Lecture Notes on Measure Theory and Integration

8 — L^p Spaces

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November 3, 2019

3 L^p Spaces

Let $1 \le p \le \infty$. Let $(X, \mathcal{M}, \mu)$ be a measure space, with X a set, $\mathcal{M}$ a σ–algebra and μ a measure. For $p < \infty$, set

$$\mathcal{L}^p(X, \mathcal{M}, \mu) = \left\{ \ \varphi : X \to \mathbb{C} \ \middle| \ \varphi \text{ is } \mathcal{M}\text{-measurable with } \int |\varphi(x)|^p \, d\mu(x) < \infty \ \right\}$$
$$\|\varphi\|_p = \left[\int |\varphi(x)|^p \, d\mu(x) \right]^{1/p}$$

For $p = \infty$, set

$$\mathcal{L}^\infty(X, \mathcal{M}, \mu) = \left\{ \ \varphi : X \to \mathbb{C} \ \middle| \ \varphi \text{ is } \mathcal{M}\text{–measurable with } \operatorname{ess\,sup} |\varphi(x)| < \infty \ \right\}$$
$$\|\varphi\|_\infty = \operatorname{ess\,sup} |\varphi(x)|$$

Here, the essential supremum of $|\varphi|$, with respect to the measure μ, is denoted $\operatorname{ess\,sup}_{x \in X} |\varphi(x)|$ and is defined by

$$\operatorname*{ess\,sup}_{x \in X} |\varphi(x)| = \inf \left\{ \ a \ge 0 \ \middle| \ |\varphi(x)| \le a \text{ almost everywhere, with respect to } \mu \ \right\}$$

For each $n \in \mathbb{N}$, there is a null set $N_n \in \mathcal{M}$ such that $|\varphi(x)| \le \frac{1}{n} + \operatorname{ess\,sup}_{x \in X} |\varphi(x)|$ for all $x \in X \setminus N_n$. Then $N = \bigcup_{n \in \mathbb{N}} N_n$ also has measure zero and

$$|\varphi(x)| \le \operatorname*{ess\,sup}_{x \in X} |\varphi(x)| \qquad \text{for all } x \in N$$

The sets $\mathcal{L}^p(X, \mathcal{M}, \mu)$ are vector spaces and $\|\varphi\|_p$ is almost a norm (explained shortly). Here is a review of the definitions.

Definition 3.1 (Vector Space)**.** A vector space over $\mathbb{C}$ is a set $\mathcal{V}$ equipped with two operations,

$$(\mathbf{v}, \mathbf{w}) \in \mathcal{V} \times \mathcal{V} \mapsto \mathbf{v} + \mathbf{w} \in \mathcal{V} \qquad\qquad (\alpha, \mathbf{v}) \in \mathbb{C} \times \mathcal{V} \mapsto \alpha \mathbf{v} \in \mathcal{V}$$

called addition and scalar multiplication, respectively, that obey the following axioms.

Additive Axioms. There is an element $\mathbf{0} \in \mathcal{V}$ and, for each $\mathbf{x} \in \mathcal{V}$, there is an element $-\mathbf{x} \in \mathcal{V}$ such that, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$,

(i) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
(ii) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
(iii) $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$
(iv) $(-\mathbf{x}) + \mathbf{x} = \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

Multiplicative Axioms. For every $\mathbf{x} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$,

(v) $0\,\mathbf{x} = \mathbf{0}$

(vi) $1\,\mathbf{x} = \mathbf{x}$

(vii) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$

Distributive Axioms. For every $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$,

(viii) $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$

(ix) $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$

Remark 3.2.

(a) The zero vector is automatically unique because if both $\mathbf{0}$ and $\mathbf{0}'$ are zero vectors, then

$$\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}'$$

(b) For each $\mathbf{x}$, the additive inverse $-\mathbf{x}$ is automatically unique because, if both $\mathbf{y}$ and $\mathbf{y}'$ are additive inverses for $\mathbf{x}$,

$$\mathbf{y} = \mathbf{0} + \mathbf{y} = (\mathbf{x} + \mathbf{y}') + \mathbf{y} = (\mathbf{x} + \mathbf{y}) + \mathbf{y}' = \mathbf{0} + \mathbf{y}' = \mathbf{y}'$$

(b) The additive inverse for $\mathbf{x}$ is $(-1)\,\mathbf{x}$ since

$$\mathbf{x} + (-1)\,\mathbf{x} = (1)\,\mathbf{x} + (-1)\,\mathbf{x} = (1-1)\,\mathbf{x} = 0\,\mathbf{x} = \mathbf{0}$$

Definition 3.3 (Norm)**.**

(a) A norm on a vector space $\mathcal{V}$ is a function $\mathbf{x} \in \mathcal{V} \mapsto \|\mathbf{x}\| \in [0, \infty)$ that obeys

(i) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(ii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$

(iii) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $\alpha \in \mathbb{C}$.

(b) A sequence $\left\{\mathbf{v}_n\right\}_{n\in\mathbb{N}} \subset \mathcal{V}$ is said to be *Cauchy* with respect to the norm $\|\cdot\|$ if

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\text{ s.t. }\; m, n > N \implies \|\mathbf{v}_n - \mathbf{v}_m\| < \varepsilon$$

(c) A sequence $\left\{\mathbf{v}_n\right\}_{n\in\mathbb{N}} \subset \mathcal{V}$ is said to *converge* to $\mathbf{v}$ in the norm $\|\cdot\|$ if

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\text{ s.t. }\; n > N \implies \|\mathbf{v}_n - \mathbf{v}\| < \varepsilon$$

(d) A normed vector space is said to be *complete* if every Cauchy sequence converges.

(e) A *Banach space* is a complete normed vector space.

Remark 3.4. If $\|\cdot\|$ is a norm, then $\rho(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$ is a metric.

Example 3.5.

(a) Let $n \in \mathbb{N}$. If $X = \{1, 2, \cdots, n\}$, $\mathcal{M} = \mathcal{P}(X)$, the set of all subsets of X, and μ is the counting measure, then, if we identify each $\varphi : \{1, 2, \cdots, n\} \to \mathbb{C}$ with $\big(\varphi(1), \varphi(2), \cdots, \varphi(n)\big)$, we have that

$$\mathcal{L}^2(X, \mathcal{M}, \mu) = \left\{ \ (z_1, z_2, \cdots, z_n) \ \middle| \ z_1, z_2, \cdots, z_n \in \mathbb{C} \ \right\}$$
$$\|(z_1, z_2, \cdots, z_n)\|_2 = \left[\sum_{\ell=1}^{n} |z_\ell|^2 \right]^{1/2}$$

is just $\mathbb{C}^n$ equipped with the usual Euclidean norm.

(b) If $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$, the set of all subsets of $\mathbb{N}$, and μ is the counting measure, then $\mathcal{L}^p(X, \mathcal{M}, \mu)$ is usually denote ℓ^p. That is, identifying each $\varphi : \mathbb{N} \to \mathbb{C}$ with $\big(\varphi(n)\big)_{n\in\mathbb{N}}$,

$$\ell^p = \left\{ \ \big(a_n\big)_{n\in\mathbb{N}} \ \middle| \ a_n \in \mathbb{C} \text{ for all } n \in \mathbb{N}, \ \big\|\big(a_n\big)_{n\in\mathbb{N}}\big\|_p < \infty \ \right\}$$
$$\|\big(a_n\big)_{n\in\mathbb{N}}\|_p = \begin{cases} \left[\sum\limits_{n=1}^{\infty} |a_n|^p \right]^{1/p} & \text{if } 1 \le p < \infty \\ \sup_{n\in\mathbb{N}} |a_n| & \text{if } p = \infty \end{cases}$$

The vector spaces $\mathcal{L}^p(X, \mathcal{M}, \mu)$ are not quite Banach spaces, because $\|\varphi\|_p$ is not quite a norm, because any function φ that is zero almost everywhere has "norm" zero. This is easily fixed. We define an equivalence relation on $\mathcal{L}^p(X, \mathcal{M}, \mu)$ by

$$\varphi \sim \psi \iff \varphi = \psi \text{ a.e.}$$

As usual, the equivalence class of $\varphi \in \mathcal{L}^p(X, \mathcal{M}, \mu)$ is

$$[\varphi] = \left\{ \ \psi \in \mathcal{L}^p(X, \mathcal{M}, \mu) \ \middle| \ \psi \sim \varphi \ \right\}$$

Then

$$L^p(X, \mathcal{M}, \mu) = \left\{ \ [\varphi] \ \middle| \ \varphi \in \mathcal{L}^p(X, \mathcal{M}, \mu) \ \right\}$$

is a Banach space (this will be justified shortly) with

$$[\varphi] + [\psi] = [\varphi + \psi] \qquad a[\varphi] = [a\varphi] \qquad \big\|[\varphi]\big\|_p = \|\varphi\|_p \tag{E1}$$

for all $\varphi, \psi \in \mathcal{L}^p(X, \mathcal{M}, \mu)$ and $a \in \mathbb{C}$. for all $\varphi, \psi \in \mathcal{L}^2(X, \mathcal{M}, \mu)$. It is standard to write φ in place of $[\varphi]$. If X is a Lebesgue measurable subset of $\mathbb{R}$, then $L^p(X)$ denotes the $L^p(X, \mathcal{M}, \mu)$ with $(\mathcal{M}, \mu)$ being Lebesgue measure.

Lemma 3.6. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \le p < \infty$. Then*

$$L^p(X, \mathcal{M}, \mu) = \left\{ \ [\varphi] \ \middle| \ \varphi \in \mathcal{L}^p(X, \mathcal{M}, \mu) \ \right\}$$

with the operations

$$[\varphi] + [\psi] = [\varphi + \psi] \qquad a[\varphi] = [a\varphi]$$

is a well-defined vector space. Furthermore

$$\big\|[\varphi]\big\|_p = \|\varphi\|_p$$

is a well-defined norm on $L^p(X, \mathcal{M}, \mu)$

Proof. Well-definedness: In order for

$$[\varphi] + [\psi] = [\varphi + \psi] \qquad a[\varphi] = [a\varphi] \qquad \big\|[\varphi]\big\|_p = \|\varphi\|_p$$

to be well-defined, we need that

$$\begin{aligned} \varphi_1 \sim \varphi_2,\ \psi_1 \sim \psi_2 &\implies \varphi_1 + \psi_1 \sim \varphi_2 + \psi_2 \\ \varphi_1 \sim \varphi_2 &\implies a\varphi_1 \sim a\varphi_2 \\ \varphi_1 \sim \varphi_2 &\implies \big\|\varphi_1\big\|_p = \big\|\varphi_2\big\|_p \end{aligned}$$

These are all obvious, or at least easy. For example, if

$$\begin{aligned} \left\{ \ x \in X \ \middle| \ \varphi_1(x) \ne \varphi_2(x) \ \right\} \subset N_1 \qquad &\text{with} \quad \mu(N_1) = 0 \\ \left\{ \ x \in X \ \middle| \ \psi_1(x) \ne \psi_2(x) \ \right\} \subset N_2 \qquad &\text{with} \quad \mu(N_2) = 0 \end{aligned}$$

then

$$\left\{ \ x \in X \ \middle| \ \varphi_1(x) + \psi_1(x) \ne \varphi_2(x) + \psi_2(x) \ \right\} \subset N_1 \cup N_2 \quad \text{and} \quad \mu(N_1 \cup N_2) = 0$$

Vector space axioms: Verification of the vector space axioms is also trivial because $\mathbb{C}$ is a field. In particular the zero vector is the function that is identically zero.

Norm axioms: As is standard, we write φ in place of $[\varphi]$.

(i) That $\|\varphi\|_p \ge 0$ with equality if and only if φ is the zero function, i.e. if and only if $\varphi(x) = 0$ a.e., is obvious.

(ii) That $\|a\varphi\|_p = |a|\|\varphi\|_p$ is also obvious.

(iii) The triangle inequality $\|\varphi + \psi\|_p \le \|\varphi\|_p + \|\psi\|_p$ will be part (a) of Theorem 3.8, which is in turn Problem Set 10, #4.

□

Lemma 3.7 (Why L^∞ is called L^∞)**.** *Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $f \in L^\infty(X, \mathcal{M}, \mu)$. Then $f \in L^p(X, \mathcal{M}, \mu)$ for all $1 \le p < \infty$ and*

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$$

Proof. We first develop a preliminary bound. Define, for each $r \ge 0$,

$$X_r = \left\{ \; x \in X \;\middle|\; |f(x)| \ge r \; \right\}$$

If $\mu(X_r) > 0$, then, for each $1 \le p < \infty$,

$$\begin{aligned}\liminf_{p\to\infty} \|f\|_p &= \liminf_{p\to\infty} \left[\int_X |f(x)|^p \, d\mu(x) \right]^{1/p} \ge \liminf_{p\to\infty} \left[\int_{X_r} |f(x)|^p \, d\mu(x) \right]^{1/p} \\ &\ge \liminf_{p\to\infty} \left[\int_{X_r} r^p \, d\mu(x) \right]^{1/p} = \liminf_{p\to\infty} \; r\, \mu(X_r)^{1/p} = r\end{aligned}$$

since $\lim_{p\to\infty} t^{1/p} = 1$ for every strictly positive real number t.

Now let $\|f\|_\infty$ and $\mu(X)$ be finite. Then

$$\begin{aligned}\limsup_{p\to\infty} \|f\|_p &= \limsup_{p\to\infty} \left[\int_X |f(x)|^p \, d\mu(x) \right]^{1/p} \le \limsup_{p\to\infty} \left[\int_X \|f\|_\infty^p \, d\mu(x) \right]^{1/p} \\ &= \liminf_{p\to\infty} \|f\|_\infty \mu(X)^{1/p} = \|f\|_\infty\end{aligned}$$

By the preliminary bound with $r = \|f\|_\infty - \varepsilon$,

$$\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty - \varepsilon$$

for every $\varepsilon > 0$. Hence $\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty$. Combining the upper and lower bounds

$$\|f\|_\infty \le \liminf_{p\to\infty} \|f\|_p \le \limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty$$

□

Theorem 3.8. *Let $(X, \mathcal{M}, \mu)$ be measure space.*

(a) (Minkowski inequality) Let $1 \le p \le \infty$ and $f, g \in L^p(X, \mathcal{M}, \mu)$. Then the sum $f + g \in L^p(X, \mathcal{M}, \mu)$ and

$$\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$$

(b) Let $0 < p < 1$ and $f, g : X \to \mathbb{R}$ be measurable. If $f(x), g(x) \ge 0$ a.e. then

$$\|f + g\|_p \ge \|f\|_p + \|g\|_p$$

(c) *(Hölder inequality) Let* $1 \le p, q \le \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in L^p(X, \mathcal{M}, \mu)$ *and* $g \in L^q(X, \mathcal{M}, \mu)$, *then we have* $fg \in L^1(X, \mathcal{M}, \mu)$ *and*

$$\int_X |f(x)\, g(x)| \; d\mu(x) \le \|f\|_{L^p} \|g\|_{L^q}$$

(d) *(Generalized Hölder inequality) Let* $1 \le r \le \infty$ *and* $1 \le p_j \le \infty$ *with* $\sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{r}$. *If* $f_j \in L^{p_j}$ *for* $1 \le j \le n$, *then* $\prod_{j=1}^{n} f_j \in L^r$ *and*

$$\Big\| \prod_{j=1}^{n} f_j \Big\|_r \le \prod_{j=1}^{n} \|f_j\|_{p_j}$$

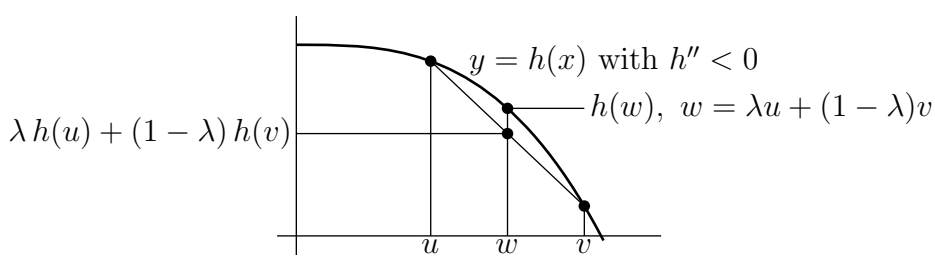
Proof. (a) Problem Set 10, #3.
(b) We are assuming that $f(x), g(x) \ge 0$. If $\|f\|_p = 0$ or $\|g\|_p = 0$, the result is trivial. So assume that $\|f\|_p, \|g\|_p > 0$. By dividing the desired inequalty by $\|f\|_p + \|g\|_p$, we may reduce consideration to $\|f\|_p, \|g\|_p \ne 0$, $\|f\|_p + \|g\|_p = 1$. With $\lambda = \|f\|_p$,

$$\begin{aligned} \|f+g\|_p^p &= \int [f(x) + g(x)]^p \; d\mu(x) \\ &= \int \left[\lambda \tfrac{f(x)}{\|f\|_p} + (1-\lambda) \tfrac{g(x)}{\|g\|_p} \right]^p \; d\mu(x) \end{aligned}$$

Define $h(y) = y^p$. Observe that for $y > 0$

$$h''(y) = p(p-1)y^{p-2} < 0$$

since $0 < p < 1$. That is, h is concave down.



Thus for all $u, v \ge 0$ and $0 \le \lambda \le 1$

$$h\big(\lambda u + (1-\lambda)v\big) \ge \big[\lambda h(u) + (1-\lambda)h(v)\big]$$

Using this with $u = \frac{f(x)}{\|f\|_p}$ and $v = \frac{g(x)}{\|g\|_p}$,

$$\begin{aligned}\|f+g\|_p^p &\geq \int \left[\lambda \tfrac{f(x)^p}{\|f\|_p^p} + (1-\lambda)\tfrac{g(x)^p}{\|g\|_p^p}\right] \, d\mu(x) \\ &= \lambda + (1-\lambda) = 1\end{aligned}$$

(c) Problem Set 10, #4.
(d) First we deal with $n = 2$. By Hölder, with $f = |f_1|^r$, $g = |f_2|^r$, $p = \frac{p_1}{r}$ and $q = \frac{p_2}{r}$,

$$\begin{aligned}\|f_1 f_2\|_r^r &= \int |f_1(x) f_2(x)|^r \, d\mu(x) \leq \||f_1|^r\|_{p_1/r} \||f_2|^r\|_{p_2/r} \\ &= \Big[\int |f_1(x)|^{r(p_1/r)} \, d\mu(x)\Big]^{r/p_1} \Big[\int |f_2(x)|^{r(p_2/r)} \, d\mu(x)\Big]^{r/p_2} \\ &= \|f_1\|_{p_1}^r \|f_2\|_{p_2}^r\end{aligned}$$

Now we proceed by induction. Once the inequality has been established for $n-1$, we apply the $n = 2$ inequality, with f_2 replaced by $\prod_{j=1}^n f_j$ and p_2 replaced by $r' = \left[\sum_{j=2}^n \frac{1}{p_j}\right]^{-1}$.

$$\Big\| \prod_{j=1}^n f_j \Big\|_r \leq \|f_1\|_{p_1} \Big\| \prod_{j=2}^n f_j \Big\|_{r'}$$

Now just apply the $n-1$ inequality. □

Our next main task is to prove that $L^p(X, \mathcal{M}, \mu)$ is complete. To do so, it is convenient to use the following test for completeness.

Lemma 3.9. *A normed vector space is complete if and only if every absolutely convergent series is convergent. That is, if and only if*

$$\sum_{n=1}^\infty \|\overbrace{f_n}^{\in V}\| < \infty \implies \lim_{N\to\infty} f_n \text{ exists in } V$$

Proof. $\implies$: Assume that V is complete and let $\sum_{n=1}^\infty \|f_n\| < \infty$. Then, for all natural numbers $N > M$

$$\Big\| \sum_{n=1}^N f_n - \sum_{n=1}^M f_n \Big\| = \Big\| \sum_{n=M+1}^N f_n \Big\| \leq \sum_{n=M+1}^N \|f_n\| \to 0 \text{ as } M \to \infty$$

So the sequence $\left\{ s_N = \sum_{n=1}^N f_n \right\}_{N\in\mathbb{N}}$ is Cauchy and the series converges.

$\Longleftarrow$: Assume that V obeys

$$\sum_{n=1}^{\infty} \|f_n\| < \infty \implies \lim_{N\to\infty} f_n \text{ exists in } V$$

and let $s_1, s_2, s_3, \cdots$ be a Cauchy sequence. Them for each $k \in \mathbb{N}$, there is an n_k such that

$$m > n_k \implies \left\|s_m - s_{n_k}\right\| \leq \frac{1}{2^k}$$

We can certainly choose $n_k > n_{k-1}$. Define $f_1 = s_{n_1}$ and, for each $k \geq 2$, $f_k = s_{n_{k+1}} - s_{n_k}$. Then

$$\sum_{n=1}^{\infty} \|f_n\| \leq \|f_1\| + \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

$$\implies s_{n_k} = \sum_{n=1}^{k-1} f_n \text{ converges as } k \to \infty$$

So the sequence $\left\{s_n\right\}_{n\in\mathbb{N}}$ is Cauchy and contains a convergent subsequence. So the full sequence $\left\{s_n\right\}_{n\in\mathbb{N}}$ converges too. $\square$

Theorem 3.10. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p \leq \infty$. Then $L^p(X, \mathcal{M}, \mu)$ is a Banach space.*

Proof. Case 1: $1 \leq p < \infty$. Let $\left\{f_n\right\}_{n\in\mathbb{N}} \subset L^p(X, \mathcal{M}, \mu)$ obey $\sum_{n=1}^{\infty} \|f_n\|_p = M < \infty$. Define

$$g_n(x) = \sum_{k=1}^{n} |f_k(x)|$$

Then

$$\|g_n\|_p \leq \sum_{k=1}^{n} \|f_k\|_p \leq M$$

For each $x \in X$, $g_n(x)$ increases monotonically with n. So

$$g(x) = \lim_{n\to\infty} g_n(x) \text{ exists in } [0, \infty]$$

By the monotone convergence theorem

$$\int_X g(x)^p \, d\mu(x) = \lim_{n\to\infty} \overbrace{\int_X g_n(x)^p \, d\mu(x)}^{\leq M^p} \leq M^p$$

Hence $g \in L^p(X, \mathcal{M}, \mu)$ and is finite a.e.. In particular, there is a set $N \in \mathcal{M}$ of measure zero such that the series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely for all $x \in X \setminus N$. As a result,

$$s_n(x) = \sum_{k=1}^{n} f_k(x)\, \chi_{N^c}(x) \text{ converges pointwise to some } s(x)$$

and $s(x)$ is measurable. Since, for all $x \in X$,

$$\begin{aligned} \big|s_n(x) - s(x)\big|^p &\to 0 \\ \big|s_n(x) - s(x)\big|^p &\leq |2g(x)^p| \end{aligned}$$

The (Lebesgue) dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_X \big|s_n(x) - s(x)\big|^p \, d\mu(x) = 0$$

which, by definition, says that $s = \lim_{n\to\infty} s_n$ in $L^p(X, \mathcal{M}, \mu)$.

Case 2: $p = \infty$. The argument is similar to that for the case $1 \leq p < \infty$, but this time we do not have access to the monotone and Lebesgue dominated convergence theorems. Let $\big\{f_n\big\}_{n\in\mathbb{N}} \subset L^\infty(X, \mathcal{M}, \mu)$ obey $\sum_{n=1}^{\infty} \|f_n\|_\infty = M < \infty$.

For each $n \in \mathbb{N}$, there is a set $N_n \in \mathcal{M}$ with $\mu(N_n) = 0$ such that

$$\sup_{x\in N_n^c} |f_n(x)| \leq \|f_n\|_\infty$$

Then $N = \bigcup_{n\in\mathbb{N}} N_n$ is also of measure zero and

$$|f_n(x)| \leq \|f_n\|_\infty \qquad \text{for all } x \in X \setminus N \text{ and all } n \in \mathbb{N}$$

Define

$$g_n(x) = \sum_{k=1}^{n} |f_k(x)|$$

For each $x \in X \setminus N$, $g_n(x)$ increases monotonically with n and is bounded above by

$$\sum_{k=1}^{n} |f_k(x)| \leq \sum_{k=1}^{n} \|f_k\|_\infty \leq M$$

So

$$\lim_{n\to\infty} g_n(x) \text{ exists in } [0, \infty) \text{ for all } x \in X \setminus N$$

Thus the series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely for each $x \in X \setminus N$. As a result,

$$s_n(x) = \sum_{k=1}^{n} f_k(x)\, \chi_{N^c}(x) \text{ converges pointwise to some } s(x)$$

and $s(x)$ is measurable. Since,

$$\sup_{x \in X} \big| s_n(x) - s(x) \big| \leq \sup_{x \in X \setminus N} \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} \|f_k\|_\infty$$

we have

$$\lim_{n \to \infty} \|s_n - s\|_\infty \leq \lim_{n \to \infty} \sum_{k=n+1}^{\infty} \|f_k\|_\infty = 0$$

which, by definition, says that $s = \lim_{n \to \infty} s_n$ in $L^\infty(X, \mathcal{M}, \mu)$. $\square$