Review of Measure Theory

Let $X$ be a nonempty set. We denote by $\mathcal{P}(X)$ the set of all subsets of $X$.

**Definition 1 (Algebras)**

(a) An algebra is a nonempty collection $\mathcal{A}$ of subsets of $X$ such that
   i) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$
   ii) $A \in \mathcal{A} \implies A^c = X \setminus A \in \mathcal{A}$

(b) A collection $\mathcal{A}$ of subsets of $X$ is a $\sigma$-algebra if it is an algebra that is closed under countable unions. That is, $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

(c) If $\mathcal{E} \subset \mathcal{P}(X)$, then the $\sigma$-algebra generated by $\mathcal{E}$ is
   $$\mathcal{M}(\mathcal{E}) = \bigcap \{ \Sigma \mid \Sigma \text{ is a } \sigma \text{-algebra containing } \mathcal{E} \}$$

(d) If $X$ is a metric space (or, more generally, a topological space) then the Borel $\sigma$-algebra on $X$, denoted $\mathcal{B}_X$, is the $\sigma$-algebra generated by the family of open subsets of $X$.

**Definition 2 (Measures)**

(a) A finitely additive measure on the algebra $\mathcal{A} \subset \mathcal{P}(X)$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that
   i) $\mu(\emptyset) = 0$
   ii) If $\{E_1, \cdots, E_n\}$ is a finite collection of disjoint subsets of $X$ with $\{E_1, \cdots, E_n\} \subset \mathcal{A}$, then
      $$\mu\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} \mu(E_j)$$

(b) A premeasure on the algebra $\mathcal{A} \subset \mathcal{P}(X)$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that
   i) $\mu(\emptyset) = 0$
   ii) If $\{E_j\}$ is a countable collection of disjoint subsets of $X$ with $\{E_j\} \subset \mathcal{A}$ and $\bigcup E_j \in \mathcal{A}$, then
      $$\mu\left(\bigcup_{j} E_j\right) = \sum_{j} \mu(E_j)$$

(c) A measure on the $\sigma$-algebra $\mathcal{M} \subset \mathcal{P}(X)$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that
   i) $\mu(\emptyset) = 0$
   ii) If $\{E_j\}$ is a countable collection of disjoint subsets of $X$ with $\{E_j\} \subset \mathcal{M}$, then
      $$\mu\left(\bigcup_{j} E_j\right) = \sum_{j} \mu(E_j)$$

If $\mu$ is a measure on the $\sigma$-algebra $\mathcal{M} \subset \mathcal{P}(X)$, then $(X, \mathcal{M}, \mu)$ is called a measure space.
(d) A measure $\mu$ on the $\sigma$-algebra $\mathcal{M} \subset \mathcal{P}(X)$ is called
   i) **finite** if $\mu(X) < \infty$
   ii) **$\sigma$-finite** if there is a countable collection $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ of subsets with $X = \bigcup_{n=1}^{\infty} E_n$ and with $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$.
   iii) **semifinite** if for each $E \in \mathcal{M}$ with $\mu(E) = \infty$, there is an $F \in \mathcal{M}$ with $0 < \mu(F) < \infty$ and $F \subset E$.
   iv) **complete** if
      $$N \in \mathcal{M}, \quad \mu(N) = 0, \quad Z \subset N \implies Z \in \mathcal{M}$$
   v) **Borel** if $X$ is a metric space (or, more generally a topological space) and $\mathcal{M}$ is $\mathcal{B}_X$, the $\sigma$-algebra of Borel subsets of $X$.

(e) An **outer measure** on $X$ is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that
   i) $\mu^*(\emptyset) = 0$
   ii) If $E \subset F$, then $\mu^*(E) \leq \mu^*(F)$.
   iii) If $\{A_j\}$ is a countable collection of subsets of $X$, then
      $$\mu^*\left(\bigcup_j A_j\right) \leq \sum_j \mu^*(A_j)$$

(f) Let $\mu^*$ be an outer measure on $X$. A subset $A \subset X$ is said to be $\mu^*$-measurable if
      $$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$ for all $E \subset X$

**Theorem 3** Let $(X, \mathcal{M}, \mu)$ be a measure space and $E, F, E_1, E_2, \ldots \in \mathcal{M}$.

(a) (**Monotonicity**) If $E \subset F$, then $\mu(E) \leq \mu(F)$.

(b) (**Subadditivity**) $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$

(c) (**Continuity from below**) If $E_1 \subset E_2 \subset E_3 \ldots$, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n)$.

(c) (**Continuity from above**) If $\mu(E_1) < \infty$ and $E_1 \supset E_2 \supset E_3 \ldots$, then $\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n)$.

**Theorem 4** (**Completion**) Let $(X, \mathcal{M}, \mu)$ be a measure space. Set

$$\mathcal{N} = \{ N \in \mathcal{M} \mid \mu(N) = 0 \}$$

$$\bar{\mathcal{M}} = \{ E \cup Z \mid E \in \mathcal{M}, \ Z \subset N \text{ for some } N \in \mathcal{N} \}$$

$$\bar{\mu} : \bar{\mathcal{M}} \to [0, \infty] \text{ with } \bar{\mu}(E \cup Z) = \mu(E) \text{ for all } E \in \mathcal{M} \text{ and } Z \subset N \text{ for some } N \in \mathcal{N}$$

Then

(a) $\bar{\mathcal{M}}$ is a $\sigma$-algebra.
(b) $\bar{\mu}$ is a well-defined, complete measure on $\bar{\mathcal{M}}$, called the completion of $\mu$.
(c) $\bar{\mu}$ is the unique extension of $\mu$ to $\bar{\mathcal{M}}$. 

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Proposition 5  Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset, X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. Define, for all $A \subset X$,

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) \left| \right. \begin{array}{l} \{E_n\}_{n \in \mathbb{N}} \subset \mathcal{E}, \ A \subset \bigcup_{n=1}^{\infty} E_n \end{array} \right\}$$

Then $\mu^*$ is an outer measure.

Theorem 6 (Carathéodory)  Let $\mu^*$ be an outer measure on $X$ and $\mathcal{M}^*$ be the set of $\mu^*$-measurable subsets of $X$. Then

(a) $\mathcal{M}^*$ is a $\sigma$-algebra.

(b) The restriction, $\mu^*|\mathcal{M}^*$, of $\mu^*$ to $\mathcal{M}^*$ is a complete measure.

Proposition 7  Let $F : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous. Define $F(\pm\infty) = \lim_{x \to \pm\infty} F(x)$. Set

$$\mathcal{A} = \{\emptyset\} \cup \left\{ \bigcup_{j=1}^{n} (a_j, b_j) \left| \begin{array}{l} n \in \mathbb{N}, \ -\infty \leq a_1 < b_1 < a_2 < b_2 < \cdots < b_n \leq \infty \end{array} \right. \right\}$$

$$\mu_0(\emptyset) = 0$$

$$\mu_0\left( \bigcup_{j=1}^{n} (a_j, b_j) \right) = \sum_{j=1}^{n} \left[ F(b_j) - F(a_j) \right] \text{ for all } n \in \mathbb{N}, \ -\infty \leq a_1 < b_1 < a_2 < b_2 < \cdots < b_n \leq \infty$$

In the above, replace $(a, b]$ by $(a, b)$ when $b = \infty$. Then $\mu_0$ is a premeasure on $\mathcal{A}$.

Theorem 8  Let

$\mathcal{A} \subset \mathcal{P}(X)$ be an algebra,

$\mathcal{M} = \mathcal{M}(\mathcal{A})$ be the $\sigma$-algebra generated by $\mathcal{A}$,

$\mu_0$ be a premeasure on $\mathcal{A}$,

$\mu^*$ be the outer measure induced by $\mu_0$ and

$\mathcal{M}^*$ be the set of $\mu^*$-measurable sets.

Recall that

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu_0(A_n) \left| \begin{array}{l} \{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A}, \ E \subset \bigcup_{n=1}^{\infty} A_n \end{array} \right\}$$

Then

(a) $\mu^*|\mathcal{A} = \mu_0$. That is, $\mu^*$ extends $\mu_0$. That is, $\mu^*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$.

(b) $\mathcal{M} \subset \mathcal{M}^*$ and $\mu \equiv \mu^*|\mathcal{M}$ is a measure that extends $\mu_0$. That is $\mu|\mathcal{A} = \mu_0$.

(c) If $\nu$ is any other measure on $\mathcal{M}$ such that $\nu|\mathcal{A} = \mu_0$, then

$\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$

$\nu(E) = \mu(E)$ if $E \in \mathcal{M}$ is $\mu$-$\sigma$-finite. That is, if $E$ is a countable union of sets of finite $\mu$-measure.
**Corollary 9** Let $F, G : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous.

(a) There is a unique Borel measure $\mu_F$ on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$ with $a < b$.

(b) $\mu_F = \mu_G$ if and only if $F - G$ is a constant function.

(c) If $\mu$ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets, then $\mu = \mu_F$ for

$$F(x) = \begin{cases} 
\mu((0, x]] & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-\mu((x, 0]) & \text{if } x < 0 
\end{cases}$$

**Definition 10** Let $F : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous.

(a) The *Lebesgue-Stieltjes measure*, $\mu_F$, associated to $F$ is the complete measure determined (by Carathéodory’s Theorem 6) from the outer measure which is, in turn, determined by Proposition 5 from the premeasure that is associated to $F$ by Proposition 7.

(b) The *Lebesgue measure*, $m$, is the Lebesgue-Stieltjes measure associated to the function $F(x) = x$.

**Theorem 11** (Regularity) Let $\mu$ be a Lebesgue-Stieltjes measure, $\mu^*$ be the corresponding outer measure and $\mathcal{M}^*$ be the set of all $\mu^*$-measurable sets. This is also the domain of $\mu$.

(a) For all $E \in \mathcal{M}^*$

$$\mu(E) = \inf \left\{ \sum_{n=1}^{\infty} [F(b_n) - F(a_n)] \mid E \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \right\}$$

$$= \inf \left\{ \mu(O) \mid O \subset \mathbb{R}, \ O \text{ open, } E \subset O \right\}$$

$$= \sup \left\{ \mu(K) \mid K \subset \mathbb{R}, \ K \text{ compact, } K \subset E \right\}$$

(b) Let $E \subset \mathbb{R}$. The following are equivalent.

(i) $E \in \mathcal{M}^*$

(ii) $E = V \setminus N_1$ where $V$ is $G_\delta$ (a countable intersection of open sets) and $\mu^*(N_1) = 0$

(iii) $E = H \cup N_2$ where $H$ is $F_\sigma$ (a countable union of compact sets) and $\mu^*(N_2) = 0$

**Proposition 12** (Invariance) Let $m$ be the Lebesgue measure and $\mathcal{L}$ be the collection of Lebesgue measurable sets. Then

(a) If $E \in \mathcal{L}$ and $y \in \mathbb{R}$, then $E + y = \{ x + y \mid x \in E \} \in \mathcal{L}$ and $m(E + y) = m(E)$.

(b) If $E \in \mathcal{L}$ and $r \in \mathbb{R}$, then $rE = \{ rx \mid x \in E \} \in \mathcal{L}$ and $m(rE) = |r| m(E)$. 

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