Review of Measure Theory

Let $X$ be a nonempty set. We denote by $\mathcal{P}(X)$ the set of all subsets of $X$.

**Definition 1 (Algebras)**

(a) An **algebra** is a nonempty collection $\mathcal{A}$ of subsets of $X$ such that
   
   i) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$  
   ii) $A \in \mathcal{A} \implies A^c = X \setminus A \in \mathcal{A}$

(b) A collection $\mathcal{A}$ of subsets of $X$ is a **$\sigma$–algebra** if it is an algebra that is closed under countable unions. That is, $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

(c) If $\mathcal{E} \subset \mathcal{P}(X)$, then the **$\sigma$–algebra generated by $\mathcal{E}$** is

$$\mathcal{M}(\mathcal{E}) = \bigcap \{ \Sigma \mid \Sigma \text{ is a } \sigma\text{-algebra containing } \mathcal{E} \}$$

(d) If $X$ is a metric space (or, more generally, a topological space) then the **Borel $\sigma$–algebra** on $X$, denoted $\mathcal{B}_X$, is the $\sigma$–algebra generated by the family of open subsets of $X$.

**Definition 2 (Measures)**

(a) A **finitely additive measure** on the algebra $\mathcal{A} \subset \mathcal{P}(X)$ is a function $\mu : \mathcal{A} \rightarrow [0, \infty]$ such that
   
   i) $\mu(\emptyset) = 0$  
   ii) If $\{E_1, \cdots, E_n\}$ is a finite collection of disjoint subsets of $X$ with $\{E_1, \cdots, E_n\} \subset \mathcal{A}$, then

$$\mu\left( \bigcup_{j=1}^{n} E_j \right) = \sum_{j=1}^{n} \mu(E_j)$$

(b) A **premeasure** on the algebra $\mathcal{A} \subset \mathcal{P}(X)$ is a function $\mu : \mathcal{A} \rightarrow [0, \infty]$ such that
   
   i) $\mu(\emptyset) = 0$  
   ii) If $\{E_j\}$ is a countable collection of disjoint subsets of $X$ with $\{E_j\} \subset \mathcal{A}$ and $\bigcup E_j \in \mathcal{A}$, then

$$\mu\left( \bigcup E_j \right) = \sum \mu(E_j)$$

(c) A **measure** on the $\sigma$–algebra $\mathcal{M} \subset \mathcal{P}(X)$ is a function $\mu : \mathcal{M} \rightarrow [0, \infty]$ such that
   
   i) $\mu(\emptyset) = 0$  
   ii) If $\{E_j\}$ is a countable collection of disjoint subsets of $X$ with $\{E_j\} \subset \mathcal{M}$, then

$$\mu\left( \bigcup E_j \right) = \sum \mu(E_j)$$

If $\mu$ is a measure on the $\sigma$–algebra $\mathcal{M} \subset \mathcal{P}(X)$, then $(X, \mathcal{M}, \mu)$ is called a measure space.
(d) A measure \( \mu \) on the \( \sigma \)-algebra \( \mathcal{M} \subset \mathcal{P}(X) \) is called

i) \textbf{finite} if \( \mu(X) < \infty \)

ii) \textbf{\( \sigma \)-finite} if there is a countable collection \( \{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M} \) of subsets with \( X = \bigcup_{n=1}^{\infty} E_n \) and with \( \mu(E_n) < \infty \) for all \( n \in \mathbb{N} \).

iii) \textbf{semifinite} if for each \( E \in \mathcal{M} \) with \( \mu(E) = \infty \), there is an \( F \in \mathcal{M} \) with \( 0 < \mu(F) < \infty \) and \( F \subset E \).

iv) \textbf{complete} if

\[
N \in \mathcal{M}, \; \mu(N) = 0, \; Z \subset N \implies Z \in \mathcal{M}
\]

v) \textbf{Borel} if \( X \) is a metric space (or, more generally a topological space) and \( \mathcal{M} \) is \( \mathcal{B}_X \), the \( \sigma \)-algebra of Borel subsets of \( X \).

(e) An \textbf{outer measure} on \( X \) is a function \( \mu^* : \mathcal{P}(X) \to [0, \infty] \) such that

i) \( \mu^*(\emptyset) = 0 \)

ii) If \( E \subset F \), then \( \mu^*(E) \leq \mu^*(F) \).

iii) If \( \{A_j\} \) is a countable collection of subsets of \( X \), then

\[
\mu^*\left( \bigcup_j A_j \right) \leq \sum_j \mu^*(A_j)
\]

(f) Let \( \mu^* \) be an outer measure on \( X \). A subset \( A \subset X \) is said to be \( \mu^* \)-\textbf{measurable} if

\[
\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)
\]

for all \( E \subset X \)

\textbf{Theorem 3} Let \( (X, \mathcal{M}, \mu) \) be a measure space and \( E, F, E_1, E_2, \ldots \in \mathcal{M} \).

(a) \textbf{(Monotonicity)} If \( E \subset F \), then \( \mu(E) \leq \mu(F) \).

(b) \textbf{(Subadditivity)} \( \mu\left( \bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \mu(E_n) \)

(c) \textbf{(Continuity from below)} If \( E_1 \subset E_2 \subset E_3 \ldots \), then \( \mu\left( \bigcup_{n=1}^{\infty} E_n \right) = \lim_{n \to \infty} \mu(E_n) \).

(c) \textbf{(Continuity from above)} If \( \mu(E_1) < \infty \) and \( E_1 \supset E_2 \supset E_3 \ldots \), then \( \mu\left( \bigcap_{n=1}^{\infty} E_n \right) = \lim_{n \to \infty} \mu(E_n) \).

\textbf{Theorem 4 (Completion)} Let \( (X, \mathcal{M}, \mu) \) be a measure space. Set

\[
\mathcal{N} = \{ N \in \mathcal{M} \mid \mu(N) = 0 \}
\]

\[
\bar{\mathcal{M}} = \{ E \cup Z \mid E \in \mathcal{M}, \; Z \subset N \text{ for some } N \in \mathcal{N} \}
\]

\( \bar{\mu} : \bar{\mathcal{M}} \to [0, \infty] \) with \( \bar{\mu}(E \cup Z) = \mu(E) \) for all \( E \in \mathcal{M} \) and \( Z \subset N \) for some \( N \in \mathcal{N} \)

Then

(a) \( \bar{\mathcal{M}} \) is a \( \sigma \)-algebra.

(b) \( \bar{\mu} \) is a well-defined, complete measure on \( \bar{\mathcal{M}} \), called the completion of \( \mu \).

(c) \( \bar{\mu} \) is the unique extension of \( \mu \) to \( \bar{\mathcal{M}} \).
Proposition 5  Let $E \subset P(X)$ and $\rho : E \to [0, \infty]$ be such that $\{\emptyset, X\} \subset E$ and $\rho(\emptyset) = 0$. Define, for all $A \subset X$,

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) \mid \{E_n\}_{n \in \mathbb{N}} \subset E, \ A \subset \bigcup_{n=1}^{\infty} E_n \right\}$$

Then $\mu^*$ is an outer measure.

Theorem 6  (Carathéodory) Let $\mu^*$ be an outer measure on $X$ and $\mathcal{M}^*$ be the set of $\mu^*$–measurable subsets of $X$. Then

(a) $\mathcal{M}^*$ is a $\sigma$–algebra.
(b) The restriction, $\mu^*|\mathcal{M}^*$, of $\mu^*$ to $\mathcal{M}^*$ is a complete measure.

Proposition 7  Let $F : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous. Define $F(\pm \infty) = \lim_{x \to \pm \infty} F(x)$. Set

$$\mathcal{A} = \{\emptyset\} \cup \left\{ \bigcup_{j=1}^{n} (a_j, b_j) \mid n \in \mathbb{N}, \ -\infty \leq a_1 < b_1 < \cdots < b_n \leq \infty \right\}$$

$$\mu_0(\emptyset) = 0$$

$$\mu_0 \left( \bigcup_{j=1}^{n} (a_j, b_j) \right) = \sum_{j=1}^{n} \left[ F(b_j) - F(a_j) \right] \text{ for all } n \in \mathbb{N}, \ -\infty \leq a_1 < b_1 < \cdots < b_n \leq \infty$$

In the above, replace $(a, b]$ by $(a, b)$ when $b = \infty$. Then $\mu_0$ is a premeasure on $\mathcal{A}$.

Theorem 8  Let

$\mathcal{A} \subset P(X)$ be an algebra,

$\mathcal{M} = \mathcal{M}(\mathcal{A})$ be the $\sigma$–algebra generated by $\mathcal{A}$,

$\mu_0$ be a premeasure on $\mathcal{A}$,

$\mu^*$ be the outer measure induced by $\mu_0$ and

$\mathcal{M}^*$ be the set of $\mu^*$–measurable sets.

Recall that

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu_0(A_n) \mid \{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A}, \ E \subset \bigcup_{n=1}^{\infty} A_n \right\}$$

Then

(a) $\mu^*|\mathcal{A} = \mu_0$. That is, $\mu^*$ extends $\mu_0$. That is, $\mu^*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$.
(b) $\mathcal{M} \subset \mathcal{M}^*$ and $\mu \equiv \mu^*|\mathcal{M}$ is a measure that extends $\mu_0$.
(c) $\mathcal{M} \subset \mathcal{M}^*$ and $\mu \equiv \mu^*|\mathcal{M}$ is a measure that extends $\mu_0$. That is $\mu|\mathcal{A} = \mu_0$.
(d) If $\nu$ is any other measure on $\mathcal{M}$ such that $\nu|\mathcal{A} = \mu_0$, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$

$$\nu(E) = \mu(E) \text{ if } E \in \mathcal{M} \text{ is } \mu-\sigma\text{–finite. That is, if } E \text{ is a countable union of sets of finite } \mu\text{–measure.}$$
Corollary 9  Let $F, G : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous.

(a) There is a unique Borel measure $\mu_F$ on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$ with $a < b$.

(b) $\mu_F = \mu_G$ if and only if $F - G$ is a constant function.

(c) If $\mu$ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets, then $\mu = \mu_F$ for

$$F(x) = \begin{cases} 
\mu((0, x]) & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-\mu((x, 0]) & \text{if } x < 0
\end{cases}$$

Definition 10  Let $F : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous. The Lebesgue–Stieltjes measure, $\mu_F$, associated to $F$ is the complete measure determined (by Carathéodory’s Theorem 6) from the outer measure which is, in turn, determined by Proposition 5 from the premeasure that is associated to $F$ by Proposition 7. The Lebesgue measure, $m$, is the Lebesgue–Stieltjes measure associated to the function $F(x) = x$.

Theorem 11 (Regularity) Let $\mu$ be a Lebesgue–Stieltjes measure, $\mu^*$ be the corresponding outer measure and $\mathcal{M}^*$ be the set of all $\mu^*$–measurable sets. This is also the domain of $\mu$.

(a) For all $E \in \mathcal{M}^*$

$$\mu(E) = \inf \left\{ \sum_{n=1}^{\infty} [F(b_n) - F(a_n)] \mid E \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \right\}$$

$$= \inf \left\{ \mu(O) \mid O \subset \mathbb{R}, \ O \text{ open}, \ E \subset O \right\}$$

$$= \sup \left\{ \mu(K) \mid K \subset \mathbb{R}, \ K \text{ compact}, \ K \subset E \right\}$$

(b) Let $E \subset \mathbb{R}$. The following are equivalent.

(i) $E \in \mathcal{M}^*$

(ii) $E = V \setminus N_1$ where $V$ is $G_\delta$ (a countable intersection of open sets) and $\mu^*(N_1) = 0$

(iii) $E = H \cup N_2$ where $H$ is $F_\sigma$ (a countable union of compact sets) and $\mu^*(N_2) = 0$

Proposition 12 (Invariance) Let $m$ be the Lebesgue measure and $\mathcal{L}$ be the collection of Lebesgue measurable sets. Then

(a) If $E \in \mathcal{L}$ and $y \in \mathbb{R}$, then $E + y = \{ x + y \mid x \in E \} \in \mathcal{L}$ and $m(E + y) = m(E)$.

(b) If $E \in \mathcal{L}$ and $r \in \mathbb{R}$, then $rE = \{ rx \mid x \in E \} \in \mathcal{L}$ and $m(rE) = |r| m(E)$. 