Review of Measurable Functions

Definition 1 (Measurable Functions) Let X and Y be nonempty sets and $\mathcal{M}$ and $\mathcal{N}$ be σ-algebras of subsets of X and Y respectively.

(a) A function $f : X \to Y$ is said to be $(\mathcal{M}, \mathcal{N})$-measurable if

$$E \in \mathcal{N} \implies f^{-1}(E) \equiv \left\{ \, x \in X \;\middle|\; f(x) \in E \, \right\} \in \mathcal{M}$$

(b) A function $f : X \to \mathbb{R}$ or $\mathbb{C}$ is said to be $\mathcal{M}$-measurable if

$$E \in \mathcal{B}_{\mathbb{R}} \text{ or } \mathcal{B}_{\mathbb{C}} \implies f^{-1}(E) \equiv \left\{ \, x \in X \;\middle|\; f(x) \in E \, \right\} \in \mathcal{M}$$

(c) A function $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ is said to be Borel-measurable if it is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}}$ or $\mathcal{B}_{\mathbb{C}})$-measurable and is said to be Lebesgue-measurable if it is $(\mathcal{L}, \mathcal{B}_{\mathbb{R}}$ or $\mathcal{B}_{\mathbb{C}})$-measurable, where $\mathcal{L}$ is the set of all Lebesgue measurable subsets of $\mathbb{R}$.

(d) A function $f : X \to [0, \infty]$ is said to be $\mathcal{M}$-measurable if

$$E \in \mathcal{B}_{\overline{\mathbb{R}}} \implies f^{-1}(E) \in \mathcal{M}$$

where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is the extended real line and $\mathcal{B}_{\overline{\mathbb{R}}}$ is the σ-algebra

$$\mathcal{B}_{\overline{\mathbb{R}}} = \left\{ \, E \subset \overline{\mathbb{R}} \;\middle|\; E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}} \, \right\}$$

Lemma 2 (Continuous Functions) *If X and Y are metric spaces and $f : X \to Y$ is continuous, then f is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.*

Lemma 3 (Real Valued Functions) *Let X be a nonempty set and $\mathcal{M}$ be a σ-algebra of subsets of X. Let $f : X \to \mathbb{R}$. Then*

$$\begin{aligned} f \text{ is } \mathcal{M}\text{-measurable} &\iff f^{-1}\big((a, \infty)\big) \in \mathcal{M} \;\forall\; a \in \mathbb{R} \\ &\iff f^{-1}\big([a, \infty)\big) \in \mathcal{M} \;\forall\; a \in \mathbb{R} \\ &\iff f^{-1}\big((-\infty, a)\big) \in \mathcal{M} \;\forall\; a \in \mathbb{R} \\ &\iff f^{-1}\big((-\infty, a]\big) \in \mathcal{M} \;\forall\; a \in \mathbb{R} \end{aligned}$$

For $f : X \to [0, \infty]$ Then

$$\begin{aligned} f \text{ is } \mathcal{M}\text{-measurable} &\iff f^{-1}\big((a, \infty]\big) \in \mathcal{M} \;\forall\; a > 0 \\ &\iff f^{-1}\big([a, \infty)\big) \in \mathcal{M} \;\forall\; a \geq 0 \end{aligned}$$

Theorem 4 (Measurable Function Toolbox) *Let $f, g : X \to \mathbb{R}$ be $\mathcal{M}$-measurable and $c \in \mathbb{R}$, or let $f, g : X \to [0, \infty]$ be $\mathcal{M}$-measurable and $c > 0$. Then*

(a) $f + c$ and cf are $\mathcal{M}$-measurable.

(b) $f + g$ is $\mathcal{M}$-measurable.

(c) fg is $\mathcal{M}$-measurable. (As usual, we use the convention that $0 \times \infty = 0$.)

(d) $\max\{f, g\}$ is $\mathcal{M}$-measurable. Here $\max\{f, g\}$ is the function from X to $\mathbb{R}$ defined by $\max\{f, g\}(x) = \max\{f(x), g(x)\}$.

(e) $\min\{f, g\}$ is $\mathcal{M}$-measurable.

(f) Let, for each $n \in \mathbb{N}$, the function $f_n : X \to \mathbb{R}$ be $\mathcal{M}$-measurable and let $h : X \to \mathbb{R}$. Or let, for each $n \in \mathbb{N}$, the function $f_n : X \to [0, \infty]$ be $\mathcal{M}$-measurable and let $h : X \to [0, \infty]$. If

$$\begin{aligned} h(x) &= \lim_{n\to\infty} f_n(x) \\ \text{or } h(x) &= \inf_{n\in\mathbb{N}} f_n(x) \\ \text{or } h(x) &= \sup_{n\in\mathbb{N}} f_n(x) \\ \text{or } h(x) &= \liminf_{n\to\infty} f_n(x) \\ \text{or } h(x) &= \limsup_{n\to\infty} f_n(x) \end{aligned}$$

for all $x \in X$, then h is $\mathcal{M}$-measurable.

(g) If $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable and $g : X \to \mathbb{R}$ is $\mathcal{M}$-measurable, then $h \circ g$ is $\mathcal{M}$ measurable. Here $h \circ g : X \to \mathbb{R}$ is the function defined by $h \circ g(x) = h\big(g(x)\big)$.

Definition 5 (Almost Everywhere) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f, g, f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$. Then

(a) $f = g$ a.e. if there is a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $f(x) = g(x)$ for all $x \notin E$.

(b) $f(x) = \lim\limits_{n\to\infty} f_n(x)$ a.e. if there is a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for all $x \notin E$.

Lemma 6 (Almost Everywhere) *Let $(X, \mathcal{M}, \mu)$ be a measure space. The following implications are true if and only if μ is complete.*

(a) Let $f, g : X \to \mathbb{R}$. If f is measurable and $f = g$ μ-a.e., then g is measurable.

(b) Let $f : X \to \mathbb{R}$ and $f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$. If f_n is measurable for all $n \in \mathbb{N}$ and $\{f_n\}$ converges μ-a.e. to f, then f is measurable.