Review of Integration

Definition 1 (Integral) Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$.

(a) $L^+(X, \mathcal{M}) = \big\{ f : X \to [0, \infty] \bigm| f \text{ is } \mathcal{M}\text{–measurable} \big\}$. If $f \in L^+(X, \mathcal{M})$, then

$$\int_E f(x)\ d\mu(x) = \sup \Big\{ \sum_{i=1}^n a_i \mu(E_i \cap E) \Bigm| n \in \mathbb{N},\ 0 \le a_i < \infty,\ E_i \in \mathcal{M} \text{ for all } 1 \le i \le n \text{ and } \sum_{i=1}^n a_i \chi_{E_i}(x) \le f(x) \text{ for all } x \in X \Big\} \in [0, \infty]$$

(b) $L^1(E, X, \mathcal{M}, \mu) = \big\{ f : X \to \mathbb{R} \bigm| f \text{ is } \mathcal{M}\text{–measurable and } \int_E |f(x)|\ d\mu(x) < \infty \big\}$

(c) If $f \in L^1(E, X, \mathcal{M}, \mu)$, then

$$\int_E f(x)\ d\mu(x) = \int_E \max\{f(x), 0\}\ d\mu(x) - \int_E \max\{-f(x), 0\}\ d\mu(x)$$

(d) We call $f : X \to \mathbb{R}$ an extended μ–integrable function if it is measurable with at least one of $\int \max\{f, 0\}\ d\mu$ and $\int \max\{-f, 0\}\ d\mu$ finite. Again

$$\int_E f(x)\ d\mu(x) = \int_E \max\{f(x), 0\}\ d\mu(x) - \int_E \max\{-f(x), 0\}\ d\mu(x)$$

Theorem 2 (Integration Toolbox – Algebra) *Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Let either $f, g \in L^+(X, \mathcal{M})$ or $f, g \in L^1(E, X, \mathcal{M}, \mu)$. Then*

(a) If $a_i \ge 0$ and $E_i \in \mathcal{M}$ for each $1 \le i \le n$, then

$$\int_E \Big[\sum_{i=1}^n a_i \chi_{E_i}(x) \Big]\ d\mu(x) = \sum_{i=1}^n a_i \mu(E_i \cap E)$$

The same formula applies if $a_i \in \mathbb{R}$ and $\mu(E_i \cap E) < \infty$ for each $1 \le i \le n$.

(b) In the case $f, g \in L^1$, we have $f + g \in L^1$. In both cases

$$\int_E (f + g)\ d\mu = \int_E f\ d\mu + \int_E g\ d\mu$$

(c) If $f \in L^1$, let $\lambda \in \mathbb{R}$. If $f \in L^+$, let $\lambda > 0$. In the case $f \in L^1$, we have $\lambda f \in L^1$. In both cases

$$\int_E (\lambda f)\ d\mu = \lambda \int_E f\ d\mu$$

Theorem 3 (Integration Toolbox – Bounds) *Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Let $f, g, h : X \to \mathbb{R}$ be measurable with $f, g \in L^1(E, X, \mathcal{M}, \mu)$. Then*

(a) If $|h| \le f$, then $h \in L^1(E, X, \mathcal{M}, \mu)$.

(b) $\left| \int_E f\ d\mu \right| \le \int_E |f|\ d\mu$

(c) If $f \le g$, then $\int_E f\ d\mu \le \int_E g\ d\mu$

(d) If h is bounded and $\mu(E) < \infty$, then $h \in L^1(E, X, \mathcal{M}, \mu)$ and

$$\left| \int_E h\ d\mu \right| \le \mu(E) \sup_{x \in E} |h(x)|$$

Lemma 4 (Fatou's Lemma) *Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Let, for each $n \in \mathbb{N}$, $f_n \in L^+(X, \mathcal{M})$. Then*

$$\int_E \liminf_{n\to\infty} f_n(x)\ d\mu(x) \le \liminf_{n\to\infty} \int_E f_n(x)\ d\mu(x)$$

Theorem 5 (Monotone Convergence Theorem) *Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Let $f \in L^+(X, \mathcal{M})$ and, for each $n \in \mathbb{N}$, $f_n \in L^+(X, \mathcal{M})$. Assume that*

$$0 \le f_n \le f \text{ a.e. on } E\ \forall\ n \in \mathbb{N} \qquad \text{and} \qquad \lim_{n\to\infty} f_n = f \text{ a.e. on } E$$

Then

$$\int_E f(x)\ d\mu(x) = \lim_{n\to\infty} \int_E f_n(x)\ d\mu(x)$$

Theorem 6 ((Lebesgue) Dominated Convergence Theorem) *Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Let $f, g : X \to \mathbb{R}$ and, for each $n \in \mathbb{N}$, $f_n : X \to \mathbb{R}$ be measurable. Assume that, $g \in L^1(E, X, \mathcal{M}, \mu)$ and*

$$|f_n| \le g \text{ a.e. on } E\ \forall\ n \in \mathbb{N} \qquad \text{and} \qquad \lim_{n\to\infty} f_n = f \text{ a.e. on } E$$

Then $f \in L^1(E, X, \mathcal{M}, \mu)$ and

$$\int_E f(x)\ d\mu(x) = \lim_{n\to\infty} \int_E f_n(x)\ d\mu(x)$$

Definition 7 (Riemann Integral) Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ be bounded.

(a) The *upper Riemann integral* of f is

$$\overline{\mathcal{R}\!\!\int_b^a} f(x)\ dx = \inf\Big\{ \sum_{i=1}^n \big(t_i - t_{i-1}\big) \sup_{t_{i-1}\le x\le t_i} f(x)\ \Big|\ n \in \mathbb{N},\ a = t_0 < t_1 < \cdots < t_n = b \Big\}$$

(b) The *lower Riemann integral* of f is

$$\underline{\mathcal{R}\!\!\int_b^a} f(x)\ dx = \sup\Big\{ \sum_{i=1}^n \big(t_i - t_{i-1}\big) \inf_{t_{i-1}\le x\le t_i} f(x)\ \Big|\ n \in \mathbb{N},\ a = t_0 < t_1 < \cdots < t_n = b \Big\}$$

(c) The function f is *Riemann integrable* if

$$\underline{\mathcal{R}\!\!\int_b^a} f(x)\ dx = \overline{\mathcal{R}\!\!\int_b^a} f(x)\ dx$$

We shall denote the common value $\mathcal{R}\!\!\int_b^a f(x)\ dx$.

Theorem 8 **(Riemann integrable $\implies$ Lebesgue integrable)** *Let $a < b$ be real numbers and let $f : [a,b] \to \mathbb{R}$ be bounded.*

(a) If f is Riemann integrable, then f is Lebesgue measurable (and hence L^1, since f is bounded and the interval $[a,b]$ is finite) and

$$\mathcal{R}\!\!\!\int_b^a f(x)\ dx = \int_{[a,b]} f(x)\ dm(x)$$

(b) f is Riemann integrable if and only if $\big\{\ x \in [a,b]\ \big|\ f \text{ is not continuous at } x\ \big\}$ has Lebesgue measure zero.

Definition 9 **(Product Measure)** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces.

(a) Define the set of finite disjoint unions of measurable rectangles in $X \times Y$ to be

$$\mathcal{R} = \Big\{\ \textstyle\bigcup_{j=1}^n A_j \times B_j\ \Big|\ n \in \mathbb{N},\ A_j \in \mathcal{M},\ B_j \in \mathcal{N},\ (A_j \times B_j) \cap (A_k \times B_k) = \emptyset, \\ \text{for all } 1 \le j,k \le n \text{ with } j \ne k\ \Big\}$$

$\mathcal{R}$ is nonempty and closed under complements and finite unions and so is an algebra.

(b) Define $\mathcal{M} \otimes \mathcal{N} = \mathcal{M}(\mathcal{R})$ to be the σ–algebra generated by $\mathcal{R}$.

(c) Define $\pi : \mathcal{R} \to [0, \infty]$ by

$$\pi\Big(\textstyle\bigcup_{j=1}^n A_j \times B_j\Big) = \displaystyle\sum_{j=1}^n \mu(A_j)\nu(B_j)$$

for all $n \in \mathbb{N}$ and all $A_j \in \mathcal{M},\ B_j \in \mathcal{N},\ 1 \le j \le n$ with $(A_j \times B_j) \cap (A_k \times B_k) = \emptyset$ for all $j \ne k$. In this definition, we use the convention that $0 \times \infty = 0$. π is a well–defined premeasure.

(d) By Theorem 8 of our "Review of Measure Theory", $\mu \times \nu = \pi^* \restriction \mathcal{M} \otimes \mathcal{N}$ is a measure which extends π. If μ and ν are σ–finite, then $\mu \times \nu$ is σ–finite and is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ such that

$$\mu \times \nu(A \times B) = \mu(A)\nu(B)\ \forall\ A \in \mathcal{M},\ B \in \mathcal{N}$$

Proposition 10 **(Slices – sets)** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces, $x \in X$ and $y \in Y$.*

(a) If $E \in \mathcal{M} \otimes \mathcal{N}$, then

$$E_x = \big\{\ y' \in Y\ \big|\ (x, y') \in E\ \big\} \in \mathcal{N}$$
$$E^y = \big\{\ x' \in X\ \big|\ (x', y) \in E\ \big\} \in \mathcal{M}$$

(b) If $f : X \times Y \to \mathbb{R}$ is $\mathcal{M} \otimes \mathcal{N}$–measurable, then the function $f_x : Y \to \mathbb{R}$ defined by $f_x(y) = f(x,y)$ is $\mathcal{N}$–measurable and the function $f^y : X \to \mathbb{R}$ defined by $f^y(x) = f(x,y)$ is $\mathcal{M}$–measurable.

Proposition 11 **(Tensor product of Borel σ–algebras)** *Let X and Y be separable metric spaces with metrics d_X and d_Y respectively. Then $X \times Y$ is a metric space with metric $D\big((x,y),(x',y')\big) = \sqrt{d_X(x,x')^2 + d_Y(y,y')^2}$ and $\mathcal{B}_{X\times Y} = \mathcal{B}_X \otimes \mathcal{B}_Y$.*

(By definition, a metric space is separable if and only if it contains a countable dense subset. For example $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$, so that $\mathbb{R}$ is separable. Applying this proposition to $X = Y = \mathbb{R}$ gives that $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.)

Proposition 12 **(Slices - measure)** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ–finite measure spaces and let $E \in \mathcal{M} \otimes \mathcal{N}$. Then the function $f : X \to [0,\infty]$ defined by $f(x) = \nu(E_x)$ is $\mathcal{M}$–measurable and the function $g : Y \to [0,\infty]$ defined by $g(y) = \mu(E^y)$ is $\mathcal{N}$–measurable and*

$$\mu \times \nu(E) = \int \nu(E_x)\ d\mu(x) = \int \mu(E^y)\ d\nu(y)$$

Theorem 13 **(Fubini–Tonelli)** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ–finite measure spaces.*

(a) (Tonelli) If $f : X \times Y \to [0,\infty]$ is $\mathcal{M} \otimes \mathcal{N}$–measurable, then the function $g : X \to [0,\infty]$ defined by $g(x) = \int f(x,y)\ d\nu(y)$ is $\mathcal{M}$–measurable and the function $h : Y \to [0,\infty]$ defined by $h(y) = \int f(x,y)\ d\mu(x)$ is $\mathcal{N}$–measurable and

$$\int f(x,y)\ d\mu \times \nu(x,y) = \int \left[\int f(x,y)\ d\nu(y) \right] d\mu(x) = \int \left[\int f(x,y)\ d\mu(x) \right] d\nu(y)$$

(b) (Fubini) If $f \in L^1(\mu \times \nu)$ then

- *the function $f_x : Y \to \mathbb{R}$ defined by $f_x(y) = f(x,y)$ is in $L^1(\nu)$ for almost all $x \in X$,*
- *$g(x) = \int f(x,y)\ d\nu(y) \in L^1(\mu)$*
- *the function $f^y : X \to \mathbb{R}$ defined by $f^y(x) = f(x,y)$ is in $L^1(\mu)$ for almost all $y \in Y$,*
- *$h(y) = \int f(x,y)\ d\mu(x) \in L^1(\nu)$*

and

$$\int f(x,y)\ d\mu \times \nu(x,y) = \int \left[\int f(x,y)\ d\nu(y) \right] d\mu(x) = \int \left[\int f(x,y)\ d\mu(x) \right] d\nu(y)$$