Review of Measure Theory

Let X be a nonempty set. We denote by $\mathcal{P}(X)$ the set of all subsets of X.

Definition 1 (Algebras)

(a) An **algebra** is a nonempty collection $\mathcal{A}$ of subsets of X such that

i) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

ii) $A \in \mathcal{A} \implies A^c = X \setminus A \in \mathcal{A}$

(b) A collection $\mathcal{A}$ of subsets of X is a **σ–algebra** if it is an algebra that is closed under countable unions. That is, $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

(c) If $\mathcal{E} \subset \mathcal{P}(X)$, then the **$\sigma$–algebra generated by $\mathcal{E}$** is

$$\mathcal{M}(\mathcal{E}) = \bigcap \left\{ \Sigma \;\middle|\; \Sigma \text{ is a } \sigma\text{–algebra containing } \mathcal{E} \right\}$$

(d) If X is a metric space (or, more generally, a topological space) then the **Borel σ–algebra** on X, denoted $\mathcal{B}_X$, is the σ–algebra generated by the family of open subsets of X.

Definition 2 (Measures)

(a) A **finitely additive measure** on the algebra $\mathcal{A} \subset \mathcal{P}(X)$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

i) $\mu(\emptyset) = 0$

ii) If $\{E_1, \cdots, E_n\}$ is a finite collection of disjoint subsets of X with $\{E_1, \cdots, E_n\} \subset \mathcal{A}$, then

$$\mu\Big(\bigcup_{j=1}^{n} E_j\Big) = \sum_{j=1}^{n} \mu(E_j)$$

(b) A **premeasure** on the algebra $\mathcal{A} \subset \mathcal{P}(X)$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

i) $\mu(\emptyset) = 0$

ii) If $\{E_j\}$ is a countable collection of disjoint subsets of X with $\{E_j\} \subset \mathcal{A}$ and $\bigcup E_j \in \mathcal{A}$, then

$$\mu\Big(\bigcup_{j} E_j\Big) = \sum_{j} \mu(E_j)$$

(c) A **measure** on the σ–algebra $\mathcal{M} \subset \mathcal{P}(X)$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

i) $\mu(\emptyset) = 0$

ii) If $\{E_j\}$ is a countable collection of disjoint subsets of X with $\{E_j\} \subset \mathcal{M}$, then

$$\mu\Big(\bigcup_{j} E_j\Big) = \sum_{j} \mu(E_j)$$

If μ is a measure on the σ–algebra $\mathcal{M} \subset \mathcal{P}(X)$, then $(X, \mathcal{M}, \mu)$ is called a measure space.

(d) A measure μ on the σ–algebra $\mathcal{M} \subset \mathcal{P}(X)$ is called

i) **finite** if $\mu(X) < \infty$

ii) **σ–finite** if there is a countable collection $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ of subsets with $X = \bigcup_{n=1}^{\infty} E_n$ and with $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$.

iii) **semifinite** if for each $E \in \mathcal{M}$ with $\mu(E) = \infty$, there is an $F \in \mathcal{M}$ with $0 < \mu(F) < \infty$ and $F \subset E$.

iv) **complete** if

$$N \in \mathcal{M},\ \mu(N) = 0,\ Z \subset N \implies Z \in \mathcal{M}$$

v) **Borel** if X is a metric space (or, more generally a topological space) and $\mathcal{M}$ is $\mathcal{B}_X$, the σ–algebra of Borel subsets of X.

(e) An **outer measure** on X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

i) $\mu^*(\emptyset) = 0$

ii) If $E \subset F$, then $\mu^*(E) \le \mu^*(F)$.

iii) If $\{A_j\}$ is a countable collection of subsets of X, then

$$\mu^*\Big(\bigcup_j A_j\Big) \le \sum_j \mu^*(A_j)$$

(f) Let μ^* be an outer measure on X. A subset $A \subset X$ is said to be μ^*–**measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subset X$$

Theorem 3 *Let $(X, \mathcal{M}, \mu)$ be a measure space and $E,\ F,\ E_1,\ E_2,\ \cdots \in \mathcal{M}$.*

*(a) (**Monotonicity**) If $E \subset F$, then $\mu(E) \le \mu(F)$.*

*(b) (**Subadditivity**) $\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) \le \sum_{n=1}^{\infty} \mu(E_n)$*

*(c) (**Continuity from below**) If $E_1 \subset E_2 \subset E_3\ \cdots$, then $\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n)$.*

*(c) (**Continuity from above**) If $\mu(E_1) < \infty$ and $E_1 \supset E_2 \supset E_3\ \cdots$, then $\mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n)$.*

Theorem 4 **(Completion)** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Set*

$$\mathcal{N} = \big\{\ N \in \mathcal{M}\ \big|\ \mu(N) = 0\ \big\}$$

$$\bar{\mathcal{M}} = \big\{\ E \cup Z\ \big|\ E \in \mathcal{M},\ Z \subset N \text{ for some } N \in \mathcal{N}\ \big\}$$

$$\bar{\mu} : \bar{\mathcal{M}} \to [0, \infty] \text{ with } \bar{\mu}(E \cup Z) = \mu(E) \text{ for all } E \in \mathcal{M} \text{ and } Z \subset N \text{ for some } N \in \mathcal{N}$$

Then

(a) $\bar{\mathcal{M}}$ is a σ–algebra.

(b) $\bar{\mu}$ is a well–defined, complete measure on $\bar{\mathcal{M}}$, called the completion of μ.

(c) $\bar{\mu}$ is the unique extension of μ to $\bar{\mathcal{M}}$.

Proposition 5 *Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0,\infty]$ be such that $\{\emptyset, X\} \subset \mathcal{E}$ and $\rho(\emptyset) = 0$. Define, for all $A \subset X$,*

$$\mu^*(A) = \inf\Big\{ \sum_{n=1}^{\infty} \rho(E_n) \ \Big|\ \{E_n\}_{n\in\mathbb{N}} \subset \mathcal{E},\ A \subset \bigcup_{n=1}^{\infty} E_n \Big\}$$

Then μ^* is an outer measure.

Theorem 6 **(Carathéodory)** *Let μ^* be an outer measure on X and $\mathcal{M}^*$ be the set of μ^*–measurable subsets of X. Then*

(a) $\mathcal{M}^*$ is a σ–algebra.

(b) The restriction, $\mu^* \restriction \mathcal{M}^*$, of μ^* to $\mathcal{M}^*$ is a complete measure.

Proposition 7 *Let $F : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous. Define $F(\pm\infty) = \lim\limits_{x\to\pm\infty} F(x)$. Set*

$$\mathcal{A} = \{\emptyset\} \cup \Big\{ \bigcup_{j=1}^{n} (a_j, b_j] \ \Big|\ n \in \mathbb{N},\ -\infty \le a_1 < b_1 < \cdots < b_n \le \infty \Big\}$$

$$\mu_0(\emptyset) = 0$$

$$\mu_0\Big(\bigcup_{j=1}^{n} (a_j, b_j]\Big) = \sum_{j=1}^{n} \big[F(b_j) - F(a_j)\big] \quad \text{for all } n \in \mathbb{N},\ -\infty \le a_1 < b_1 < \cdots < b_n \le \infty$$

In the above, replace $(a,b]$ by (a,b) when $b = \infty$. Then μ_0 is a premeasure on $\mathcal{A}$.

Theorem 8 *Let*

$\mathcal{A} \subset \mathcal{P}(X)$ be an algebra,

$\mathcal{M} = \mathcal{M}(\mathcal{A})$ be the σ–algebra generated by $\mathcal{A}$,

μ_0 be a premeasure on $\mathcal{A}$,

μ^* be the outer measure induced by μ_0 and

$\mathcal{M}^*$ be the set of μ^*–measurable sets.

Recall that

$$\mu^*(E) = \inf\Big\{ \sum_{n=1}^{\infty} \mu_0(A_n) \ \Big|\ \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A},\ E \subset \bigcup_{n=1}^{\infty} A_n \Big\}$$

Then

(a) $\mu^* \restriction \mathcal{A} = \mu_0$. That is, μ^* extends μ_0. That is, $\mu^*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$.

(b) $\mathcal{M} \subset \mathcal{M}^*$ and $\mu \equiv \mu^* \restriction \mathcal{M}$ is a measure that extends μ_0.

(c) $\mathcal{M} \subset \mathcal{M}^*$ and $\mu \equiv \mu^* \restriction \mathcal{M}$ is a measure that extends μ_0. That is $\mu \restriction \mathcal{A} = \mu_0$.

(d) If ν is any other measure on $\mathcal{M}$ such that $\nu \restriction \mathcal{A} = \mu_0$, then

$\nu(E) \le \mu(E)$ for all $E \in \mathcal{M}$

$\nu(E) = \mu(E)$ if $E \in \mathcal{M}$ is μ–σ–finite. That is, if E is a countable union of sets of finite μ–measure.

Corollary 9 *Let $F, G : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous.*

(a) There is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F\big((a,b]\big) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$ with $a < b$.

(b) $\mu_F = \mu_G$ if and only if $F - G$ is a constant function.

(c) If μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets, then $\mu = \mu_F$ for

$$F(x) = \begin{cases} \mu\big((0,x]\big) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu\big((x,0]\big) & \text{if } x < 0 \end{cases}$$

Definition 10 Let $F : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous. The **Lebesgue–Stieltjes measure**, μ_F, associated to F is the complete measure determined (by Carathéodory's Theorem 6) from the outer measure which is, in turn, determined by Proposition 5 from the premeasure that is associated to F by Proposition 7. The **Lebesgue measure**, m, is the Lebesgue–Stieltjes measure associated to the function $F(x) = x$.

Theorem 11 (Regularity) *Let μ be a Lebesgue–Stieltjes measure, μ^* be the corresponding outer measure and $\mathcal{M}^*$ be the set of all μ^*–measurable sets. This is also the domain of μ.*

(a) For all $E \in \mathcal{M}^*$

$$\begin{aligned} \mu(E) &= \inf\Big\{ \sum_{n=1}^{\infty} \big[F(b_n) - F(a_n)\big] \ \Big|\ E \subset \textstyle\bigcup_{n=1}^{\infty} (a_n, b_n) \Big\} \\ &= \inf\big\{ \mu(\mathcal{O}) \ \big|\ \mathcal{O} \subset \mathbb{R},\ \mathcal{O} \text{ open},\ E \subset \mathcal{O} \big\} \\ &= \sup\big\{ \mu(K) \ \big|\ K \subset \mathbb{R},\ K \text{ compact},\ K \subset E \big\} \end{aligned}$$

(b) Let $E \subset \mathbb{R}$. The following are equivalent.

- *(i) $E \in \mathcal{M}^*$*
- *(ii) $E = V \setminus N_1$ where V is G_δ (a countable intersection of open sets) and $\mu^*(N_1) = 0$*
- *(iii) $E = H \cup N_2$ where H is F_σ (a countable union of compact sets) and $\mu^*(N_2) = 0$*

Proposition 12 (Invariance) *Let m be the Lebesgue measure and $\mathcal{L}$ be the collection of Lebesgue measurable sets. Then*

(a) If $E \in \mathcal{L}$ and $y \in \mathbb{R}$, then $E + y = \big\{ x + y \ \big|\ x \in E \big\} \in \mathcal{L}$ and $m(E + y) = m(E)$.

(b) If $E \in \mathcal{L}$ and $r \in \mathbb{R}$, then $rE = \big\{ rx \ \big|\ x \in E \big\} \in \mathcal{L}$ and $m(rE) = |r|\, m(E)$.