Lecture Notes on Measure Theory and Integration

2 — Measures

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2 Measures

2.1 A Problem

In an ideal world, we would like to be able to extend the definition "for all $a \le b$, the length of the interval $[a,b]$ is $b-a$" by assigning a size $\mu(E)$ to every subset $E \subset \mathbb{R}$ with the properties that

(i) $\mu : \mathcal{P}(\mathbb{R}) = \left\{ \; E \;\middle|\; E \subset \mathbb{R} \; \right\} \to [0,\infty]$

(ii) *Countable additivity*: If $\{E_i\}_{i\in\mathbb{N}}$ is a sequence of disjoint subsets of $\mathbb{R}$, then

$$\mu\Big(\bigcup_{i\in\mathbb{N}} E_i\Big) = \sum_{i\in\mathbb{N}} \mu(E_i)$$

(To get a union of finitely many sets, just make $E_i = \emptyset$ for most i's.)

(iii) *Translation invariance*: For all $E \subset \mathbb{R}$ and $y \in \mathbb{R}$

$$\mu(E+y) = \mu(E) \qquad \text{where} \quad E+y = \left\{ \; x+y \;\middle|\; x \in E \; \right\}$$

(iv) For any $a \le b$,

$$\mu\big([a,b]\big) = \mu\big((a,b]\big) = \mu\big([a,b)\big) = \mu\big((a,b)\big) = b-a$$

There is a problem with this. It is impossible to simultaneously satisfy all of these conditions. They are mutually inconsistent. Here is an example which shows this.

Example 2.1. In this example we will express

$$[0,1) = \bigcup_{r\in\mathbb{Q}\cap[0,1)} N_r$$

with the N_r's

- all disjoint and
- all having the same size. Say $\mu(N_r) = \mu_N$, for all r.

Once we have done so, we will have

$$\begin{aligned} 1 = \mu\big([0,1)\big) &= \sum_{r\in\mathbb{Q}\cap[0,1)} \mu\big(N_r\big) = \sum_{r\in\mathbb{Q}\cap[0,1)} \mu_N \\ &= \begin{cases} 0 & \text{if } \mu_N = 0 \\ \infty & \text{if } \mu_N > 0 \end{cases} \end{aligned}$$

which is a contradiction.

Now we construct the N_r's. Define an equivalence relation[1] on $[0,1)$ by

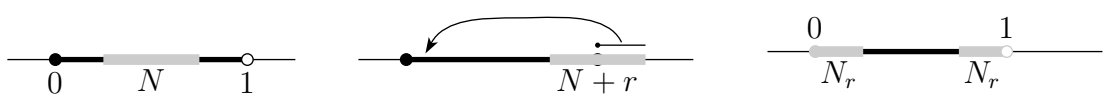
$$x \sim y \quad \Longleftrightarrow \quad x - y \in \mathbb{Q}$$

and define, for each $x \in [0,1)$, the equivalence class of x to be

$$[x] = \left\{ \; y \in [0,1) \;\middle|\; y \sim x, \text{ i.e. } y - x \in \mathbb{Q} \; \right\}$$

As is the case for all equivalence relations, for each pair $x, y \in [0,1)$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$. Consequently $[0,1)$ is the disjoint union of all equivalence classes. Now, let[2] N contain exactly one member of each equivalence class and define, for each $r \in \mathbb{Q} \cap [0,1)$,

$$\begin{aligned}
\tilde{N}_r &= N + r = \left\{ \; x + r \;\middle|\; x \in N \; \right\} \\
N_r &= \left(\tilde{N}_r \cap [0,1)\right) \bigcup \left(\tilde{N}_r \cap [1,2) - 1\right) \\
&= \left\{ \; \overbrace{x + r}^{\geq r} \;\middle|\; \overbrace{x \in N}^{\geq 0}, \; 0 \leq x + r < 1 \; \right\} \bigcup \left\{ \; \overbrace{x + r - 1}^{< r} \;\middle|\; x \in N, \; \overbrace{1 \leq x + r < 2}^{0 \leq x + r - 1 < 1} \; \right\}
\end{aligned}$$



Then

(a) $N_r \subset [0,1)$ for each $r \in \mathbb{Q} \cap [0,1)$

(b) if $r, s \in \mathbb{Q} \cap [0,1)$ with $r \neq s$, then $N_r \cap N_s = \emptyset$ because
  if $y \in N_r$, (i.e. $y = x + r$ or $x + r - 1$ with $x \in N$)
  and $y \in N_s$ (i.e. $y = x' + s$ or $x' + s - 1$ with $x' \in N$)
  then $x - x' = s - r + (-1 \text{ or } 0 \text{ or } 1) \in \mathbb{Q}$
  which implies that $x \sim x'$
  and $x \neq x'$ (since $r \neq s$ and $0 \leq r, s < 1$ so that $-1 < s - r < 1$)
  so that N contains two different elements from the equivalence class $[x]$
  which is a contradiction

[1] An equivalence relation $x \sim y$ has the properties

$$\begin{aligned}
&x \sim x \\
&x \sim y \implies y \sim x \\
&x \sim y, \; y \sim z \implies x \sim z
\end{aligned}$$

[2] We're using the axiom of choice here.

(c) $\mu(N_r) = \mu(N)$ because

$$\begin{aligned}\mu(N_r) &= \mu\big(\tilde{N}_r \cap [0,1)\big) + \mu\big(\tilde{N}_r \cap [1,2) - 1\big)\\ &= \mu\big(\tilde{N}_r \cap [0,1)\big) + \mu\big(\tilde{N}_r \cap [1,2)\big)\\ &= \mu\big(\tilde{N}_r = N + r\big)\\ &= \mu(N)\end{aligned}$$

(d) $[0,1) = \bigcup_{r \in \mathbb{Q} \cap [0,1)} N_r$ because

$$\begin{aligned} y \in [0,1) &\implies \exists\, x \in N \text{ with } x \in [y] \text{ i.e. } x \sim y\\ &\implies y = x + r \text{ for some } r \in \mathbb{Q} \cap (-1,1)\\ &\implies \quad y \in N_r \text{ if } r \in [0,1)\\ &\qquad\quad \text{or } y = \overbrace{(\underbrace{x+r}_{\in[0,1)}+1)}^{\in[1,2)} - 1 \in N_{r+1} \text{ if } r \in (-1,0) \text{ i.e. } r+1 \in (0,1)\end{aligned}$$

So these N_r's satisfy all of the required properties.

We have just seen that it is impossible to define a size function $\mu(E)$ which simultaneously has all four of the properties (i), (ii), (iii), (iv) above. We have to weaken one or more of those properties. In this course, we will deal with this problem in two different ways.

- *the σ-algebra approach:* under this approach, we weaken property (i) and only require that $\mu(E)$ be defined for certain sets E (that include all sets that we are likely to encounter in practice).
- *the outer measure approach:* under this approach, we weaken property (ii) and only require that $\mu\big(\bigcup_i E_i\big) = \sum_i \mu(E_i)$ hold for certain collections of sets (that include all sets that we are likely to encounter in practice).

2.2 σ-algebras

Let X be a nonempty set. We denote by $\mathcal{P}(X)$ the set of all subsets of X.

Definition 2.2 (Algebras)**.** A nonempty collection $\mathcal{A}$ of subsets of X is an **algebra** if

(i) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

(ii) $A \in \mathcal{A} \implies A^c = X \setminus A \in \mathcal{A}$

Remark 2.3 (More about algebras)**.** An algebra $\mathcal{A}$ of subsets of X also (trivially) obeys

(i') $A_1, \cdots, A_n \in \mathcal{A} \implies A_1 \cup \cdots \cup A_n \in \mathcal{A}$

(iii) $A_1, \cdots, A_n \in \mathcal{A} \implies A_1 \cap \cdots \cap A_n = \left(A_1^c \cup \cdots \cup A_n^c\right)^c \in \mathcal{A}$

(iv) $\emptyset = A \cap A^c \in \mathcal{A}$

(v) $X = \emptyset^c \in \mathcal{A}$

Definition 2.4 (σ-algebras)**.** A collection $\mathcal{A}$ of subsets of X is a **$\boldsymbol{\sigma}$-algebra** if it is an algebra that is closed under countable unions. That is,

$$\left\{A_n\right\}_{n\in\mathbb{N}} \subset \mathcal{A} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$$

We shall eventually define a measure on a σ-algebra $\mathcal{A}$ to be a function $\mu : \mathcal{A} \to [0, \infty]$ that obeys certain properties.

Example 2.5. For any nonempty set X,

- $\mathcal{P}(X)$ is a σ-algebra
- $\left\{\emptyset\,,\, X\right\}$ is a σ-algebra

Example 2.6. For any uncountable[3] set X,

$$\mathcal{A} = \left\{\ E \subset X \ \middle|\ E \text{ is countable or } E^c \text{ is countable}\ \right\}$$

is a σ-algebra. If E^c is countable[4], E is said to be co-countable. That $\mathcal{A}$ is a σ-algebra follows easily from the observation that, if $E_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, then

- either at least one E_m is co-countable, in which case $\bigcup_{n\in\mathbb{N}} E_n \supset E_m$ is co-countable
- or every E_n is countable, in which case $\bigcup_{n\in\mathbb{N}} E_n$ is countable.

2.3 Techniques for Building σ-algebras

- If $\mathcal{I}$ is any (index) set and, for each $i \in \mathcal{I}$, $\mathcal{A}_i$ is a σ-algebra of subsets of X then $\bigcap_{i\in\mathcal{I}} \mathcal{A}_i$ is a σ-algebra.

  Proof. ∘ For each $i \in \mathcal{I}$, we have $\emptyset \in \mathcal{A}_i$, so $\bigcap_{i\in\mathcal{I}} \mathcal{A}_i$ is not empty.

[3]This example is not interesting if X is not uncountable — then $\mathcal{A}$ reduces to $\mathcal{P}(X)$.

[4]For a review of countablility, see the notes "Cardinality". By definition, a set is *countable* if it has the same number of elements as a subset of the natural numbers. In particular, finite sets are countable.

- If $\{E_n\}_{n\in\mathbb{N}} \subset \bigcap_{i\in\mathcal{I}} \mathcal{A}_i$ then,

$$\text{for each } i \in \mathcal{I}, \{E_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_i \implies \bigcup_{n\in\mathbb{N}} E_n \subset \mathcal{A}_i$$
$$\implies \bigcup_{n\in\mathbb{N}} E_n \subset \bigcap_{i\in\mathcal{I}} \mathcal{A}_i$$

  So $\bigcap_{i\in\mathcal{I}} \mathcal{A}_i$ is closed under countable unions.
- Similarly for complements.

□

- If X is any nonempty set and $\mathcal{E} \subset \mathcal{P}(X)$, then the **$\boldsymbol{\sigma}$-algebra generated by $\mathcal{E}$** is

$$\mathcal{M}(\mathcal{E}) = \bigcap \left\{ \mathcal{A} \;\middle|\; \mathcal{A} \text{ is a } \sigma\text{-algebra containing } \mathcal{E} \right\}$$

  (Note that $\left\{ \mathcal{A} \;\middle|\; \mathcal{A} \text{ is a } \sigma\text{-algebra containing } \mathcal{E} \right\}$ is not empty — $\mathcal{P}(X)$ is always an element.) That is, $\mathcal{M}(\mathcal{E})$ is the smallest σ-algebra that contains $\mathcal{E}$. That smallest σ-algebra always exists.

Example 2.7 (Borel σ-algebra)**.** If X is a metric space (or, more generally, a topological space) then the **Borel $\boldsymbol{\sigma}$-algebra** on X, denoted $\mathcal{B}_X$, is the σ-algebra generated by the family of open subsets of X (or the family of closed subsets of X). $\mathcal{B}_X$ always contains

- all open subsets of X
- all closed subsets of X
- all countable intersections of open subsets of X (these are called[5] G_δ sets)
- all countable unions of closed subsets of X (these are called[6] F_σ sets)

Lemma 2.8. *Let X be a nonempty set.*
(a) If $\mathcal{E}, \mathcal{F} \subset \mathcal{P}(X)$ and $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$.
(b) If $\mathcal{E}, \mathcal{F} \subset \mathcal{P}(X)$ and $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$ and $\mathcal{F} \subset \mathcal{M}(\mathcal{E})$, then $\mathcal{M}(\mathcal{E}) = \mathcal{M}(\mathcal{F})$.

Proof. (a) is obvious as $\mathcal{M}(\mathcal{F})$ is a σ-algebra containing $\mathcal{E}$.
(b) is trivial from (a).

□

[5]The "G" stands for "Gebeit", which is German for "neighbourhood" and the δ stands for "Durchschnitt", which is German for "intersection".

[6]The "F" stands for "Fermé", which is French for "closed" and the σ stands for "somme", which is French for "sum" or "union".

Proposition 2.9. *$\mathcal{B}_{\mathbb{R}}$, the Borel σ-algebra on $\mathbb{R}$, is generated by*
(a) $\left\{ (a,b) \;\middle|\; a < b \right\}$
(b) $\left\{ [a,b] \;\middle|\; a < b \right\}$
(c) $\left\{ (a,b] \;\middle|\; a < b \right\}$ *or* $\left\{ [a,b) \;\middle|\; a < b \right\}$
(d) $\left\{ (a,\infty) \;\middle|\; a \in \mathbb{R} \right\}$ *or* $\left\{ (-\infty,a) \;\middle|\; a \in \mathbb{R} \right\}$
(e) $\left\{ [a,\infty) \;\middle|\; a \in \mathbb{R} \right\}$ *or* $\left\{ (-\infty,a] \;\middle|\; a \in \mathbb{R} \right\}$

Proof. (a) Let $\mathcal{O} = \left\{ A \subset \mathbb{R} \;\middle|\; A \text{ open} \right\}$ and $\mathcal{E} = \left\{ (a,b) \;\middle|\; a < b \right\}$.

- $\mathcal{M}(\mathcal{E}) \subset \mathcal{B}_{\mathbb{R}}$:

$$\mathcal{E} \subset \mathcal{O} \subset \mathcal{M}(\mathcal{O}) = \mathcal{B}_{\mathbb{R}} \implies \mathcal{M}(\mathcal{E}) \subset \mathcal{B}_{\mathbb{R}}$$

- $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E})$: As $\mathcal{M}(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$ it suffices to prove that $\mathcal{O} \subset \mathcal{M}(\mathcal{E})$. Let $A \in \mathcal{O}$. It now suffices to prove that A is a countable union of open intervals, since that implies that $A \in \mathcal{M}(\mathcal{E})$.

  Proof that A is a countable union of open intervals. Let $x \in A$.

$$\begin{aligned} A \text{ open} &\implies \exists\ a < x \text{ and } b > x \text{ such that } x \in (a,b) \subset A \\ &\implies \exists\ \rho, \sigma \in \mathbb{Q} \text{ with } a < \rho < x,\ x < \sigma < b \text{ so that } x \in (\rho,\sigma) \subset A \\ &\implies A = \bigcup_{\substack{\rho,\sigma \in \mathbb{Q} \\ \rho < \sigma \\ (\rho,\sigma) \subset A}} (\rho,\sigma) \\ &\implies A \in \mathcal{M}(\mathcal{E}) \end{aligned}$$

  since $\bigcup_{\substack{\rho,\sigma \in \mathbb{Q} \\ \rho < \sigma \\ (\rho,\sigma) \subset A}}$ is a countable union. □

(b), (c) The proofs of all three parts of (b) and (c) are very similar. The proof of one of the three parts is problem #8 of Problem Set 1.

(d), (e) The proofs of all four parts (d) and (e) are very similar. Here is the proof of the first part of (d). Let $\mathcal{O} = \left\{ A \subset \mathbb{R} \;\middle|\; A \text{ open} \right\}$ and $\mathcal{E}' = \left\{ (a,\infty) \;\middle|\; a \in \mathbb{R} \right\}$.

- $\mathcal{M}(\mathcal{E}') \subset \mathcal{B}_{\mathbb{R}}$:

$$\mathcal{E}' \subset \mathcal{O} \subset \mathcal{M}(\mathcal{O}) = \mathcal{B}_{\mathbb{R}} \implies \mathcal{M}(\mathcal{E}') \subset \mathcal{B}_{\mathbb{R}}$$

- $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}')$: Let $\mathcal{E} = \left\{ (a,b) \;\middle|\; a < b \right\}$. We already know, by part (a), that $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$, so it suffices to prove that $\mathcal{E} \subset \mathcal{M}(\mathcal{E}')$. Let $\alpha < \beta$. Then

$$(\alpha,\beta) = (-\infty,\beta) \cap (\alpha,\infty) = [\beta,\infty)^c \cap (\alpha,\infty) = \left(\bigcap_{n=1}^{\infty} \left(\beta - \tfrac{1}{n}, \infty\right) \right)^c \cap (\alpha,\infty)$$

  So $(\alpha,\beta) \in \mathcal{M}(\mathcal{E}')$. Hence $\mathcal{E} \subset \mathcal{M}(\mathcal{E}')$, so that, part (a), $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{E}')$.

□

2.4 Measures

Let X be a set.

Definition 2.10. A **measure** on the σ-algebra $\mathcal{M} \subset \mathcal{P}(X)$ is a function

$$\mu : \mathcal{M} \to [0, \infty]$$

such that

(i) $\mu(\emptyset) = 0$

(ii) If $\{E_j\}_{j \in \mathbb{N}} \subset \mathcal{M}$ is a countable collection of disjoint subsets of X, then

$$\mu\Big(\bigcup_{j \in \mathbb{N}} E_j\Big) = \sum_{j=1}^{\infty} \mu(E_j)$$

(ii) is called countable additivity.

Definition 2.11. A **finitely additive measure** on the algebra $\mathcal{A} \subset \mathcal{P}(X)$ is a function

$$\mu : \mathcal{A} \to [0, \infty]$$

such that

(i) $\mu(\emptyset) = 0$

(ii') If $\{E_1, \cdots, E_n\}$ is a finite collection of disjoint sets in $\mathcal{A}$, then

$$\mu\Big(\bigcup_{j=1}^{n} E_j\Big) = \sum_{j=1}^{n} \mu(E_j)$$

(ii') is called finite additivity.

Definition 2.12. A measure μ on the σ-algebra $\mathcal{M} \subset \mathcal{P}(X)$ is called

(i) **finite** if $\mu(X) < \infty$

(ii) **$\boldsymbol{\sigma}$-finite** if there is a countable collection $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ of subsets with $X = \bigcup_{n=1}^{\infty} E_n$ and with $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$.

(iii) **semifinite** if for each $E \in \mathcal{M}$ with $\mu(E) = \infty$, there is an $F \in \mathcal{M}$ with $0 < \mu(F) < \infty$ and $F \subset E$.

(iv) **Borel** if X is a metric space (or, more generally a topological space) and $\mathcal{M}$ is $\mathcal{B}_X$, the σ-algebra of Borel subsets of X.

Remark 2.13. (a) If μ is a finite measure on the σ-algebra $\mathcal{M} \subset \mathcal{P}(X)$ and if $E \in \mathcal{M}$, then

$$\mu(X) = \mu(E \cup E^c) = \mu(E) + \overbrace{\mu(E^c)}^{\geq 0} \implies \mu(E) \leq \mu(X) < \infty$$

(b) Here is some terminology and notation.

$$\begin{array}{rl} (X, \mathcal{M}) & \text{measurable space} \\ (X, \mathcal{M}, \mu) & \text{measure space} \end{array}$$

Example 2.14 (counting measure)**.** If X is any set and $\mathcal{M} = \mathcal{P}(X)$, then

$$\mu(E) = \#\text{ points in } E$$

is called the counting measure on X.

Example 2.15 (point mass)**.** If X is any set, $x_0 \in X$, and $\mathcal{M}$ is any σ-algebra for X, then

$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$$

is called a point mass or Dirac measure at x_0.

Example 2.16. Let X be any set, $\mathcal{M}$ be any σ-algebra for X, and $f : X \to [0, \infty]$. Then

$$\mu(E) = \sum_{x \in E} f(x) \qquad \text{for all } E \in \mathcal{M}$$

is a measure, if $\sum_{x \in E}$ is interpreted appropriately. See the definition below.

Definition 2.17 (the meaning of $\sum\limits_{x \in E} f(x)$)**.** Let E be any set and $f : E \to [0, \infty]$.

- If $f(x) = \infty$ for at least one $x \in E$, then $\displaystyle\sum_{x \in E} f(x) = \infty$

For the rest of this definition, assume that $f : E \to [0, \infty)$. Set

$$P = \left\{ \; x \in E \;\middle|\; f(x) > 0 \; \right\}$$

- If P is a finite set, then $\displaystyle\sum_{x \in E} f(x)$ is, of course, the sum of the finite number of (strictly positive) numbers in $\left\{ \; f(x) \;\middle|\; x \in P \; \right\}$.

- If P is a countable set, and $P = \{x_1, x_2, x_3, \cdots\}$ is *any* ordering for P, then

$$\sum_{x\in E} f(x) = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)$$

  The right hand side is independent of the choice of ordering. *Note* that the fact that $f(x_n) \geq 0$, for all n, is critical for this independence of ordering. See problem #1 on Problem Set 2.
- If P is an uncountable set, then $\sum_{x\in E} f(x) = \infty$. This is reasonable because there must exist an $N \in \mathbb{N}$ with $P_N = \left\{ \, x \in E \;\middle|\; f(x) \geq \frac{1}{N} \, \right\}$ infinite (because otherwise $P = \bigcup_{N\in\mathbb{N}} P_N$ is countable) and any reasonable definition of $\sum\limits_{x\in E} f(x)$ should obey

$$\sum_{x\in E} f(x) \geq \sum_{x\in P_N} f(x) \geq \sum_{x\in P_N} \frac{1}{N} = \infty$$

Example 2.18. Let X be any uncountable set and set

$$\mathcal{M} = \left\{ \, E \in \mathcal{P}(X) \;\middle|\; E \text{ is countable or co-countable} \, \right\}$$

See Example 2.5. Then

$$\mu(E) = \begin{Bmatrix} 0 & \text{if } E \text{ is countable} \\ 1 & \text{if } E \text{ is co-countable} \end{Bmatrix} \qquad \text{for all } E \in \mathcal{M}$$

is a measure. This is because

- If E_1, E_2, E_3, $\cdots$ are all countable, then $\bigcup_{j\in\mathbb{N}} E_j$ is also countable so that

$$\sum_{j\in\mathbb{N}} \overbrace{\mu(E_j)}^{=0} = 0 \qquad \text{and} \qquad \mu\Big(\bigcup\nolimits_{j\in\mathbb{N}} E_j\Big) = 0$$

- If E_1, E_2, E_3, $\cdots$ are all countable, except for E_{j_0}, which is co-countable, then $\bigcup_{j\in\mathbb{N}} E_j$ is co-countable so that

$$\sum_{j\in\mathbb{N}} \mu(E_j) = \overbrace{\mu(E_{j_0})}^{=1} + \sum_{\substack{j\in\mathbb{N} \\ j\neq j_0}} \overbrace{\mu(E_j)}^{=0} = 1 \qquad \text{and} \qquad \mu\Big(\bigcup\nolimits_{j\in\mathbb{N}} E_j\Big) = 1$$

- It is impossible to have two disjoint co-countable sets E_1, E_2 because $E_2 \subset E_1^c$.

Example 2.19. Let X be any infinite set and set $\mathcal{M} = \mathcal{P}(X)$ and

$$\mu(E) = \begin{Bmatrix} 0 & \text{if } E \text{ is finite} \\ \infty & \text{if } E \text{ is infinite} \end{Bmatrix} \qquad \text{for all } E \in \mathcal{M}$$

This is a finitely additive measure, but not a measure.

Theorem 2.20. *Let $(X, \mathcal{M}, \mu)$ be a measure space and E, F, E_1, E_2, $\cdots \in \mathcal{M}$.*

*(a) (***Monotonicity***) If $E \subset F$, then $\mu(E) \leq \mu(F)$.*

*(b) (***Subadditivity***) $\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq \sum_{n=1}^{\infty} \mu(E_n)$ (Note that the E_n's need not be disjoint.)*

*(c) (***Continuity from below***) If $E_1 \subset E_2 \subset E_3 \ \cdots$, then*

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n)$$

*(d) (***Continuity from above***) If $\mu(E_1) < \infty$ and $E_1 \supset E_2 \supset E_3 \ \cdots$, then*

$$\mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n)$$

Proof. (a) $\quad \mu(F) = \mu\big(E \cup (F \setminus E)\big) = \mu(E) + \overbrace{\mu(F \setminus E)}^{\geq 0} \geq \mu(E)$

(b) Set $F_1 = E_1$, $F_2 = E_2 \setminus E_1$, $F_3 = E_3 \setminus (E_1 \cup E_2)$, $\cdots$, $F_k = E_k \setminus \big(\cup_{n=1}^{k-1} E_n\big)$, $\cdots$. Then the F_k's are all disjoint and, for all $N \in \mathbb{N}$, we have $\cup_{n=1}^{N} F_n = \cup_{n=1}^{N} E_n$, so that

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n \implies \mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \mu\Big(\bigcup_{n=1}^{\infty} F_n\Big) = \sum_{n=1}^{\infty} \mu(F_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

(c) Writing

$$\bigcup_{n=1}^{\infty} E_n = E_1 \cup \Big(\bigcup_{j=2}^{\infty} E_j \setminus E_{j-1}\Big)$$

we have

$$\begin{aligned} \mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) &= \mu(E_1) + \sum_{j=2}^{\infty} \mu\big(E_j \setminus E_{j-1}\big) \\ &= \lim_{n\to\infty} \Bigg[\overbrace{\mu(E_1) + \sum_{j=2}^{n} \mu\big(E_j \setminus E_{j-1}\big)}^{\mu\left(\bigcup_{j=1}^{n} E_j\right)} \Bigg] \\ &= \lim_{n\to\infty} \mu(E_n) \end{aligned}$$

(d) Set, for $n \in \mathbb{N}$, $F_n = E_1 \setminus E_n$. Then $F_1 \subset F_2 \subset F_3 \subset \cdots$ so that, by part (c),

$$\mu\Big(\bigcup_{n=1}^{\infty} F_n\Big) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu(E_1 \setminus E_n)$$

Subtracting $\mu(E_n)$ from both sides of $\mu(E_1) = \mu(E_n) + \mu(E_1 \setminus E_n)$ (which we are allowed to do *only* because we know that $\mu(E_n)$ is finite), we have $\mu(E_1 \setminus E_n) = \mu(E_1) - \mu(E_n)$ and hence

$$\mu\Big(\bigcup_{n=1}^{\infty} F_n\Big) = \lim_{n\to\infty} \big[\mu(E_1) - \mu(E_n)\big] = \mu(E_1) - \lim_{n\to\infty} \mu(E_n) \qquad (*)$$

As $F_n = E_1 \cap E_n^c$, we have

$$\bigcup_{n=1}^{\infty} F_n = E_1 \cap \Big(\bigcup_{n=1}^{\infty} E_n^c\Big) = E_1 \cap \Big(\bigcap_{n=1}^{\infty} E_n\Big)^c = E_1 \setminus \Big(\bigcap_{n=1}^{\infty} E_n\Big)$$

So the left hand side of $(*)$ is

$$\mu\Big(\bigcup_{n=1}^{\infty} F_n\Big) = \mu\Big(E_1 \setminus \Big(\bigcap_{n=1}^{\infty} E_n\Big)\Big) = \mu(E_1) - \mu\Big(\bigcap_{n=1}^{\infty} E_n\Big)$$

and the claim follows. □

Example 2.21. The hypothesis that $\mu(E_1) < \infty$ in part (d) of Theorem 2.20 (continuity from above) really is needed. For example if

$$\begin{aligned} X &= \mathbb{N} \\ \mathcal{M} &= \mathcal{P}(X) \\ \mu &= \text{counting measure} \\ E_n &= \big\{\ j \in \mathbb{N} \ \big|\ j \geq n\ \big\} \qquad \text{for each } n \in \mathbb{N} \end{aligned}$$

then

$$\mu(E_n) = \infty \text{ for all } n \in \mathbb{N} \quad \text{so that} \quad \lim_{n\to\infty} \mu(E_n) = \infty$$

but

$$\bigcap_{n=1}^{\infty} E_n = \emptyset \quad \text{so that} \quad \mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = 0$$

Definition 2.22. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) A **null set** is a set $E \in \mathcal{M}$ with $\mu(E) = 0$.

(b) If $f : X \to \{\text{true}, \text{false}\}$ is a statement about points of X and

$$\mu\big(\big\{\ x \in X \ \big|\ f(x) = \text{false}\ \big\}\big) = 0$$

then we say that f is true **almost everywhere** (with respect to μ). It is standard to abbreviate "almost everywhere" by "**a.e.**".

Remark 2.23. The following "silliness" is completely consistent with all of our definitions so far.

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $N \in \mathcal{M}$ obey $\mu(N) = 0$ (i.e. N is a null set). As N has "size" zero, we would certainly expect that any smaller subset $Z \subset N$ would also have "size" zero. *But* we have *not* required that

$$N \in \mathcal{M},\ \mu(N) = 0,\ Z \subset N \implies Z \in \mathcal{M}$$

So it is possible that $\mu(Z)$ is not defined, even though it ought to be zero. That's silly.

Here is an extreme example of this.

$$X = \text{any nonempty set} \qquad \mathcal{M} = \{\emptyset, X\} \qquad \mu(\emptyset) = \mu(X) = 0$$

The entire world has size zero, and yet not a single nontrivial subset has a defined size.

Definition 2.24. A measure space $(X, \mathcal{M}, \mu)$ is said to be **complete** if

$$Z \subset N \in \mathcal{M},\ \mu(N) = 0 \implies Z \in \mathcal{M}$$

The following theorem says that the silliness of Remark 2.23 can always be fixed — any measure can be completed.

Theorem 2.25 (Completion)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Set*

$$\begin{aligned}
\mathcal{N} &= \left\{\ N \in \mathcal{M} \ \middle|\ \mu(N) = 0\ \right\} \\
\overline{\mathcal{M}} &= \left\{\ E \cup Z \ \middle|\ E \in \mathcal{M},\ Z \subset N \text{ for some } N \in \mathcal{N}\ \right\} \\
\bar{\mu} &: \overline{\mathcal{M}} \to [0, \infty] \text{ with} \\
\bar{\mu}(E \cup Z) &= \mu(E) \text{ for all } E \in \mathcal{M} \text{ and } Z \subset N \text{ for some } N \in \mathcal{N}
\end{aligned}$$

Then

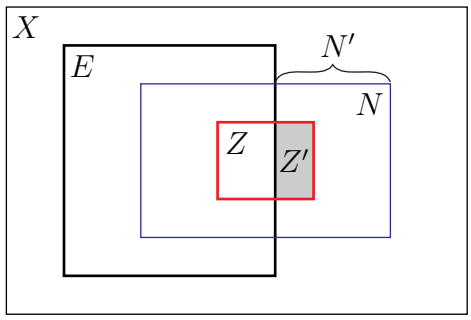
(a) $\overline{\mathcal{M}}$ is a σ-algebra.

(b) $\bar{\mu}$ is a well-defined, complete measure on $\overline{\mathcal{M}}$, called the completion of μ.

(c) $\bar{\mu}$ is the unique extension of μ to $\overline{\mathcal{M}}$. That is, $\bar{\mu}$ is the unique measure with domain of definition $\overline{\mathcal{M}}$ that obeys $\bar{\mu}(E) = \mu(E)$ for all $E \in \mathcal{M}$.

Proof.

- $\overline{\mathcal{M}}$ *is a σ-algebra*:
  - Both $\mathcal{M}$ and $\mathcal{N}$ are closed under countable unions. So $\overline{\mathcal{M}}$ is closed under countable unions.
  - Let $E \in \mathcal{M}$ and $Z \subset N \in \mathcal{N}$. We must show that $(E \cup Z)^c \in \overline{\mathcal{M}}$. That is, we must express $(E \cup Z)^c$ as the union of a set in $\mathcal{M}$ and a subset of a null set. Define
  $$N' = N \setminus E \in \mathcal{M} \qquad Z' = Z \setminus E \subset N'$$
  In the figure below, Z is outlined in red and Z' is shaded.



  Then X is the disjoint union of E, Z', $N' \setminus Z'$ and $(E \cup N')^c$ and $\mu(N') = 0$ so that
  $$(E \cup Z)^c = (E \cup Z')^c = \overbrace{(E \cup N')^c}^{\in \mathcal{M}} \cup \overbrace{(N' \setminus Z)}^{\subset N'} \in \overline{\mathcal{M}}$$

- $\bar{\mu}$ *is well-defined*:
  Let
  $$E, E' \in \mathcal{M}, \quad Z \subset N \in \mathcal{N}, \quad Z' \subset N' \in \mathcal{N} \quad \text{with} \quad E \cup Z = E' \cup Z'$$
  We need to prove that $\bar{\mu}(E \cup Z) = \bar{\mu}(E' \cup Z')$, i.e. that $\mu(E) = \mu(E')$. As $E \cup Z = E' \cup Z'$,
  $$\mu(E) = \mu(E \cap E') + \overbrace{\mu(E \setminus E')}^{\subset Z' \subset N'} = \mu(E \cap E')$$
  $$\mu(E') = \mu(E' \cap E) + \underbrace{\mu(E' \setminus E)}_{\subset Z \subset N} = \mu(E \cap E')$$

- *$\bar{\mu}$ is a complete measure*:

  This is problem #8 on Problem Set 2.

- *Unique extension*:

  Let μ' be any extension of μ to $\overline{\mathcal{M}}$. We have to show that $\mu' = \bar{\mu}$.

  - If $E \in \mathcal{M}$, then $\mu'(E) = \mu(E)$ by hypothesis.
  - If $Z \subset N \in \mathcal{N}$, then $\mu'(Z) \leq \mu'(N) = \mu(N) = 0$.
  - In general, if $\bar{E} = E \cup Z \in \overline{\mathcal{M}}$, with $E \in \mathcal{M}$, $Z \subset N \in \mathcal{N}$, then

$$\begin{aligned} &\mu(E) = \mu'(E) \leq \mu'(\bar{E}) \leq \mu'(E \cup N) \leq \overbrace{\mu'(E)}^{=\mu(E)} + \overbrace{\mu'(N)}^{=0} = \mu(E) \\ &\implies \mu'(\bar{E}) = \mu(E) = \bar{\mu}(\bar{E}) \end{aligned}$$

$\square$

2.5 Outer Measures

We now have a reasonable abstraction of "volume" — namely measure. But, so far, we have only pretty trivial and artificial examples. We now start building the procedure that we'll use to define the "Lebesgue measure", which is our usual notion of volume in $\mathbb{R}^d$. The strategy that we'll use to define the volume of a set $A \subset \mathbb{R}^d$ is

- approximate the volume of A from above by covering A by a union of (at most countably many) rectangles. Certainly the volume of A is at most the sum of the volumes of the rectangles.
- Make the rectangles finer and finer. Define the

$$\text{“outer volume of } A\text{”} = \inf \left\{ \ \mathrm{vol}(S) \ \middle| \ S \text{ simple}\,,\ A \subset S \ \right\}$$

- approximate the volumes of A from below by the volumes of simple sets contained in A. Define the

$$\text{“inner volume of } A\text{”} = \sup \left\{ \ \mathrm{vol}(S) \ \middle| \ S \text{ simple}\,,\ S \subset A \ \right\}$$

- If the "inner volume" equals the "outer volume", we have the volume of A.

We now abstract these ideas.

Definition 2.26. An **outer measure** on X is a function

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

such that

(i) $\mu^*(\emptyset) = 0$

(ii) If $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$.

(iii) If $\{A_j\}_{j\in\mathbb{N}}$ is a countable collection of (not necessarily disjoint) subsets of X, then

$$\mu^*\Big(\textstyle\bigcup_{j=1}^{\infty} A_j\Big) \leq \displaystyle\sum_{j=1}^{\infty} \mu^*(A_j)$$

This is called countable subadditivity.

Proposition 2.27. *Let X be a set, $\mathcal{S} \subset \mathcal{P}(X)$ and $\rho : \mathcal{S} \to [0, \infty]$ be such that*

$$\emptyset \in \mathcal{S} \qquad X \in \mathcal{S} \qquad \rho(\emptyset) = 0$$

Define, for each $A \subset X$,

$$\mu^*(A) = \inf \left\{ \textstyle\sum_{j=1}^{\infty} \rho(S_j) \,\middle|\, \{S_j\}_{j\in\mathbb{N}} \subset \mathcal{S},\ A \subset \bigcup_{j=1}^{\infty} S_j \right\}$$

Then μ^* is an outer measure.

Example 2.28 (Lebesgue outer measure on $\mathbb{R}$)**.**

$$X = \mathbb{R}$$
$$\mathcal{S} = \{\emptyset, \mathbb{R}\} \cup \left\{ \ (a,b) \ \middle| \ a,b \in \mathbb{R},\ a < b \ \right\} \cup \left\{ \ (a,\infty),\ (-\infty,a) \ \middle| \ a \in \mathbb{R} \ \right\}$$
$$\rho(\emptyset) = 0 \qquad \rho\big((a,b)\big) = b - a \qquad \rho\big((a,\infty)\big) = \rho\big((-\infty,a)\big) = \rho\big(\mathbb{R}\big) = \infty$$

Proof of Proposition 2.27.

- That $\mu^*(\emptyset) = 0$ is obvious since $\emptyset \in \mathcal{S}$ and $\rho(\emptyset) = 0$.
- Let $A \subset B$. We need to show that $\mu^*(A) \le \mu^*(B)$.
  If $\{S_j\}_{j\in\mathbb{N}} \subset \mathcal{S}$ with $B \subset \bigcup_{j=1}^\infty S_j$. Then we also have $A \subset \bigcup_{j=1}^\infty S_j$. Consequently,

$$\left\{ \sum_{j=1}^\infty \rho(S_j) \,\middle|\, \{S_j\}_{j\in\mathbb{N}} \subset \mathcal{S},\ B \subset \bigcup_{j=1}^\infty S_j \right\} \subset \left\{ \sum_{j=1}^\infty \rho(S_j) \,\middle|\, \{S_j\}_{j\in\mathbb{N}} \subset \mathcal{S},\ A \subset \bigcup_{j=1}^\infty S_j \right\}$$

  so that

$$\mu^*(B) = \inf \left\{ \sum_{j=1}^\infty \rho(S_j) \,\middle|\, \{S_j\}_{j\in\mathbb{N}} \subset \mathcal{S},\ B \subset \bigcup_{j=1}^\infty S_j \right\} \ge \inf \left\{ \sum_{j=1}^\infty \rho(S_j) \,\middle|\, \{S_j\}_{j\in\mathbb{N}} \subset \mathcal{S},\ A \subset \bigcup_{j=1}^\infty S_j \right\} = \mu^*(A)$$

- Countable subadditivity
  - Let $\varepsilon > 0$. Let, for each $n \in \mathbb{N}$, $A^{(n)} \subset X$ with $\mu^*(A^{(n)}) < \infty$. (The other case is trivial.)
  - For each $n \in \mathbb{N}$, there exists $\{S_j^{(n)}\}_{j\in\mathbb{N}} \subset \mathcal{S}$ such that $A^{(n)} \subset \bigcup_{j=1}^\infty S_j^{(n)}$ and $\sum_{j=1}^\infty \rho(S_j^{(n)}) \le \mu^*(A^{(n)}) + \frac{\varepsilon}{2^n}$.
  - Then $\{S_j^{(n)}\}_{j,n\in\mathbb{N}} \subset \mathcal{S}$ and $\bigcup_{n\in\mathbb{N}} A^{(n)} \subset \bigcup_{j,n\in\mathbb{N}} S_j^{(n)}$ so that

$$\mu^*\Big(\bigcup_{n=1}^\infty A^{(n)}\Big) \le \sum_{j,n\in\mathbb{N}} \rho\big(S_j^{(n)}\big) \le \sum_{n=1}^\infty \Big(\mu^*(A^{(n)}) + \frac{\varepsilon}{2^n}\Big) = \varepsilon + \sum_{n=1}^\infty \mu^*(A^{(n)})$$

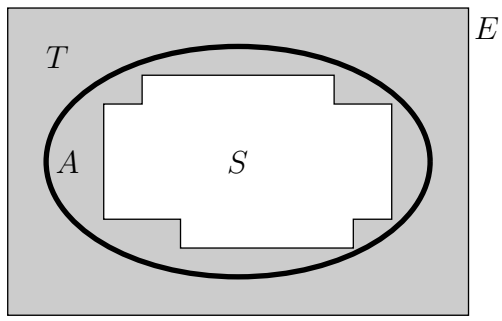
    As this is true for all $\varepsilon > 0$, we have

$$\mu^*\Big(\bigcup_{n=1}^\infty A^{(n)}\Big) \le \sum_{n=1}^\infty \mu^*(A^{(n)})$$

$\square$

Next we'll have some motivation for a trick which will allow us to avoid repeating the previous arguments for "inner measure". Let $A \subset \mathbb{R}^n$. Choose a definition of a "simple set" (for example a countable union of rectangles) covering A. Fix a simple set E with $A \subset E \subset \mathbb{R}^n$. For this motivation, we'll restrict to bounded A's and E's. Then define

$$S \text{ to be a simple set with } S \subset A \iff T = E \setminus S \text{ is a simple set covering } E \setminus A$$



Then we'll define

$$\begin{aligned}
\text{inner measure}(A) &= \sup\left\{ \ \mathrm{vol}(S) \ \middle| \ S \text{ simple set with } S \subset A \ \right\} \\
&= \sup\left\{ \ \mathrm{vol}(E \setminus T) \ \middle| \ T \text{ simple set with } E \setminus A \subset T \ \right\} \\
&= \sup\left\{ \ \mathrm{vol}(E) - \mathrm{vol}(T) \ \middle| \ T \text{ simple set with } E \setminus A \subset T \ \right\} \\
&= \mathrm{vol}(E) - \inf\left\{ \ \mathrm{vol}(T) \ \middle| \ T \text{ simple set with } E \setminus A \subset T \ \right\} \\
&= \mu^*(E) - \mu^*(E \setminus A)
\end{aligned}$$

and A will have a well–defined volume if

$$\begin{aligned}
&\text{inner measure}(A) = \text{outer measure}(A) \\
\text{i.e.} \quad &\mu^*(E) - \mu^*(E \setminus A) = \mu^*(A) \\
\text{i.e.} \quad &\mu^*(A) + \mu^*(E \setminus A) = \mu^*(E)
\end{aligned}$$

We'll use something like this as the definition of "A is measurable". Technically, it is convenient to *not* restrict to a single[7] E or even to E's that are nice sets or that contain all of A. Here is the standard definition.

[7]For example, if we were restrict to a single E, there would be the danger that whether or not "inner measure(A) = outer measure(A)" is true could depend on the choice of E.

Definition 2.29. Let μ^* be an outer measure on a set X. A subset $A \subset X$ is said to be μ^*-**measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subset X$$

Remark 2.30.

(a) That $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ is always true, by axiom (iii) of Definition 2.26.

(b) That $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ is trivial for $\mu^*(E) = \infty$. So, to verify that A is measurable, it suffices to verify $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all E with $\mu^*(E) < \infty$.

(c) We will show, in Problem Set 4 #2, that if μ^* is induced from a premeasure (to be defined - most of our measures will be) and if $\mu^*(X)$ is finite, then A is μ^*-measurable if and only if $\mu^*(A) + \mu^*(X \setminus A) = \mu^*(X)$.

THEOREM 2.31 (Carathéodory[8]). *Let μ^* be an outer measure on the set X and*

$$\mathcal{M}^* = \left\{ \ A \subset X \ \middle| \ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \forall \ E \subset X \ \right\}$$

be the set of μ^*-measurable subsets of X. Then

(a) $\mathcal{M}^*$ is a σ-algebra.

(b) The restriction, $\mu^* \restriction \mathcal{M}^*$, of μ^* to $\mathcal{M}^*$ is a complete measure.

(c) If $N \subset X$ obeys $\mu^*(N) = 0$, then $N \in \mathcal{M}^*$.

Proof.

- *Step 1*: $\mathcal{M}^*$ is nonempty
  since $\emptyset \in \mathcal{M}^*$ since $\mu^*(E) = \underbrace{\mu^*(E \cap \emptyset)}_{=0} + \underbrace{\mu^*(E \cap \emptyset^c)}_{=\mu^*(E)}$ for all $E \subset X$.
- *Step 2*: $\mathcal{M}^*$ is closed under complements.
  This is obvious since

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \iff \mu^*(E) = \mu^*(E \cap A^c) + \mu^*(E \cap \underbrace{\{A^c\}^c}_{A})$$

- *Step 3*: $\mathcal{M}^*$ is closed under finite unions.
  It suffices to prove that $A, B \in \mathcal{M}^* \implies A \cup B \in \mathcal{M}^*$.

[8] Constantin Carathéodory (1873–1950) was the son of a Greek diplomat who spent most of his career in Germany.

To do so, it suffices to prove that

$$A, B \in \mathcal{M}^*,\ E \subset X \implies \mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap A^c \cap B^c)$$

But, if $A, B \in \mathcal{M}^*$ and $E \subset X$, then

$$\begin{aligned}
\mu^*(E) &\overset{A \in \mathcal{M}^*}{=} \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&\overset{B \in \mathcal{M}^*}{=} \overbrace{\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c)}^{\text{using } E \cap A \text{ as } E} + \overbrace{\mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c)}^{\text{using } E \cap A^c \text{ as } E} \\
&\geq \ \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)
\end{aligned}$$

since $A \cup B = (A \cap B) \ \cup \ (A \cap B^c) \ \cup \ (A^c \cap B)$ and $(A \cup B)^c = A^c \cap B^c$.

- *Step 4*: μ^* is finitely additive on $\mathcal{M}^*$.

  Let $A, B \in \mathcal{M}^*$ with $A \cap B = \emptyset$. We have to prove that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$. For later use, we prove more generally, that, for any $\tilde{E} \subset X$,

$$\begin{aligned}
\mu^*\big(\tilde{E} \cap (A \cup B)\big) &\overset{A \in \mathcal{M}^*}{=} \mu^*\big(\tilde{E} \cap \overbrace{(A \cup B) \cap A}^{=A}\big) + \mu^*\big(\tilde{E} \cap \overbrace{(A \cup B) \cap A^c}^{=B}\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(using } \tilde{E} \cap (A \cup B) \text{ as } E) \\
&= \mu^*(\tilde{E} \cap A) + \mu^*(\tilde{E} \cap B) \qquad\qquad\qquad\qquad\qquad\qquad (*)
\end{aligned}$$

  Just choosing $\tilde{E} = X$ gives $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

- *Step 5*: $\mathcal{M}^*$ is a σ-algebra.

  By Problem Set 1, #2, it suffices to prove that $\mathcal{M}^*$ is closed under countable disjoint unions. Let $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{M}^*$ with $A_n \cap A_m = \emptyset$ for all $m \neq n$. Let $E \subset X$. As $\mathcal{M}^*$ is closed under finite unions, we have that, for each $n \in \mathbb{N}$, $\bigcup_{j=1}^n A_j \in \mathcal{M}^*$ so that

$$\begin{aligned}
\mu^*(E) &= \mu^*\Big(E \cap \Big(\bigcup_{j=1}^n A_j\Big)\Big) + \mu^*\Big(E \cap \Big(\bigcup_{j=1}^n A_j\Big)^c\Big) \\
&= \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*\Big(E \cap \Big(\bigcup_{j=1}^n A_j\Big)^c\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by } (*) \text{ in step 4)} \\
&\geq \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*\Big(E \cap \Big(\bigcup_{j=1}^\infty A_j\Big)^c\Big)
\end{aligned}$$

Taking the limit $n \to \infty$,

$$\begin{aligned}
\mu^*(E) &\geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*\Big(E \cap \Big(\textstyle\bigcup_{j=1}^{\infty} A_j\Big)^c\Big) \\
&\geq \mu^*\Big(E \cap \Big(\textstyle\bigcup_{j=1}^{\infty} A_j\Big)\Big) + \mu^*\Big(E \cap \Big(\textstyle\bigcup_{j=1}^{\infty} A_j\Big)^c\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by subadditivity)} \\
&\geq \mu^*(E)
\end{aligned}$$

So both of the $\geq$'s in the last displayed equation must be equalities and $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}^*$.

- *Step 6*: $\mu^* \restriction \mathcal{M}^*$ is countably additive.

  Just take $E = X$ in Problem Set 3, #2.

- *Step 7*: $\mu^* \restriction \mathcal{M}^*$ is complete.

  Let $N \in \mathcal{M}^*$ with $\mu^*(N) = 0$ and let $Z \subset N$. We must show that $Z \in \mathcal{M}^*$, i.e. that $\mu^*(E) = \mu^*(E \cap Z) + \mu^*(E \cap Z^c)$ for all $E \subset X$. So let $E \subset X$. Then

$$\begin{aligned}
\mu^*(E) &\leq \mu^*\big(\overbrace{E \cap Z}^{\subset N}\big) + \mu^*\big(\overbrace{E \cap Z^c}^{\subset E}\big) \\
&\leq \mu^*(N) + \mu^*(E) \\
&= \mu^*(E)
\end{aligned} \tag{$**$}$$

  which forces the first $\leq$ above to be $=$.

- *Step 8*: $\mu^*(N) = 0 \implies N \in \mathcal{M}^*$.

  Note that, in Step 7, we did not use that $N \in \mathcal{M}^*$ in $(**)$. We only used that $\mu^*(N) = 0$. So taking $Z = N$ in $(**)$ shows that if $\mu^*(N) = 0$, then $N \in \mathcal{M}^*$.

$\square$

2.6 Premeasures and Lebesgue Measure

Typically, when you wish to construct a measure, you have

- a collection of really nice sets and
- you know what measure (i.e. volume) you want to assign to each of these really nice sets and

you want to extend this to a full measure. Here are two examples. After the examples, we'll formalize this framework.

Example 2.32.

(a) (will lead to "Lebesgue measure on $\mathbb{R}$")

$$\begin{aligned}
\{\text{really nice sets}\} &= \mathcal{A} \\
&= \{\emptyset\} \cup \Big\{ \bigcup_{j=1}^{n} (a_j, b_j] \,\Big|\, -\infty \le a_1 < b_1 < a_2 < \cdots < b_n \le \infty,\ n \in \mathbb{N} \Big\} \\
\mu_0(\emptyset) &= 0 \\
\mu_0\Big(\textstyle\bigcup_{j=1}^{n} (a_j, b_j]\Big) &= \sum_{j=1}^{n} (b_j - a_j) \quad \text{for all } -\infty \le a_1 < b_1 < a_2 < \cdots < b_n \le \infty
\end{aligned}$$

with $(a, \infty]$ defined to be (a, ∞)

(b) (will lead to "Lebesgue-Stieltjes measures on $\mathbb{R}$") Let $F : \mathbb{R} \to \mathbb{R}$ be a function with some properties that we will specify at the end of this example.

$$\begin{aligned}
\{\text{really nice sets}\} &= \mathcal{A} \\
&= \{\emptyset\} \cup \Big\{ \bigcup_{j=1}^{n} (a_j, b_j] \,\Big|\, -\infty \le a_1 < b_1 < a_2 < \cdots < b_n \le \infty,\ n \in \mathbb{N} \Big\} \\
\mu_0(\emptyset) &= 0 \\
\mu_0\Big(\textstyle\bigcup_{j=1}^{n} (a_j, b_j]\Big) &= \sum_{j=1}^{n} \big(F(b_j) - F(a_j)\big) \quad \text{for all } -\infty \le a_1 < b_1 < a_2 < \cdots < b_n \le \infty
\end{aligned}$$

Let's think about what conditions we need to impose on the function F. Suppose that we succeed in extending μ_0 to a measure μ. I.e. suppose that we find a σ-algebra $\mathcal{M}$ and a measure μ with

(i) $\mathcal{A} \subset \mathcal{M}$

(ii) $\mu(A) = \mu_0(A)$ for all $A \in \mathcal{A}$.

Then

- $\mathcal{M}$ must contain all Borel sets (since the σ-algebra generated by the $(a,b]$'s is $\mathcal{B}_{\mathbb{R}}$)
- $F(b)-F(a)=\mu_0\big((a,b]\big)=\mu\big(a,b]\big)\geq 0$ for all $b>a$. That is, F must be nondecreasing.
- By continuity from above

$$\begin{aligned}\lim_{n\to\infty}\big[F\big(a+\tfrac{1}{n}\big)-F(a)\big]&=\lim_{n\to\infty}\mu_0\big(\big(a,a+\tfrac{1}{n}\big]\big)\\&=\lim_{n\to\infty}\mu\big(\big(a,a+\tfrac{1}{n}\big]\big)\\&=\mu\Big(\textstyle\bigcap_{n=1}^{\infty}\big(a,a+\tfrac{1}{n}\big]\Big)\\&=\mu(\emptyset)=0\end{aligned}$$

  That is, F must be continuous from the right.
- Recall that, in $\mu_0\big((a,b]\big)=F(b)-F(a)$, $a=-\infty$ and $b=+\infty$ are allowed. So we need to define $F(-\infty)$ and $F(+\infty)$. By countable additivity,

$$\begin{aligned}F(\infty)-F(0)=\mu_0\big((0,\infty]\big)=\mu\big((0,\infty]\big)&=\sum_{j=1}^{\infty}\mu\big((j-1,j]\big)\\&=\sum_{j=1}^{\infty}\mu_0\big((j-1,j]\big)=\sum_{j=1}^{\infty}\big(F(j)-F(j-1)\big)\\&=\lim_{n\to\infty}\sum_{j=1}^{n}\big(F(j)-F(j-1)\big)\\&=\lim_{n\to\infty}\big(F(n)-F(0)\big)\end{aligned}$$

  and

$$\begin{aligned}F(0)-F(-\infty)=\mu_0\big((-\infty,0]\big)=\mu\big(-\infty,0]\big)&=\sum_{j=0}^{\infty}\mu\big((-j-1,-j]\big)\\&=\sum_{j=0}^{\infty}\mu_0\big((-j-1,-j]\big)=\sum_{j=0}^{\infty}\big(F(-j)-F(-j-1)\big)\\&=\lim_{n\to\infty}\sum_{j=0}^{n-1}\big(F(-j)-F(-j-1)\big)\\&=\lim_{n\to\infty}\big(F(0)-F(-n)\big)\end{aligned}$$

So we define

$$F(\infty) = \lim_{x\to\infty} F(x) \qquad F(-\infty) = \lim_{x\to-\infty} F(x)$$

Note that, as F is nondecreasing, both of these limits exist, though the first could be $+\infty$ and the second could be $-\infty$.

We can now fill in the needed properties of F. We should have started this example (b) like this:

(b) (will lead to "Lebesgue-Stieltjes measures on $\mathbb{R}$") Let $F : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and right continuous function, and define

$$F(\infty) = \lim_{x\to\infty} F(x) \qquad \text{(possibly } +\infty)$$
$$F(-\infty) = \lim_{x\to-\infty} F(x) \qquad \text{(possibly } -\infty)$$

Here is a definition which formalizes the above setting.

Definition 2.33. Let X be a set. A **premeasure** on the algebra $\mathcal{A} \subset \mathcal{P}(X)$ is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ such that

(i) $\mu_0(\emptyset) = 0$

(ii) If $\{A_j\}_{j\in\mathbb{N}}$ is a countable collection of disjoint subsets of X with $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{A}$ *and if* $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$, then

$$\mu_0\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu_0(A_j)$$

(This includes finite additivity, since we can choose many $A_j = \emptyset$.)

Remark 2.34. Our goal here is to rig the definitions of algebra and premeasure so that

- it is easy to construct algebras $\mathcal{A}$ and premeasures μ_0 on $\mathcal{A}$ and yet
- it is possible to enlarge an algebra $\mathcal{A}$ to a σ-algebra $\mathcal{M}$ and to extend the premeasure μ_0 to a full measure μ on $\mathcal{M}$.

Now a countable union $\bigcup_{j=1}^{\infty} A_j$ of disjoint subsets of X with $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{A}$ is not required to be in $\mathcal{A}$. But there is no law that says that it can't be in $\mathcal{A}$. And *if* $\bigcup_{j=1}^{\infty} A_j$ does happen to be in $\mathcal{A}$ and *if* μ_0 can be extended to a measure μ on a σ-algebra $\mathcal{M} \supset \mathcal{A}$, then we do have to have

$$\mu_0\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j)$$

That's the reason for part (ii) of Definition 2.33.

Proposition 2.35. *Let $F : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous. Define $F(\pm\infty) = \lim_{x\to\pm\infty} F(x)$. Set*

$$\mathcal{A} = \{\emptyset\} \cup \left\{ \bigcup_{j=1}^{n} (a_j, b_j] \,\middle|\, n \in \mathbb{N},\ -\infty \le a_1 < b_1 < a_2 < \cdots < b_n \le \infty \right\}$$

$$\mu_0(\emptyset) = 0$$

$$\mu_0\Big(\textstyle\bigcup_{j=1}^{n}(a_j, b_j]\Big) = \sum_{j=1}^{n} \big[F(b_j) - F(a_j)\big]$$

$$\textit{for all } n \in \mathbb{N},\ -\infty \le a_1 < b_1 < a_2 < b_2 < \cdots < b_n \le \infty$$

In the above, replace $(a, b]$ by (a, b) when $b = \infty$. Then μ_0 is a premeasure on $\mathcal{A}$.

Proof.

- *Step 1*: $\mathcal{A}$ is an algebra
  - $\mathcal{A}$ is closed under complements

    since $\emptyset^c = (-\infty, \infty]$ and

    $$\Big(\textstyle\bigcup_{j=1}^{n}(a_j, b_j]\Big)^c = (-\infty, a_1] \cup (b_1, a_2] \cup \cdots (b_{n-1}, a_n] \cup (b_n, \infty]$$

  - $\mathcal{A}$ is closed under finite unions.

    To verify that any finite union of $(\alpha, \beta]$'s can be expressed as a disjoint (with gaps) finite union, repeatedly apply

    $$\begin{aligned}&\text{if } a < b,\ a' < b' \text{ and } (a, b] \cap (a', b'] \ne \emptyset \text{ or } b = a' \text{ ar } a = b' \\ &\text{then } (a, b] \cup (a', b'] = \big(\min\{a, a'\}\,,\ \max\{b, b'\}\big]\end{aligned}$$

- *Step 2*: $\mu_0(\emptyset) = 0$

  by hypothesis.

- *Step 3*: μ_0 is countably additive where defined.

  Let $\{I_m\}_{m\in\mathbb{N}} \subset \mathcal{A}$ with $I_m \cap I_{m'} = \emptyset$ for $m \ne m'$, and with $I = \bigcup_{m=1}^{\infty} I_m$ being in $\mathcal{A}$. We have to show that $\mu_0(I) = \sum_{m=1}^{\infty} \mu_0(A_m)$.

  The reason that this is not trivial is that $\{I_m\}_{m\in\mathbb{N}}$ could contain a countable collection $\big(\alpha + \frac{1}{2^{n+1}}, \alpha + \frac{1}{2^n}\big]$, $n \ge n_0$, of shorter and shorter intervals that "accumulate at $\alpha \in I$". Furthermore, there can be a countable number of different such "accumulation" points in I.

First we make some reductions.

It suffices to consider the case in which none of I_m's is empty. Otherwise, drop them.

It suffices to consider the case in which each I_m is a single interval. Otherwise replace $\{I_m\}_{m\in\mathbb{N}}$ by $\{\ J \mid J \text{ is an interval in some } I_m\ \}$.

It suffices to consider the case in which I is a single interval. Otherwise consider separately each interval $\bar{I}$ of I together with $\{\ I_m \mid I_m \subset \bar{I}\ \}$. (Recall that, by hypothesis, $I \in \mathcal{A}$.)

The case in which there are only finitely many I_m's is easy. So let $I = (a, b]$ and $I_m = (a_m, b_m]$, $m \in \mathbb{N}$, with $I_m \cap I_{m'} = \emptyset$ for $m \neq m'$ and with $I = \bigcup_{m=1}^{\infty} I_m$. We have to show that $\sum_{m=1}^{\infty} \big(F(b_m) - F(a_m)\big) = F(b) - F(a)$.

- *Step 3a*: $\mu_0(I) \geq \sum_{m=1}^{\infty} \mu_0(I_m)$

  Let $n \in \mathbb{N}$. By relabelling $I_1, \cdots, I_n$, we may assume that

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq a_3 \cdots \leq a_{n-1} < b_{n-1} \leq a_n < b_n \leq b$$

  So

$$\begin{aligned}\sum_{m=1}^{n} \mu_0(I_m) &= \overbrace{F(b_n)}^{\leq F(b)}\overbrace{-F(a_n) + F(b_{n-1})}^{\leq 0}\overbrace{-F(a_{n-1}) + F(b_{n-2})}^{\leq 0} \\ &\qquad \cdots \overbrace{-F(a_2) + F(b_1)}^{\leq 0}\overbrace{-F(a_1)}^{\leq -F(a)} \\ &\leq F(b) - F(a) = \mu_0(I)\end{aligned}$$

  This is true for all n, so that $\mu_0(I) \geq \sum_{m=1}^{\infty} \mu_0(I_m)$.

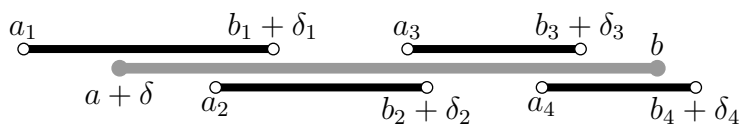
- *Step 3b*: $\mu_0(I) \leq \sum_{m=1}^{\infty} \mu_0(I_m)$ if a, b are finite

  Here is our strategy. Let $\varepsilon > 0$.

  - We want to use compactness to replace the infinite sum by a finite sum. By compactness, any cover by open sets of a compact set has a finite subcover.
  - But $I = (a, b]$ is not compact, so replace it by $[a+\delta, b]$ with $\delta > 0$ chosen so that $F(a + \delta) - F(a) < \varepsilon$.
  - And $I_m = (a_m, b_m]$ is not open. So replace it by $\big(a_m, b_m + \delta_m\big)$ with $\delta_m > 0$ chosen so that $F(b + \delta_m) - F(b_m) < \frac{\varepsilon}{2^m}$.
  - Using compactness, we will extract a finite subsum from $\sum_{m=1}^{\infty} \mu_0(I_m)$ that obeys $\mu_0(I) \leq \sum_j \mu_0(I_j) + 2\varepsilon$. That will do it.

  Here we go.

- Since F is right continuous, there exists a $\delta > 0$ with $F(a+\delta) - F(a) < \varepsilon$ and, for each $m \in \mathbb{N}$, there exists a $\delta_m > 0$ with $F(b+\delta_m) - F(b_m) < \frac{\varepsilon}{2^m}$.
- Then $\bigcup_{m\in\mathbb{N}}(a_m, b_m+\delta_m)$ covers $\bigcup_{m\in\mathbb{N}} I_m = \bigcup_{m\in\mathbb{N}}(a_m, b_m]$, which in turn covers $I = (a, b]$, which in turn covers $[a+\delta, b]$. So there is a cover of $[a+\delta, b]$ by a finite number of $(a_m, b_m+\delta_m)$'s. Pick one.
- Relabel the $(a_m, b_m+\delta_m)$'s (and possibly discard some unnecessary $(a_m, b_m+\delta_m)$'s) as follows. Choose as $(a_1, b_1+\delta_1)$ the $(a_m, b_m+\delta_m)$ that contains $a+\delta$ (and in particular has $a_1 < a+\delta$) with $b_m+\delta_m$ as large as possible. Then choose as $(a_2, b_2+\delta_2)$ the $(a_m, b_m+\delta_m)$ that contains $b_1+\delta_1$ (and in particular has $a_2 < a_1+\delta_1$) with $b_m+\delta_m$ as large as possible. And so on.



So we may assume that

$$a_1 < a+\delta \quad \leq \quad a_2 < b_1+\delta_1 \quad \leq \quad a_3 < b_2+\delta_2 \quad \cdots \quad < b < b_n+\delta_n$$

Then

$$\begin{aligned}
\mu_0(I) = F(b) - F(a) &\leq F(b) - F(a+\delta) + \varepsilon \leq F(b_n+\delta_n) - F(a_1) + \varepsilon \\
&= F(b_n+\delta_n) - F(a_n) + \sum_{j=1}^{n-1} \big[F(a_{j+1}) - F(a_j)\big] + \varepsilon \\
&\leq F(b_n+\delta_n) - F(a_n) + \sum_{j=1}^{n-1} \big[F(b_j+\delta_j) - F(a_j)\big] + \varepsilon \\
&\leq \sum_{j=1}^{n} \big[F(b_j) - F(a_j) + \tfrac{\varepsilon}{2^j}\big] + \varepsilon \\
&\leq \sum_{j=1}^{n} \mu_0(I_j) + 2\varepsilon
\end{aligned}$$

- *Step 3c*: $\mu_0(I) \leq \sum_{m=1}^{\infty} \mu_0(I_m)$ if $a = -\infty$ and/or $b = \infty$

  Let $M > 0$. By the same argument as in Step 3b

$$F\big(\min\{M, b\}\big) - F\big(\max\{-M, a\}\big) \leq \sum_{m=1}^{\infty} \mu_0(I_m)$$

  Take the limit $M \to \infty$.

$\square$

THEOREM 2.36. *Let X be a nonempty set and*

$\mathcal{A} \subset \mathcal{P}(X)$ be an algebra,

$\mathcal{M} = \mathcal{M}(\mathcal{A})$ be the σ-algebra generated by $\mathcal{A}$,

μ_0 be a premeasure on $\mathcal{A}$,

μ^* be the outer measure induced by μ_0 and

$\mathcal{M}^*$ be the set of μ^*-measurable sets.

Recall that

$$\mu^*(E) = \inf\left\{ \sum_{n=1}^{\infty} \mu_0(A_n) \;\middle|\; \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A},\ E \subset \bigcup_{n=1}^{\infty} A_n \right\}$$

Then

(a) $\mu^* \restriction \mathcal{A} = \mu_0$. That is, μ^* extends μ_0. That is, $\mu^*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$.

(b) $\mathcal{M} \subset \mathcal{M}^*$ and $\mu \equiv \mu^* \restriction \mathcal{M}$ is a measure that extends μ_0. That is, all sets in $\mathcal{M}(\mathcal{A})$ are μ^*-measurable and $\mu(A) = \mu_0(A)$ for all $A \in \mathcal{A}$.

(c) If ν is any other[9] measure on $\mathcal{M}$ such that $\nu \restriction \mathcal{A} = \mu_0$, then

$\nu(B) \le \mu(B)$ for all $B \in \mathcal{M}$

$\nu(B) = \mu(B)$ if $B \in \mathcal{M}$ is μ-σ-finite. That is, if B is a countable union of sets of finite μ-measure.

Proof. (a) Let $A \in \mathcal{A}$.

- $\mu^*(A) \le \mu_0(A)$ by the definition of μ^*. (Choose $A_1 = A$ and $A_n = \emptyset$ for all $n > 1$.)
- If $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ with $A \subset \bigcup_{n=1}^{\infty} A_n$, then set $B_n = A \cap \left\{A_n \setminus \bigcup_{j=1}^{n-1} A_j\right\}$. The B_n's are disjoint sets in $\mathcal{A}$ with union A. So, as $A \in \mathcal{A}$, and μ_0 is a premeasure,

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(\overbrace{B_n}^{\subset A_n}) \le \sum_{n=1}^{\infty} \mu_0(A_n)$$

This is true for any choice of $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ with $A \subset \bigcup_{n=1}^{\infty} A_n$, so $\mu_0(A) \le \mu^*(A)$.

(b) By Carathéodory (Theorem 2.31), $\mathcal{M}^*$ is a σ-algebra and the restriction of μ^* to $\mathcal{M}^*$ is a measure. So it suffices to prove that every $A \in \mathcal{A}$ is μ^*-measurable.

Let $A \in \mathcal{A}$ and $E \in \mathcal{P}(X)$. Let $\varepsilon > 0$. By the definition of μ^*, there exist

[9] See Problem Set 4, #4 for an example of multiple extensions.

$\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ with $E \subset \bigcup_{n=1}^{\infty} B_n$ and

$$\begin{aligned}
\mu^*(E) &\geq \sum_{n=1}^{\infty} \mu_0(B_n) - \varepsilon = \sum_{n=1}^{\infty} \left[\mu_0(B_n \cap A) + \mu_0(B_n \cap A^c)\right] - \varepsilon \\
&\qquad\qquad \text{(since } \mu_0 \text{ is finitely additive on } \mathcal{A}) \\
&= \sum_{n=1}^{\infty} \mu_0(B_n \cap A) + \sum_{n=1}^{\infty} \mu_0(B_n \cap A^c) - \varepsilon \\
&\geq \mu^*(E \cap A) + \mu^*(E \cap A^c) - \varepsilon \\
&\qquad\qquad \text{(since } E \cap A \subset \cup_n (B_n \cap A) \text{ and } E \cap A^c \subset \cup_n (B_n \cap A^c))
\end{aligned}$$

As this is true for all $\varepsilon > 0$, we have $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. So A is μ^*-measurable by part (b) of Remark 2.30.

(c) *First part*: Let $B \in \mathcal{M}$ and $B \subset \bigcup_{n=1}^{\infty} A_n$ with $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$. Then

$$\nu(B) \leq \sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} \mu_0(A_n)$$

Taking the inf over the choice of $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ gives

$$\nu(B) \leq \mu^*(B) = \mu(B)$$

(c) *Second part*:

- Let $B \in \mathcal{M}$. First consider the case that $\mu(B) < \infty$. As a preliminary step, we approximate B by a union of A_n's (since we know $\mu(A_n) = \nu(A_n)$). Let $\varepsilon > 0$. By the definition of μ^*, we may choose $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $B \subset A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ and

  $$\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu_0(A_n) \leq \mu^*(B) + \varepsilon = \mu(B) + \varepsilon$$

  As $A = B \cup (A \cap B^c)$, we have $\mu(A) = \mu(B) + \mu(A \cap B^c) \leq \mu(B) + \varepsilon$ and hence

  $$\mu(A \cap B^c) \leq \varepsilon$$

  Hence

  $$\begin{aligned}
  \mu(B) \leq \mu(A) &= \lim_{n\to\infty} \mu\Big(\overbrace{\textstyle\bigcup_{j=1}^{n} A_j}^{\in\mathcal{A}}\Big) = \lim_{n\to\infty} \mu_0\Big(\textstyle\bigcup_{j=1}^{n} A_j\Big) = \lim_{n\to\infty} \nu\Big(\textstyle\bigcup_{j=1}^{n} A_j\Big) \\
&= \nu(A) = \nu(B) + \nu(A \cap B^c) \leq \nu(B) + \mu(A \cap B^c) \\
&\leq \nu(B) + \varepsilon
\end{aligned}$$

  As this is true for all $\varepsilon > 0$, we have $\mu(B) = \nu(B)$.

- Finally consider the case that $B = \bigcup_{n=1}^{\infty} B_n$ with $B_n \in \mathcal{M}$ and $\mu(B_n) < \infty$ for all $n \in \mathbb{N}$. We may assume that the B_n's are disjoint. Otherwise replace B_n by $B_n \setminus \left(\bigcup_{j=1}^{n-1} B_j\right)$. So

$$\mu(B) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \nu(B_n) = \nu(B)$$

□

Corollary 2.37. *Let $F, G : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous.*

(a) There is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F\big((a,b]\big) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$ with $a < b$.

(b) $\mu_F = \mu_G$ if and only if $F - G$ is a constant function.

(c) If μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets, then $\mu = \mu_F$ for

$$F(x) = \begin{cases} \mu\big((0,x]\big) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu\big((x,0]\big) & \text{if } x < 0 \end{cases}$$

Proof. (a) We have already shown, in Proposition 2.35, that $\mu_F\big((a,b]\big) = F(b) - F(a)$ gives a premeasure and, in Proposition 2.9, that the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ is generated by the $(a,b]$'s. The premeasure is σ-finite since $\mathbb{R} = \bigcup_{n=-\infty}^{\infty}(n, n+1]$, so Theorem 2.36 gives both the existence and the uniqueness of the corresponding Borel measure.

(b) We have

$$\begin{aligned} \mu_F = \mu_G &\iff \text{the corresponding premeasures are the same} \\ &\iff \mu_F\big((a,b]\big) = \mu_G\big((a,b]\big) \qquad \text{for all } a < b \\ &\iff F(b) - F(a) = G(b) - G(a) \qquad \text{for all } a < b \\ &\iff F(a) - G(a) = F(b) - G(b) \qquad \text{for all } a < b \end{aligned}$$

(c) For the specified F,

$$\begin{aligned} F(b) - F(a) &= \begin{cases} \mu\big((0,b]\big) - \mu\big((0,a]\big) & \text{if } 0 \le a < b \\ \mu\big((0,b]\big) + \mu\big((a,0]\big) & \text{if } a < 0 \le b \\ -\mu\big((b,0]\big) + \mu\big((a,0]\big) & \text{if } a < b < 0 \end{cases} \\ &= \mu\big((a,b]\big) \qquad \text{for all } a < b \end{aligned}$$

We have already checked, in Example 2.32(b), that F is nondecreasing and right continuous. So μ_F and μ give the same premeasure, and so are the same. $\square$

Remark 2.38. Here is some terminology and notation.

- A Borel measure is a measure on the σ-algebra of Borel sets (i.e. the σ-algebra generated by the open sets).
- We have just seen that every Borel measure on $\mathbb{R}$ that is finite on bounded sets is μ_F for some nondecreasing, right continuous F.
- For each nondecreasing, right continuous $F : \mathbb{R} \to \mathbb{R}$, the complete measure
  - determined (via Carathéodory) by the outer measure, which is, in turn,
  - determined by the premeasure with $\mu_0\big((a,b]\big) = F(b) - F(a)$, $-\infty \le a < b \le \infty$,

  is usually *also* denoted μ_F and is called the **Lebesgue-Stieltjes measure** associated to F.
- The (complete) Lebesgue-Stieltjes measure associated to the function $F(x) = x$ is called the **Lebesgue measure**.
  - The Lebesgue measure of the set A is denoted $m(A)$.
  - The domain of the Lebesgue measure is denoted $\mathcal{L}$. It is a σ-algebra that contains $\mathcal{B}_{\mathbb{R}}$.
  - The Lebesgue measure is translation invariant. That is, if $E \in \mathcal{L}$ and $t \in \mathbb{R}$, then $E + t = \left\{ \, x + t \;\middle|\; x \in E \, \right\} \in \mathcal{L}$ and $m(E+t) = m(E)$. That's Problem Set 6, #1.

Example 2.39. As a simple example, consider

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$$

Then $\mu_F(S) = 0$ for all $S \subset (-\infty, 0)$. In particular every $S \subset (-\infty, 0)$ is null and hence measurable. So there are lots of sets that are measurable, but not Borel. On the other hand the set $N \subset [0,1)$ of Problem set 3, #4 is not measurable. So not all subsets are measurable.

2.7 The Cantor Set Example

Each $x \in [0,1]$ can be represented by the ternary expansion (as opposed to the decimal expansion) $x = \sum_{n=0}^{\infty} x_n 3^{-n}$ with each $x_n \in \{0,1,2\}$. Such representations are not unique. For example

$$0.2222\cdots = \sum_{n=1}^{\infty} 2 \times 3^{-n} = \frac{2}{3}\,\frac{1}{1-\frac{1}{3}} = 1 = 1.0000\cdots$$

When faced with such a choice, in this example, we always avoid terminal 1's (so we express $\frac{1}{3}$ as $0.0222\cdots$ rather than $0.1000\cdots$) and keep terminal 2's (so we express $\frac{2}{3}$ as $0.2000\cdots$ rather than $0.1222\cdots$). Given $x \in [0,1]$ we can generate the x_n's inductively by the algorithm

Set $n = 0$, $x_n = 0$.

$\downarrow$

Set $\varepsilon_n = x - \sum_{p=0}^{n} x_p 3^{-p}$.

$\downarrow$

Observe that $0 \le \varepsilon_n \le \frac{1}{3^n}$.

$\downarrow$

Set $x_{n+1} = \begin{cases} 0 & \text{if } 0 \le \varepsilon_n \le \frac{1}{3^{n+1}} \\ 1 & \text{if } \frac{1}{3^{n+1}} < \varepsilon_n < \frac{2}{3^{n+1}} \\ 2 & \text{if } \frac{2}{3^{n+1}} \le \varepsilon_n \le \frac{3}{3^{n+1}} = \frac{1}{3^n} \end{cases}$

$\downarrow$

$n = 0 \to n+1$.

The Cantor set is

$$C = \left\{\ x \in [0,1] \ \middle|\ x_n \neq 1 \text{ for all } n \in \mathbb{N}\ \right\}$$

For example

$$1/3 = \overbrace{0.0222\cdots}^{\substack{\text{rather than} \\ 0.1000\cdots}} \in C$$

$$x \in (1/3, 2/3) = 0.1\overbrace{x_2x_3x_4\cdots}^{\substack{\text{not } 000\cdots \\ \text{or } 2222\cdots}} \notin C$$

$$2/3 = 0.2000\cdots \in C$$

So

$$C = [0,1] \setminus \overbrace{\left(\tfrac{1}{3}, \tfrac{2}{3}\right)}^{x_1=1} \setminus \underbrace{\left(\tfrac{1}{9}, \tfrac{2}{9}\right)}_{\substack{x_1=0\\x_2=1}} \cup \underbrace{\left(\tfrac{7}{9}, \tfrac{8}{9}\right)}_{\substack{x_1=2\\x_2=1}} \cdots$$

We have that C is

- compact (C is $[0,1]$ with an open set removed)
- nowhere dense, i.e. the interior of its closure is empty, i.e. if $x \in \bar{C} = C$, $\varepsilon > 0$, then $(x-\varepsilon, x+\varepsilon) \not\subset C$
- totally disconnected, i.e. the only connected subsets are single points (if $\alpha < \beta$ were both in a connected subset, then we would have $(\alpha, \beta) \subset C$, which doesn't happen)
- uncountable (by the same argument as for $\mathbb{R}$)
- of Lebesgue measure zero because

$$\begin{aligned} m(C) &= m\big([0,1]\big) - m\big((1/3, 2/3)\big) - \big\{m\big((1/9, 2/9)\big) + m\big((7/9, 8/9)\big)\big\} - \cdots \\ &= 1 - \frac{1}{3} - \frac{2}{9} - \frac{4}{27} - \cdots \\ &= 1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = 1 - \frac{1}{3}\,\frac{1}{1-\frac{2}{3}} \\ &= 0 \end{aligned}$$

Remark 2.40.

(a) We can generalize this. Let $\{\alpha_j\}_{j\in\mathbb{N}} \subset (0,1)$. Then

(1) Start with $[0,1]$. (measure 1)

(2) Remove the middle α_1^{th}. (measure $1-\alpha_1$)

(3) Remove the middle α_2^{th} of the remaining pieces. (measure $(1-\alpha_1)(1-\alpha_2)$)

(4) And so on.

This gives a set of measure

$$\prod_{j=1}^{\infty}(1-\alpha_j) = \lim_{n\to\infty}\prod_{j=1}^{n}(1-\alpha_j) = \lim_{n\to\infty}\exp\left\{\sum_{j=1}^{n}\ln(1-\alpha_j)\right\}$$

We can get *any* measure in $(0,1)$ in this way.

(b) The Cantor function is defined by

$$f : x = \sum_{n=0}^{\infty} x_n 3^{-n} \mapsto \begin{cases} \sum_{n=0}^{\infty} \overbrace{\frac{x_n}{2}}^{\in\{0,1\}} 2^{-n} & \text{if } x \in C \\ \sum_{n=0}^{N} \frac{x_n}{2} 2^{-n} + \frac{1}{2^{N+1}} & \text{if } x \notin C,\ x_{N+1} = 1 \\ & \quad x_n \in \{0,2\} \text{ for all } n \le N \end{cases}$$

This function is

· increasing,

· continuous (it takes all values in $[0,1]$ and so cannot have jump discontinuities),

· constant on each subinterval we removed while building C.

It can be used to construct a subset of $\mathbb{R}$ that is Lebesgue measurable but not Borel. See Exercise 9 of Chapter 2 of Folland.

2.8 Regularity Properties of Lebesgue-Stieltjes Measures

Fix a (complete) Lebesgue-Stieltjes measure μ_F and call it μ. Call its domain $\mathcal{M}_\mu$. By definition, for any $E \in \mathcal{M}_\mu$,

$$\begin{aligned} \mu(E) = \mu^*(E) &= \inf\Big\{ \sum_{n=1}^{\infty} \big[F(b_n) - F(a_n)\big] \;\Big|\; E \subset \textstyle\bigcup_{n=1}^{\infty}(a_n, b_n] \Big\} \\ &= \inf\Big\{ \sum_{n=1}^{\infty} \mu\big((a_n, b_n]\big) \;\Big|\; E \subset \textstyle\bigcup_{n=1}^{\infty}(a_n, b_n] \Big\} \end{aligned}$$

Theorem 2.41 (Regularity)**.** *For all $E \in \mathcal{M}_\mu$,*

$$\begin{aligned} \mu(E) &= \inf\Big\{ \sum_{n=1}^{\infty} \mu\big((a_n, b_n)\big) \;\Big|\; E \subset \textstyle\bigcup_{n=1}^{\infty}\big(a_n, b_n\big) \Big\} && \textit{(a)} \\ &= \inf\big\{ \mu(\mathcal{O}) \;\big|\; \mathcal{O} \subset \mathbb{R},\ \mathcal{O} \textit{ open},\ E \subset \mathcal{O} \big\} && \textit{(b)} \\ &= \sup\big\{ \mu(K) \;\big|\; K \subset \mathbb{R},\ K \textit{ compact},\ K \subset E \big\} && \textit{(c)} \end{aligned}$$

Proof. (a) Call the right hand side $\nu(E)$. Fix any $E \in \mathcal{M}_\mu$.

- *Proof that $\mu(E) \le \nu(E)$:*

  Let $a < b$. Choose $c_0 = a < c_1 < c_2 < \cdots$ with $\lim_{n\to\infty} c_n = b$. Then $(a,b) = \bigcup_{n-1}^{\infty}(c_{n-1}, c_n]$ with the intervals $(c_{n-1}, c_n]$ all disjoint, so that

$$\mu\big((a,b)\big) = \sum_{n=1}^{\infty} \mu\big((c_{n-1}, c_n]\big)$$

So every $\sum_{n=1}^{\infty} \mu\big((a_n, b_n)\big)$ can be expressed in the form $\sum_{p=1}^{\infty} \mu\big((a'_p, b'_p]\big)$ and, consequently,

$$\mu(E) \leq \nu(E)$$

- *Proof that* $\mu(E) \geq \nu(E)$:

  Let $\varepsilon > 0$ and let $E \subset \bigcup_{n=1}^{\infty} \big(a_n, b_n\big]$ with $\sum_{n=1}^{\infty} \mu\big((a_n, b_n]\big) \leq \mu(E) + \varepsilon$. Since F is right continuous there is, for each $n \in \mathbb{N}$, a $b'_n > b_n$ with $F(b'_n) \leq F(b_n) + \frac{\varepsilon}{2^n}$. Then

$$E \subset \bigcup_{n=1}^{\infty} \big(a_n, b_n\big] \subset \bigcup_{n=1}^{\infty} \big(a_n, b'_n\big)$$

  so that

$$\begin{aligned}
\nu(E) &\leq \sum_{n=1}^{\infty} \mu\big((a_n, b'_n)\big) \leq \sum_{n=1}^{\infty} \mu\big((a_n, b'_n]\big) \leq \sum_{n=1}^{\infty} \left\{\mu\big((a_n, b_n]\big) + \frac{\varepsilon}{2^n}\right\} \\
&= \left\{\sum_{n=1}^{\infty} \mu\big((a_n, b_n]\big)\right\} + \varepsilon \\
&\leq \mu(E) + 2\varepsilon
\end{aligned}$$

  This is true for all $\varepsilon > 0$, so $\nu(E) \leq \mu(E)$.

(b) Call the right hand side $\tilde{\nu}(E)$. Fix any $E \in \mathcal{M}_\mu$.

- *Proof that* $\mu(E) \leq \tilde{\nu}(E)$:

  If $E \subset \mathcal{O}$, with $\mathcal{O}$ open, then $\mu(E) \leq \mu(\mathcal{O})$. Just taking the inf over open $\mathcal{O}$'s containing E gives $\mu(E) \leq \tilde{\nu}(E)$.

- *Proof that* $\tilde{\nu}(E) \leq \mu(E)$:

  Let $\varepsilon > 0$. By part (a), there exist $\big\{a_n, b_n\big\}_{n \in \mathbb{N}}$ such that

$$E \subset \mathcal{O} = \bigcup_{n=1}^{\infty} \big(a_n, b_n\big)$$

  and

$$\tilde{\nu}(E) \leq \mu(\mathcal{O}) \leq \sum_{n=1}^{\infty} \mu\big((a_n, b_n)\big) \leq \mu(E) + \varepsilon$$

  This is true for all $\varepsilon > 0$, so $\tilde{\nu}(E) \leq \mu(E)$.

(c) This is Problem Set 5, #4(a). $\square$

Corollary 2.42. *Let $E \subset \mathbb{R}$. The following are equivalent.*
(a) $E \in \mathcal{M}_\mu$
(b) $E = V \setminus N_1$ *where* V *is* G_δ *(a countable intersection of open sets) and* $\mu^*(N_1) = 0$
(c) $E = H \cup N_2$ *where* H *is a countable union of compact sets and* $\mu^*(N_2) = 0$
Here μ^* *is the outer measure for* μ.

Proof.

- *Proof that* $(a) \iff (c)$:
  This is Problem Set 5, #4(b).
- *Proof that* $(b) \implies (a)$:
  As V is G_δ it is in $\mathcal{B}_\mathbb{R} \subset \mathcal{M}_\mu$. As $\mu^*(N_1) = 0$, N_1 is μ^*-measurable (see part (c) of Carathéodory, Theorem 2.31) and hence is in M_μ. So $E = V \cap N_1^c \in \mathcal{M}_\mu$.
- *Proof that* $(a) \implies (b)$:
  By Theorem 2.41(b), there is, for each $k \in \mathbb{Z}$ and $j \in \mathbb{N}$, an open set $\mathcal{O}_{j,k}$ such that $E \cap (k, k+1] \subset \mathcal{O}_{j,k}$ and

$$\begin{aligned}
&\mu(\mathcal{O}_{j,k}) \le \mu\big(E \cap (k, k+1]\big) + \frac{1}{2^{j+|k|}} \\
&\implies \mu\big(E \cap (k, k+1]\big) + \mu\left(\mathcal{O}_{j,k} \setminus \big[E \cap (k, k+1]\big]\right) \le \mu\big(E \cap (k, k+1]\big) + \frac{1}{2^{j+|k|}} \\
&\implies \mu\left(\mathcal{O}_{j,k} \setminus \big[E \cap (k, k+1]\big]\right) \le \frac{1}{2^{j+|k|}}
\end{aligned}$$

  Set

$$V = \bigcap_{j=1}^{\infty} \Bigg(\overbrace{\bigcup_{k \in \mathbb{Z}} \mathcal{O}_{j,k}}^{\substack{\text{open} \\ \text{covers } E}} \Bigg) \in G_\delta$$

  Then $E \subset V$ and, setting $N_1 = V \setminus E$,

$$\begin{aligned}
\mu(N_1) &\le \mu\Bigg(\Big(\bigcup_{k \in \mathbb{Z}} \mathcal{O}_{j,k} \Big) \setminus E \Bigg) \le \sum_{k \in \mathbb{Z}} \mu\big(\mathcal{O}_{j,k} \setminus E\big) \le \sum_{k \in \mathbb{Z}} \frac{1}{2^{j+|k|}} \\
&\le \frac{3}{2^j}
\end{aligned}$$

  for each $j \in \mathbb{N}$. So $\mu(E \setminus V) = 0$.

$\square$