Review of Measurable Functions

**Definition 1 (Measurable Functions)** Let $X$ and $Y$ be nonempty sets and $\mathcal{M}$ and $\mathcal{N}$ be $\sigma$-algebras of subsets of $X$ and $Y$ respectively.

(a) A function $f : X \to Y$ is said to be $(\mathcal{M}, \mathcal{N})$-measurable if

$$E \in \mathcal{N} \implies f^{-1}(E) \equiv \{ x \in X \mid f(x) \in E \} \in \mathcal{M}$$

(b) A function $f : X \to \mathbb{R}$ or $\mathbb{C}$ is said to be $\mathcal{M}$-measurable if

$$E \in \mathcal{B}_\mathbb{R} \text{ or } \mathcal{B}_\mathbb{C} \implies f^{-1}(E) \equiv \{ x \in X \mid f(x) \in E \} \in \mathcal{M}$$

(c) A function $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ is said to be Borel-measurable if it is $(\mathcal{B}_\mathbb{R}, \mathcal{B}_\mathbb{R})$-measurable and is said to be Lebesgue-measurable if it is $(\mathcal{L}, \mathcal{B}_\mathbb{R})$-measurable, where $\mathcal{L}$ is the set of all Lebesgue measurable subsets of $\mathbb{R}$.

(d) A function $f : X \to [0, \infty]$ is said to be $\mathcal{M}$-measurable if

$$E \in \mathcal{B}_{\overline{\mathbb{R}}} \implies f^{-1}(E) \in \mathcal{M}$$

where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is the extended real line and $\mathcal{B}_{\overline{\mathbb{R}}}$ is the $\sigma$-algebra

$$\mathcal{B}_{\overline{\mathbb{R}}} = \{ E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}_\mathbb{R} \}$$

**Lemma 2 (Continuous Functions)** If $X$ and $Y$ are metric spaces and $f : X \to Y$ is continuous, then $f$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

**Lemma 3 (Real Valued Functions)** Let $X$ be a nonempty set and $\mathcal{M}$ be a $\sigma$-algebra of subsets of $X$. Let $f : X \to \mathbb{R}$. Then

$$f \text{ is } \mathcal{M}\text{-measurable} \iff f^{-1}((a, \infty)) \in \mathcal{M} \ \forall \ a \in \mathbb{R}$$

$$\iff f^{-1}([a, \infty)) \in \mathcal{M} \ \forall \ a \in \mathbb{R}$$

$$\iff f^{-1}((\infty, a)) \in \mathcal{M} \ \forall \ a \in \mathbb{R}$$

$$\iff f^{-1}((\infty, a]) \in \mathcal{M} \ \forall \ a \in \mathbb{R}$$

For $f : X \to [0, \infty]$ Then

$$f \text{ is } \mathcal{M}\text{-measurable} \iff f^{-1}((a, \infty]) \in \mathcal{M} \ \forall \ a > 0$$

$$\iff f^{-1}([a, \infty)) \in \mathcal{M} \ \forall \ a \geq 0$$
Theorem 4 (Measurable Function Toolbox) Let \( f, g : X \to \mathbb{R} \) be \( \mathcal{M} \)-measurable and \( c \in \mathbb{R} \), or let \( f, g : X \to [0, \infty] \) be \( \mathcal{M} \)-measurable and \( c > 0 \). Then

(a) \( f + c \) and \( cf \) are \( \mathcal{M} \)-measurable.
(b) \( f + g \) is \( \mathcal{M} \)-measurable.
(c) \( fg \) is \( \mathcal{M} \)-measurable. (As usual, we use the convention that \( 0 \times \infty = 0 \).)
(d) \( \max\{f, g\} \) is \( \mathcal{M} \)-measurable. Here \( \max\{f, g\} \) is the function from \( X \) to \( \mathbb{R} \) defined by \( \max\{f, g\}(x) = \max\{f(x), g(x)\} \).
(e) \( \min\{f, g\} \) is \( \mathcal{M} \)-measurable.
(f) Let, for each \( n \in \mathbb{N} \), the function \( f_n : X \to \mathbb{R} \) be \( \mathcal{M} \)-measurable and let \( h : X \to \mathbb{R} \). Or let, for each \( n \in \mathbb{N} \), the function \( f_n : X \to [0, \infty] \) be \( \mathcal{M} \)-measurable and let \( h : X \to [0, \infty] \). If

\[
h(x) = \lim_{n \to \infty} f_n(x)
\]

or

\[
h(x) = \inf_{n \in \mathbb{N}} f_n(x)
\]

or

\[
h(x) = \sup_{n \in \mathbb{N}} f_n(x)
\]

or

\[
h(x) = \liminf_{n \to \infty} f_n(x)
\]

or

\[
h(x) = \limsup_{n \to \infty} f_n(x)
\]

for all \( x \in X \), then \( h \) is \( \mathcal{M} \)-measurable.

(g) If \( h : \mathbb{R} \to \mathbb{R} \) is Borel measurable and \( g : X \to \mathbb{R} \) is \( \mathcal{M} \)-measurable, then \( h \circ g \) is \( \mathcal{M} \) measurable. Here \( h \circ g : X \to \mathbb{R} \) is the function defined by \( h \circ g(x) = h(g(x)) \).

Definition 5 (Almost Everywhere) Let \((X, \mathcal{M}, \mu)\) be a measure space and \( f, g, f_n : X \to \mathbb{R} \) for all \( n \in \mathbb{N} \). Then

(a) \( f = g \) a.e. if there is a set \( E \in \mathcal{M} \) such that \( \mu(E) = 0 \) and \( f(x) = g(x) \) for all \( x \notin E \).
(b) \( f = \lim_{n \to \infty} f_n(x) \) a.e. if there is a set \( E \in \mathcal{M} \) such that \( \mu(E) = 0 \) and \( f = \lim_{n \to \infty} f_n(x) \) for all \( x \notin E \).

Lemma 6 (Almost Everywhere) Let \((X, \mathcal{M}, \mu)\) be a measure space. The following implications are true if and only if \( \mu \) is complete.

(a) Let \( f, g : X \to \mathbb{R} \). If \( f \) is measurable and \( f = g \) \( \mu \)-a.e., then \( g \) is measurable.
(b) Let \( f : X \to \mathbb{R} \) and \( f_n : X \to \mathbb{R} \) for all \( n \in \mathbb{N} \). If \( f_n \) is measurable for all \( n \in \mathbb{N} \) and \( \{f_n\} \) converges \( \mu \)-a.e. to \( f \), then \( f \) is measurable.