)$  is closed under countable decreasing intersections. To see this, let  $\{F_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(E)$  obey  $F_1 \supset F_2 \supset F_3 \supset \dots$  and set  $F = \bigcap_{n=1}^{\infty} F_n$ . As in part (c)

$$\begin{aligned} E \setminus F &= E \cap \left( \bigcap_{n=1}^{\infty} F_n \right)^c = E \cap \left( \bigcup_{n=1}^{\infty} F_n^c \right) = \bigcup_{n=1}^{\infty} (E \cap F_n^c) \in \mathcal{C}(\mathcal{A}) \\ F \setminus E &= \left( \bigcap_{n=1}^{\infty} F_n \right) \cap E^c = \bigcap_{n=1}^{\infty} (F_n \cap E^c) = \bigcap_{n=1}^{\infty} (F_n \setminus E) \in \mathcal{C}(\mathcal{A}) \\ E \cap F &= E \cap \left( \bigcap_{n=1}^{\infty} F_n \right) = \bigcap_{n=1}^{\infty} (E \cap F_n) \in \mathcal{C}(\mathcal{A}) \end{aligned}$$

We are now ready to prove (1), or equivalently, that  $\mathcal{C}(\mathcal{A}) \subset \mathcal{D}(E)$  for all  $E \in \mathcal{C}(\mathcal{A})$ . So let  $E \in \mathcal{C}(\mathcal{A})$ . By properties (c) and (d),  $\mathcal{D}(E)$  is a monotone class, so it suffices to prove that  $\mathcal{A} \subset \mathcal{D}(E)$ . But

$$\begin{aligned} F \in \mathcal{A} &\implies \mathcal{A} \subset \mathcal{D}(F) && \text{by the definition of } \mathcal{D}(F), \text{ since } \mathcal{A} \text{ is an algebra} \\ &\implies \mathcal{C}(\mathcal{A}) \subset \mathcal{D}(F) && \text{since } \mathcal{D}(F) \text{ is a monotone class} \\ &\implies E \in \mathcal{D}(F) && \text{since } E \in \mathcal{C}(\mathcal{A}) \\ &\implies F \in \mathcal{D}(E) && \text{by property (b)} \end{aligned}$$

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