Lecture Notes on Measure Theory and Integration
8 — $L^p$ Spaces

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3 \( L^p \) Spaces

Let \( 1 \leq p \leq \infty \). Let \((X, \mathcal{M}, \mu)\) be a measure space, with \( X \) a set, \( \mathcal{M} \) a \( \sigma \)-algebra and \( \mu \) a measure. For \( p < \infty \), set

\[
L^p(X, \mathcal{M}, \mu) = \{ \varphi : X \to \mathbb{C} \mid \varphi \text{ is } \mathcal{M}\text{-measurable with } \int |\varphi(x)|^p \, d\mu(x) < \infty \}
\]

\[
\|\varphi\|_p = \left[ \int |\varphi(x)|^p \, d\mu(x) \right]^{1/p}
\]

For \( p = \infty \), set

\[
L^\infty(X, \mathcal{M}, \mu) = \{ \varphi : X \to \mathbb{C} \mid \varphi \text{ is } \mathcal{M}\text{-measurable with } \text{ess sup}_x |\varphi(x)| < \infty \}
\]

\[
\|\varphi\|_\infty = \text{ess sup}_x |\varphi(x)|
\]

Here, the essential supremum of \(|\varphi|\), with respect to the measure \( \mu \), is denoted \( \text{ess sup}_x |\varphi(x)| \) and is defined by

\[
\text{ess sup}_x |\varphi(x)| = \inf \{ \, a \geq 0 \mid |\varphi(x)| \leq a \text{ almost everywhere, with respect to } \mu \, \}
\]

For each \( n \in \mathbb{N} \), there is a null set \( N_n \in \mathcal{M} \) such that \(|\varphi(x)| \leq \frac{1}{n} + \text{ess sup}_x |\varphi(x)|\) for all \( x \in X \setminus N_n \). Then \( N = \bigcup_{n \in \mathbb{N}} N_n \) also has measure zero and

\[
|\varphi(x)| \leq \text{ess sup}_x |\varphi(x)| \quad \text{for all } x \in N
\]

The sets \( L^p(X, \mathcal{M}, \mu) \) are vector spaces and \( \|\varphi\|_p \) is almost a norm (explained shortly). Here is a review of the definitions.

**Definition 3.1 (Vector Space).** A vector space over \( \mathbb{C} \) is a set \( V \) equipped with two operations,

\[
(v, w) \in V \times V \mapsto v + w \in V \quad (\alpha, v) \in \mathbb{C} \times V \mapsto \alpha v \in V
\]

called addition and scalar multiplication, respectively, that obey the following axioms.

*Additive Axioms.* There is an element \( 0 \in V \) and, for each \( x \in V \), there is an element \( -x \in V \) such that, for all \( x, y, z \in V \),

(i) \( x + y = y + x \)

(ii) \( (x + y) + z = x + (y + z) \)

(iii) \( 0 + x = x + 0 = x \)

(iv) \( (-x) + x = x + (-x) = 0 \)
**Multiplicative Axioms.** For every $x \in V$ and $\alpha, \beta \in \mathbb{C}$,

(v) $0 \cdot x = 0$

(vi) $1 \cdot x = x$

(vii) $(\alpha \beta) \cdot x = \alpha (\beta \cdot x)$

**Distributive Axioms.** For every $x, y \in V$ and $\alpha, \beta \in \mathbb{C}$,

(viii) $\alpha (x + y) = \alpha x + \alpha y$

(ix) $(\alpha + \beta) \cdot x = \alpha x + \beta x$

**Remark 3.2.**

(a) The zero vector is automatically unique because if both $0$ and $0'$ are zero vectors, then

$$0 = 0 + 0' = 0'$$

(b) For each $x$, the additive inverse $-x$ is automatically unique because, if both $y$ and $y'$ are additive inverses for $x$,

$$y = 0 + y = (x + y') + y = (x + y) + y' = 0 + y' = y'$$

(b) The additive inverse for $x$ is $(-1) \cdot x$ since

$$x + (-1) \cdot x = (1) \cdot x + (-1) \cdot x = (1 - 1) \cdot x = 0 \cdot x = 0$$

**Definition 3.3** (Norm).

(a) A norm on a vector space $V$ is a function $x \in V \mapsto \|x\| \in [0, \infty)$ that obeys

(i) $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\| = |\alpha| \|x\|$

(iii) $\|x + y\| \leq \|x\| + \|y\|$

for all $x, y \in V$ and $\alpha \in \mathbb{C}$.

(b) A sequence $\{v_n\}_{n \in \mathbb{N}} \subset V$ is said to be **Cauchy** with respect to the norm $\| \cdot \|$ if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } m, n > N \implies \|v_n - v_m\| < \varepsilon$$

(c) A sequence $\{v_n\}_{n \in \mathbb{N}} \subset V$ is said to **converge** to $v$ in the norm $\| \cdot \|$ if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } n > N \implies \|v_n - v\| < \varepsilon$$

(d) A normed vector space is said to be **complete** if every Cauchy sequence converges. 

(e) A Banach space is a complete normed vector space.

**Remark 3.4.** If $\| \cdot \|$ is a norm, then $\rho(v, w) = \|v - w\|$ is a metric.
Example 3.5.
(a) Let \( n \in \mathbb{N} \). If \( X = \{1, 2, \cdots, n\}, \mathcal{M} = \mathcal{P}(X) \), the set of all subsets of \( X \), and \( \mu \) is the counting measure, then, if we identify each \( \varphi : \{1, 2, \cdots, n\} \to \mathbb{C} \) with \( (\varphi(1), \varphi(2), \cdots, \varphi(n)) \), we have that

\[
L^2(X, \mathcal{M}, \mu) = \left\{ (z_1, z_2, \cdots , z_n) \mid z_1, z_2, \cdots , z_n \in \mathbb{C} \right\}
\]

\[
\| (z_1, z_2, \cdots , z_n) \|_2 = \left[ \sum_{\ell=1}^{n} |z\ell|^2 \right]^{1/2}
\]

is just \( \mathbb{C}^n \) equipped with the usual Euclidean norm.

(b) If \( X = \mathbb{N}, \mathcal{M} = \mathcal{P}(\mathbb{N}) \), the set of all subsets of \( \mathbb{N} \), and \( \mu \) is the counting measure, then \( L^p(X, \mathcal{M}, \mu) \) is usually denote \( \ell^p \). That is, identifying each \( \varphi : \mathbb{N} \to \mathbb{C} \) with \( (\varphi(n))_{n \in \mathbb{N}} \),

\[
\ell^p = \left\{ (a_n)_{n \in \mathbb{N}} \mid a_n \in \mathbb{C} \text{ for all } n \in \mathbb{N}, \| (a_n)_{n \in \mathbb{N}} \|_p < \infty \right\}
\]

\[
\| (a_n)_{n \in \mathbb{N}} \|_p = \left\{ \begin{array}{ll}
\sum_{n=1}^{\infty} |a_n|^p & \text{if } 1 \leq p < \infty \\
\sup_{n \in \mathbb{N}} |a_n| & \text{if } p = \infty
\end{array} \right.
\]

The vector spaces \( \ell^p(X, \mathcal{M}, \mu) \) are not quite Banach spaces, because \( \| \varphi \|_p \) is not quite a norm, because any function \( \varphi \) that is zero almost everywhere has “norm” zero. This is easily fixed. We define an equivalence relation on \( \ell^p(X, \mathcal{M}, \mu) \) by

\[
\varphi \sim \psi \iff \varphi = \psi \text{ a.e.}
\]

As usual, the equivalence class of \( \varphi \in \ell^p(X, \mathcal{M}, \mu) \) is

\[
[\varphi] = \left\{ \psi \in \ell^p(X, \mathcal{M}, \mu) \mid \psi \sim \varphi \right\}
\]

Then

\[
L^p(X, \mathcal{M}, \mu) = \left\{ [\varphi] \mid \varphi \in \ell^p(X, \mathcal{M}, \mu) \right\}
\]

is a Banach space (this will be justified shortly) with

\[
[\varphi] + [\psi] = [\varphi + \psi] \quad a[\varphi] = [a\varphi] \quad \|[\varphi]\|_p = \|\varphi\|_p \quad (E1)
\]

for all \( \varphi, \psi \in \ell^p(X, \mathcal{M}, \mu) \) and \( a \in \mathbb{C} \). for all \( \varphi, \psi \in L^2(X, \mathcal{M}, \mu) \). It is standard to write \( \varphi \) in place of \([\varphi]\). If \( X \) is a Lebesgue measurable subset of \( \mathbb{R} \), then \( L^p(X) \) denotes the \( L^p(X, \mathcal{M}, \mu) \) with \((\mathcal{M}, \mu)\) being Lebesgue measure.
Lemma 3.6. Let \((X, \mathcal{M}, \mu)\) be a measure space and \(1 \leq p < \infty\). Then
\[
L^p(X, \mathcal{M}, \mu) = \{ [\varphi] \mid \varphi \in L^p(X, \mathcal{M}, \mu) \}
\]
with the operations
\[ [\varphi] + [\psi] = [\varphi + \psi] \quad a[\varphi] = [a\varphi] \]
is a well-defined vector space. Furthermore
\[
\|[\varphi]\|_p = \|\varphi\|_p
\]
is a well-defined norm on \(L^p(X, \mathcal{M}, \mu)\).

Proof. Well-definedness: In order for
\[
[\varphi] + [\psi] = [\varphi + \psi] \quad a[\varphi] = [a\varphi] \quad \|[\varphi]\|_p = \|\varphi\|_p
\]
to be well-defined, we need that
\[
\varphi_1 \sim \varphi_2, \ \psi_1 \sim \psi_2 \implies \varphi_1 + \psi_1 \sim \varphi_2 + \psi_2
\]
\[
\varphi_1 \sim \varphi_2 \implies a\varphi_1 \sim a\varphi_2
\]
\[
\varphi_1 \sim \varphi_2 \implies \|\varphi_1\|_p = \|\varphi_2\|_p
\]
These are all obvious, or at least easy. For example, if
\[
\{ x \in X \mid \varphi_1(x) \neq \varphi_2(x) \} \subset N_1 \quad \text{with} \quad \mu(N_1) = 0
\]
\[
\{ x \in X \mid \psi_1(x) \neq \psi_2(x) \} \subset N_2 \quad \text{with} \quad \mu(N_2) = 0
\]
then
\[
\{ x \in X \mid \varphi_1(x) + \psi_1(x) \neq \varphi_2(x) + \psi_2(x) \} \subset N_1 \cup N_2 \quad \text{and} \quad \mu(N_1 \cup N_2) = 0
\]

Vector space axioms: Verification of the vector space axioms is also trivial because \(\mathbb{C}\) is a field. In particular the zero vector is the function that is identically zero.

Norm axioms: As is standard, we write \(\varphi\) in place of \([\varphi]\).

(i) That \(\|\varphi\|_p \geq 0\) with equality if and only if \(\varphi\) is the zero function, i.e. if and only if \(\varphi(x) = 0\) a.e., is obvious.

(ii) That \(\|a\varphi\|_p = |a|\|\varphi\|_p\) is also obvious.

(iii) The triangle inequality \(\|\varphi + \psi\|_p \leq \|\varphi\|_p + \|\psi\|_p\) will be part (a) of Theorem 3.8, which is in turn Problem Set 10, #4.
Lemma 3.7 (Why $L^\infty$ is called $L^\infty$). Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $f \in L^\infty(X, \mathcal{M}, \mu)$. Then $f \in L^p(X, \mathcal{M}, \mu)$ for all $1 \leq p < \infty$ and

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$$

Proof. We first develop a preliminary bound. Define, for each $r \geq 0$,

$$X_r = \{ x \in X \mid |f(x)| \geq r \}$$

If $\mu(X_r) > 0$, then, for each $1 \leq p < \infty$,

$$\liminf_{p \to \infty} \|f\|_p = \liminf_{p \to \infty} \left[ \int_X |f(x)|^p d\mu(x) \right]^{1/p} \geq \liminf_{p \to \infty} \left[ \int_{X_r} |f(x)|^p d\mu(x) \right]^{1/p} \geq \liminf_{p \to \infty} \left[ \int_{X_r} r^p d\mu(x) \right]^{1/p} = \liminf_{p \to \infty} r \mu(X_r)^{1/p} = r$$

since $\lim_{p \to \infty} t^{1/p} = 1$ for every strictly positive real number $t$.

Now let $\|f\|_\infty$ and $\mu(X)$ be finite. Then

$$\limsup_{p \to \infty} \|f\|_p = \limsup_{p \to \infty} \left[ \int_X |f(x)|^p d\mu(x) \right]^{1/p} \leq \limsup_{p \to \infty} \left[ \int_X \|f\|_\infty^p d\mu(x) \right]^{1/p} = \liminf_{p \to \infty} \|f\|_\infty \mu(X)^{1/p} = \|f\|_\infty$$

By the preliminary bound with $r = \|f\|_\infty - \varepsilon$,

$$\liminf_{p \to \infty} \|f\|_p \geq \|f\|_\infty - \varepsilon$$

for every $\varepsilon > 0$. Hence $\liminf_{p \to \infty} \|f\|_p \geq \|f\|_\infty$. Combining the upper and lower bounds

$$\|f\|_\infty \leq \liminf_{p \to \infty} \|f\|_p \leq \limsup_{p \to \infty} \|f\|_p \leq \|f\|_\infty$$

Theorem 3.8. Let $(X, \mathcal{M}, \mu)$ be measure space.

(a) (Minkowski inequality) Let $1 \leq p \leq \infty$ and $f, g \in L^p(X, \mathcal{M}, \mu)$. Then the sum $f + g \in L^p(X, \mathcal{M}, \mu)$ and

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

(b) Let $0 < p < 1$ and $f, g : X \to \mathbb{R}$ be measurable. If $f(x), g(x) \geq 0$ a.e. then

$$\|f + g\|_p \geq \|f\|_p + \|g\|_p$$
(c) (Hölder inequality) Let \(1 \leq p, q \leq \infty\) with \(\frac{1}{p} + \frac{1}{q} = 1\). If \(f \in L^p(X, \mathcal{M}, \mu)\) and \(g \in L^q(X, \mathcal{M}, \mu)\), then we have \(fg \in L^1(X, \mathcal{M}, \mu)\) and
\[
\int_X |f(x)g(x)| \, d\mu(x) \leq \|f\|_{L^p}\|g\|_{L^q}.
\]

(d) (Generalized Hölder inequality) Let \(1 \leq r \leq \infty\) and \(1 \leq p_j \leq \infty\) with \(\sum_{j=1}^n \frac{1}{p_j} = \frac{1}{r}\). If \(f_j \in L^{p_j}\) for \(1 \leq j \leq n\), then \(\prod_{j=1}^n f_j \in L^r\) and
\[
\left\|\prod_{j=1}^n f_j\right\|_r \leq \prod_{j=1}^n \|f_j\|_{p_j}.
\]

Proof. (a) Problem Set 10, #3.
(b) We are assuming that \(f(x), g(x) \geq 0\). If \(\|f\|_p = 0\) or \(\|g\|_p = 0\), the result is trivial. So assume that \(\|f\|_p, \|g\|_p > 0\). By dividing the desired inequality by \(\|f\|_p + \|g\|_p\), we may reduce consideration to \(\|f\|_p, \|g\|_p \neq 0, \|f\|_p + \|g\|_p = 1\). With \(\lambda = \|f\|_p\),
\[
\|f + g\|_p^p = \int [f(x) + g(x)]^p \, d\mu(x)
\]
\[
= \int \left[\lambda \frac{f(x)}{\|f\|_p} + (1 - \lambda) \frac{g(x)}{\|g\|_p}\right]^p \, d\mu(x)
\]

Define \(h(y) = y^p\). Observe that for \(y > 0\)
\[
h''(y) = p(p-1)y^{p-2} < 0
\]
since \(0 < p < 1\). That is, \(h\) is concave down.

Thus for all \(u, v \geq 0\) and \(0 \leq \lambda \leq 1\)
\[
h(\lambda u + (1 - \lambda)v) \geq [\lambda h(u) + (1 - \lambda)h(v)]
\]
Using this with \( u = \frac{f(x)}{\|f\|_p} \) and \( v = \frac{g(x)}{\|g\|_p} \),

\[
\|f + g\|_p^p \geq \int \left[ \lambda (\|f\|_p^p) + (1 - \lambda) (\|g\|_p^p) \right] \, d\mu(x) \\
= \lambda + (1 - \lambda) = 1
\]

(c) Problem Set 10, #4.

(d) First we deal with \( n = 2 \). By Hölder, with \( f = |f_1|^r \), \( g = |f_2|^r \), \( p = \frac{p_1}{r} \) and \( q = \frac{p_2}{r} \),

\[
\|f_1 f_2\|_r^r = \int |f_1(x) f_2(x)|^r \, d\mu(x) \leq \|f_1\|_r^{p_1/r} \|f_2\|_r^{p_2/r} \\
= \left[ \int |f_1(x)|^{r(p_1/r)} \, d\mu(x) \right]^{r/p_1} \left[ \int |f_2(x)|^{r(p_2/r)} \, d\mu(x) \right]^{r/p_2} \\
= \|f_1\|_{p_1} \|f_2\|_{p_2}
\]

Now we proceed by induction. Once the inequality has been established for \( n - 1 \), we apply the \( n = 2 \) inequality, with \( f_2 \) replaced by \( \prod_{j=1}^n f_j \) and \( p_2 \) replaced by \( r' = \left[ \sum_{j=2}^n \frac{1}{p_j} \right]^{-1} \).

\[
\left\| \prod_{j=1}^n f_j \right\|_r \leq \|f_1\|_{p_1} \left\| \prod_{j=2}^n f_j \right\|_{r'}
\]

Now just apply the \( n - 1 \) inequality.

Our next main task is to prove that \( L^p(X, \mathcal{M}, \mu) \) is complete. To do so, it is convenient to use the following test for completeness.

**Lemma 3.9.** A normed vector space is complete if and only if every absolutely convergent series is convergent. That is, if and only if

\[
\sum_{n=1}^{\infty} \left\| f_n \right\| < \infty \implies \lim_{N \to \infty} f_n \text{ exists in } V
\]

**Proof.** \( \implies: \) Assume that \( V \) is complete and let \( \sum_{n=1}^{\infty} \|f_n\| < \infty \). Then, for all natural numbers \( N > M \)

\[
\left\| \sum_{n=1}^N f_n - \sum_{n=1}^M f_n \right\| = \left\| \sum_{n=M+1}^N f_n \right\| \leq \sum_{n=M+1}^N \|f_n\| \to 0 \text{ as } M \to \infty
\]

So the sequence \( \left\{ s_N = \sum_{n=1}^N f_n \right\}_{N \in \mathbb{N}} \) is Cauchy and the series converges.
\[ \sum_{n=1}^{\infty} ||f_n|| < \infty \implies \lim_{N \to \infty} f_n \text{ exists in } V \]

and let \( s_1, s_2, s_3, \cdots \) be a Cauchy sequence. Then for each \( k \in \mathbb{N} \), there is an \( n_k \) such that

\[ m > n_k \implies ||s_m - s_{n_k}|| \leq \frac{1}{2^k} \]

We can certainly choose \( n_k > n_{k-1} \). Define \( f_1 = s_{n_1} \) and, for each \( k \geq 2 \), \( f_k = s_{n_{k+1}} - s_{n_k} \). Then

\[ \sum_{n=1}^{\infty} ||f_n|| \leq ||f_1|| + \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty \]

\[ \implies s_{n_k} = \sum_{n=1}^{k-1} f_n \text{ converges as } k \to \infty \]

So the sequence \( \{s_n\}_{n \in \mathbb{N}} \) is Cauchy and contains a convergent subsequence. So the full sequence \( \{s_n\}_{n \in \mathbb{N}} \) converges too. \( \square \)

**Theorem 3.10.** Let \((X, \mathcal{M}, \mu)\) be a measure space and let \(1 \leq p \leq \infty \). Then \( L^p(X, \mathcal{M}, \mu) \) is a Banach space.

**Proof.** Case 1: \( 1 \leq p < \infty \). Let \( \{f_n\}_{n \in \mathbb{N}} \subset L^p(X, \mathcal{M}, \mu) \) obey \( \sum_{n=1}^{\infty} ||f_n||_p = M < \infty \). Define

\[ g_n(x) = \sum_{k=1}^{n} |f_k(x)| \]

Then

\[ ||g_n||_p \leq \sum_{k=1}^{n} ||f_k||_p \leq M \]

For each \( x \in X \), \( g_n(x) \) increases monotonically with \( n \). So

\[ g(x) = \lim_{n \to \infty} g_n(x) \text{ exists in } [0, \infty] \]

By the monotone convergence theorem

\[ \int_X g(x)^p \, d\mu(x) = \lim_{n \to \infty} \int_X g_n(x)^p \, d\mu(x) \leq M^p \]
Hence $g \in L^p(X, \mathcal{M}, \mu)$ and is finite a.e. In particular, there is a set $N \in \mathcal{M}$ of measure zero such that the series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely for all $x \in X \setminus N$. As a result,

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \chi_{N^c}(x)$$

converges pointwise to some $s(x)$ and $s(x)$ is measurable. Since, for all $x \in X$,

$$|s_n(x) - s(x)|^p \to 0$$
$$|s_n(x) - s(x)|^p \leq |2g(x)^p|$$

The (Lebesgue) dominated convergence theorem implies that

$$\lim_{n \to \infty} \int_X |s_n(x) - s(x)|^p \, d\mu(x) = 0$$

which, by definition, says that $s = \lim_{n \to \infty} s_n$ in $L^p(X, \mathcal{M}, \mu)$.

**Case 2: $p = \infty$.** The argument is similar to that for the case $1 \leq p < \infty$, but this time we do not have access to the monotone and Lebesgue dominated convergence theorems. Let $\{f_n\}_{n \in \mathbb{N}} \subset L^\infty(X, \mathcal{M}, \mu)$ obey $\sum_{n=1}^{\infty} \|f_n\|_\infty = M < \infty$.

For each $n \in \mathbb{N}$, there is a set $N_n \in \mathcal{M}$ with $\mu(N_n) = 0$ such that

$$\sup_{x \in N_n^c} |f_n(x)| \leq \|f_n\|_\infty$$

Then $N = \bigcup_{n \in \mathbb{N}} N_n$ is also of measure zero and

$$|f_n(x)| \leq \|f_n\|_\infty \quad \text{for all } x \in X \setminus N \text{ and all } n \in \mathbb{N}$$

Define

$$g_n(x) = \sum_{k=1}^{n} |f_k(x)|$$

For each $x \in X \setminus N$, $g_n(x)$ increases monotonically with $n$ and is bounded above by

$$\sum_{k=1}^{n} |f_k(x)| \leq \sum_{k=1}^{n} \|f_k\|_\infty \leq M$$

So

$$\lim_{n \to \infty} g_n(x) \text{ exists in } [0, \infty) \text{ for all } x \in X \setminus N$$
Thus the series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely for each $x \in X \setminus N$. As a result,

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \chi_{N^c}(x)$$

converges pointwise to some $s(x)$ and $s(x)$ is measurable. Since,

$$\sup_{x \in X} |s_n(x) - s(x)| \leq \sup_{x \in X \setminus N} \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} \|f_k\|_{\infty}$$

we have

$$\lim_{n \to \infty} \|s_n - s\|_{\infty} \leq \lim_{n \to \infty} \sum_{k=n+1}^{\infty} \|f_k\|_{\infty} = 0$$

which, by definition, says that $s = \lim_{n \to \infty} s_n$ in $L^\infty(X, \mathcal{M}, \mu)$.

$\square$