Monotone Classes

Definition 1  Let $X$ be a nonempty set. A collection $\mathcal{C} \subset \mathcal{P}(X)$ of subsets of $X$ is called a monotone class if it is closed under countable increasing unions (that is, if $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{C}$ and $E_1 \subset E_2 \subset E_3 \subset \cdots$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$) and it is closed under countable decreasing intersections (that is, if $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{C}$ and $E_1 \supset E_2 \supset E_3 \supset \cdots$, then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{C}$).

Remark 2  
(a) Monotone classes are closely related to $\sigma$–algebras. In fact, for us, their only use will be to help verify that a certain collection of subsets is a $\sigma$–algebra.
(b) Every $\sigma$–algebra is a monotone class, because $\sigma$–algebras are closed under arbitrary countable unions and intersections.
(c) If, for every index $i$ in some index set $\mathcal{I}$, $\mathcal{C}_i$ is a monotone class, then $\bigcap_{i \in \mathcal{I}} \mathcal{C}_i$ is also a monotone class. In particular, for any $\mathcal{E} \subset \mathcal{P}(X)$, the collection
\[
\mathcal{C}(\mathcal{E}) = \bigcap_{\mathcal{C} \text{ monotone class} \atop \mathcal{E} \subset \mathcal{C}} \mathcal{C}
\]
is a monotone class, called the monotone class generated by $\mathcal{E}$. It is the smallest monotone class that contains $\mathcal{E}$. So if $\mathcal{C}$ is any monotone class that contains $\mathcal{E}$, then $\mathcal{C}(\mathcal{E}) \subset \mathcal{C}$.

Lemma 3  Let $X$ be a nonempty set. If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, then $\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$. That is, the monotone class generated by $\mathcal{A}$ is the same as the $\sigma$–algebra generated by $\mathcal{A}$.

Proof:

$\mathcal{C}(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$: By Remark 2.b, $\mathcal{M}(\mathcal{A})$ is a monotone class that contains $\mathcal{A}$. So, this follows by Remark 2.c.

$\mathcal{M}(\mathcal{A}) \subset \mathcal{C}(\mathcal{A})$: It suffices to prove that $\mathcal{C}(\mathcal{A})$ is a $\sigma$–algebra, because then we will know that $\mathcal{C}(\mathcal{A})$ is a $\sigma$–algebra containing $\mathcal{A}$ and hence $\mathcal{M}(\mathcal{A})$, which is the smallest $\sigma$–algebra containing $\mathcal{A}$.

By question # 6 of Problem Set 1, any algebra that is closed under countable increasing unions is a $\sigma$–algebra. So it suffices to prove that $\mathcal{C}(\mathcal{A})$ is an algebra (i.e. that $\mathcal{C}(\mathcal{A})$ is nonempty and closed under complements and finite intersections). So it suffices to prove
\[
E, F \in \mathcal{C}(\mathcal{A}) \implies E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(\mathcal{A})
\]
(1)

(since $X$ is automatically in $\mathcal{A}$, which is an algebra, and hence is automatically in $\mathcal{C}(\mathcal{A})$ and is an allowed choice for $E$). Define, for each $E \in \mathcal{C}(\mathcal{A})$,
\[
\mathcal{D}(E) = \{ F \in \mathcal{C}(\mathcal{A}) \mid E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(\mathcal{A}) \}
\]
We wish to show that $\mathcal{C}(\mathcal{A}) \subset \mathcal{D}(E)$. To do so, it suffices to show that $\mathcal{D}(E)$ is a monotone class that contains $\mathcal{A}$. We first prove some properties of $\mathcal{D}(E)$.
(a) \( \emptyset, E \in \mathcal{D}(E) \).
(b) \( F \in \mathcal{D}(E) \iff E \in \mathcal{D}(F) \).
(c) \( \mathcal{D}(E) \) is closed under countable increasing unions. To see this, let \( \{ F_n \}_{n \in \mathbb{N}} \subset \mathcal{D}(E) \) obey \( F_1 \subset F_2 \subset F_3 \subset \cdots \) and set \( F = \bigcup_{n=1}^{\infty} F_n \). Then \( \{ E \setminus F_n = E \cap F_n^c \}_{n \in \mathbb{N}} \subset \mathcal{C}(\mathcal{A}) \) is decreasing, \( \{ F_n \setminus E = F_n \cap E^c \}_{n \in \mathbb{N}} \subset \mathcal{C}(\mathcal{A}) \) is increasing and \( \{ E \cap F_n \}_{n \in \mathbb{N}} \subset \mathcal{C}(\mathcal{A}) \) is increasing, so that

\[
E \setminus F = E \cap \left( \bigcup_{n=1}^{\infty} F_n \right)^c = E \cap \left( \bigcap_{n=1}^{\infty} F_n^c \right) = \bigcap_{n=1}^{\infty} (E \cap F_n^c) \in \mathcal{C}(\mathcal{A})
\]
\[
F \setminus E = \left( \bigcup_{n=1}^{\infty} F_n \right) \cap E^c = \bigcup_{n=1}^{\infty} (F_n \cap E^c) = \bigcup_{n=1}^{\infty} (F_n \setminus E) \in \mathcal{C}(\mathcal{A})
\]
\[
E \cap F = E \cap \left( \bigcap_{n=1}^{\infty} F_n \right) = \bigcup_{n=1}^{\infty} (E \cap F_n) \in \mathcal{C}(\mathcal{A})
\]

since \( \mathcal{C}(\mathcal{A}) \) is closed under countable decreasing intersections and countable increasing unions.
(d) \( \mathcal{D}(E) \) is closed under countable decreasing intersections. To see this, let \( \{ F_n \}_{n \in \mathbb{N}} \subset \mathcal{D}(E) \) obey \( F_1 \supset F_2 \supset F_3 \supset \cdots \) and set \( F = \bigcap_{n=1}^{\infty} F_n \). As in part (c)

\[
E \setminus F = E \cap \left( \bigcap_{n=1}^{\infty} F_n \right)^c = E \cap \left( \bigcup_{n=1}^{\infty} F_n^c \right) = \bigcup_{n=1}^{\infty} (E \cap F_n^c) \in \mathcal{C}(\mathcal{A})
\]
\[
F \setminus E = \left( \bigcap_{n=1}^{\infty} F_n \right) \cap E^c = \bigcap_{n=1}^{\infty} (F_n \cap E^c) = \bigcap_{n=1}^{\infty} (F_n \setminus E) \in \mathcal{C}(\mathcal{A})
\]
\[
E \cap F = E \cap \left( \bigcap_{n=1}^{\infty} F_n \right) = \bigcap_{n=1}^{\infty} (E \cap F_n) \in \mathcal{C}(\mathcal{A})
\]

We are now ready to prove (1), or equivalently, that \( \mathcal{C}(\mathcal{A}) \subset \mathcal{D}(E) \) for all \( E \in \mathcal{C}(\mathcal{A}) \). So let \( E \in \mathcal{C}(\mathcal{A}) \). By properties (c) and (d), \( \mathcal{D}(E) \) is a monotone class, so it suffices to prove that \( \mathcal{A} \subset \mathcal{D}(E) \). But

\[
F \in \mathcal{A} \implies \mathcal{A} \subset \mathcal{D}(F) \quad \text{by the definition of } \mathcal{D}(F), \text{ since } \mathcal{A} \text{ is an algebra}
\]
\[
\implies \mathcal{C}(\mathcal{A}) \subset \mathcal{D}(F) \quad \text{since } \mathcal{D}(\mathcal{F}) \text{ is a monotone class}
\]
\[
\implies E \in \mathcal{D}(F) \quad \text{since } E \in \mathcal{C}(\mathcal{A})
\]
\[
\implies F \in \mathcal{D}(E) \quad \text{by property (b)}
\]