Review of Measurable Functions

Definition 1 (Measurable Functions) Let $X$ and $Y$ be nonempty sets and $\mathcal{M}$ and $\mathcal{N}$ be $\sigma$-algebras of subsets of $X$ and $Y$ respectively.

(a) A function $f : X \to Y$ is said to be $(\mathcal{M}, \mathcal{N})$-measurable if

$$E \in \mathcal{N} \implies f^{-1}(E) = \{ x \in X \mid f(x) \in E \} \in \mathcal{M}$$

(b) A function $f : X \to \mathbb{R}$ or $\mathbb{C}$ is said to be $\mathcal{M}$-measurable if

$$E \in \mathcal{B}_{\mathbb{R}} \text{ or } \mathcal{B}_{\mathbb{C}} \implies f^{-1}(E) = \{ x \in X \mid f(x) \in E \} \in \mathcal{M}$$

(c) A function $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ is said to be Borel-measurable if it is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}}$ or $\mathcal{B}_{\mathbb{C}})$-measurable and is said to be Lebesgue-measurable if it is $(\mathcal{L}, \mathcal{B}_{\mathbb{R}}$ or $\mathcal{B}_{\mathbb{C}})$-measurable, where $\mathcal{L}$ is the set of all Lebesgue measurable subsets of $\mathbb{R}$.

(d) A function $f : X \to [0, \infty]$ is said to be $\mathcal{M}$-measurable if

$$E \in \mathcal{B}_{\overline{\mathbb{R}}} \implies f^{-1}(E) \in \mathcal{M}$$

where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is the extended real line and $\mathcal{B}_{\overline{\mathbb{R}}}$ is the $\sigma$-algebra

$$\mathcal{B}_{\overline{\mathbb{R}}} = \{ E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}} \}$$

Lemma 2 (Continuous Functions) If $X$ and $Y$ are metric spaces and $f : X \to Y$ is continuous, then $f$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

Lemma 3 (Real Valued Functions) Let $X$ be a nonempty set and $\mathcal{M}$ be a $\sigma$-algebra of subsets of $X$. Let $f : X \to \mathbb{R}$. Then

$$f \text{ is } \mathcal{M}\text{-measurable} \iff f^{-1}((a, \infty)) \in \mathcal{M} \ \forall \ a \in \mathbb{R}$$

$$\iff f^{-1}([a, \infty)) \in \mathcal{M} \ \forall \ a \in \mathbb{R}$$

$$\iff f^{-1}((-\infty, a)) \in \mathcal{M} \ \forall \ a \in \mathbb{R}$$

$$\iff f^{-1}((-\infty, a]) \in \mathcal{M} \ \forall \ a \in \mathbb{R}$$

For $f : X \to [0, \infty]$ Then

$$f \text{ is } \mathcal{M}\text{-measurable} \iff f^{-1}((a, \infty]) \in \mathcal{M} \ \forall \ a > 0$$

$$\iff f^{-1}([a, \infty)) \in \mathcal{M} \ \forall \ a \geq 0$$
Theorem 4 (Measurable Function Toolbox) Let $f, g : X \to \mathbb{R}$ be $\mathcal{M}$-measurable and $c \in \mathbb{R}$, or let $f, g : X \to [0, \infty]$ be $\mathcal{M}$-measurable and $c > 0$. Then

(a) $f + c$ and $cf$ are $\mathcal{M}$-measurable.
(b) $f + g$ is $\mathcal{M}$-measurable.
(c) $fg$ is $\mathcal{M}$-measurable. (As usual, we use the convention that $0 \times \infty = 0$.)
(d) $\max\{f, g\}$ is $\mathcal{M}$-measurable. Here $\max\{f, g\}$ is the function from $X$ to $\mathbb{R}$ defined by $\max\{f, g\}(x) = \max\{f(x), g(x)\}$.
(e) $\min\{f, g\}$ is $\mathcal{M}$-measurable.
(f) Let, for each $n \in \mathbb{N}$, the function $f_n : X \to \mathbb{R}$ be $\mathcal{M}$-measurable and let $h : X \to \mathbb{R}$. Or let, for each $n \in \mathbb{N}$, the function $f_n : X \to [0, \infty]$ be $\mathcal{M}$-measurable and let $h : X \to [0, \infty]$. If

$$h(x) = \lim_{n \to \infty} f_n(x)$$
or

$$h(x) = \inf_{n \in \mathbb{N}} f_n(x)$$
or

$$h(x) = \sup_{n \in \mathbb{N}} f_n(x)$$
or

$$h(x) = \liminf_{n \to \infty} f_n(x)$$
or

$$h(x) = \limsup_{n \to \infty} f_n(x)$$

for all $x \in X$, then $h$ is $\mathcal{M}$-measurable.

(g) If $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable and $g : X \to \mathbb{R}$ is $\mathcal{M}$-measurable, then $h \circ g$ is $\mathcal{M}$ measurable. Here $h \circ g : X \to \mathbb{R}$ is the function defined by $h \circ g(x) = h(g(x))$.

Definition 5 (Almost Everywhere) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f, g, f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$. Then

(a) $f = g$ a.e. if there is a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $f(x) = g(x)$ for all $x \notin E$.
(b) $f(x) = \lim_{n \to \infty} f_n(x)$ a.e. if there is a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $f(x) = \lim_{n \to \infty} f_n(x)$ for all $x \notin E$.

Lemma 6 (Almost Everywhere) Let $(X, \mathcal{M}, \mu)$ be a measure space. The following implications are true if and only if $\mu$ is complete.

(a) Let $f, g : X \to \mathbb{R}$. If $f$ is measurable and $f = g \mu$-a.e., then $g$ is measurable.
(b) Let $f : X \to \mathbb{R}$ and $f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$. If $f_n$ is measurable for all $n \in \mathbb{N}$ and $\{f_n\}$ converges $\mu$-a.e. to $f$, then $f$ is measurable.