Incompleteness of L^2 with the Riemann Integral

In these notes we shall see that the "normed" vector space

$$\left\{ \; f : [0,1] \to \mathbb{C} \;\middle|\; f \text{ is Riemann integrable} \; \right\}$$

with the "norm"

$$\|f\| = \left[\int_0^1 |f(x)|^2 \, dx \right]^{1/2}$$

is not complete, by exhibiting an example of a Cauchy sequence that does not converge.

The reason that I have put "norm(ed)" above in quotes is that, to be picky, $\| \cdot \|$ is not a norm, because there exist nonzero functions f with $\|f\| = 0$. Indeed, if $f : [0,1] \to \mathbb{C}$ takes nonzero values at only a finite number of points in $[0,1]$, then $\|f\| = 0$. This is a technicality that is relatively easy to deal with*, but, for pedagogical reasons, I am simply going to ignore it (and its consequences).

Here is an example of a Cauchy sequence that does not converge.

Example 1 Consider the sequence

$$\left\{ f_n(x) = \frac{\chi_{[1/n\,,\,1]}}{x^{1/4}} \right\}_{n=1}^{\infty}$$

Then, for all $m, n \in \mathbb{N}$ with $m > n$,

$$f_m(x) - f_n(x) = \frac{\chi_{[1/m\,,\,1/n)}}{x^{1/4}}$$

so that

$$\left\| f_m - f_n \right\|^2 = \int_{1/m}^{1/n} \frac{dx}{x^{1/2}} = \frac{2}{\sqrt{n}} - \frac{2}{\sqrt{m}}$$

converges to zero as $m, n \to \infty$. So this sequence is Cauchy. But there is no Riemann integrable function g on $[0,1]$ such that $\lim\limits_{n\to\infty} f_n = g$, because, if there were,

- for each $0 < a < 1$, the sequence $\left\{ \chi_{[a,1]}(x)\, f_n(x) \right\}_{n=1}^{\infty}$ would have to converge to $\chi_{[a,1]}(x)\, g(x)$. For all $n > \frac{1}{a}$, $\chi_{[a,1]}(x)\, f_n(x) = \chi_{[a\,,\,1]}(x)\, x^{-1/4}$ so we would necessarily have $\chi_{[a,1]}(x) g(x) = \chi_{[a\,,\,1]}(x)\, x^{-1/4}$.
- As this is the case for all $0 < a < 1$, we must have $g(x) = x^{-1/4}$ for all $0 < x \le 1$.
- So $g(x)$ is not bounded and consequently is not Riemann integrable.

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* The idea is to define two functions, f and g, to be equivalent if $\int_0^1 |f(x) - g(x)|^2 \, dx = 0$ and then work in the vector space of equivalence classes of functions.

Next, we have a second example of a Cauchy sequence that does not converge. However the justification for this second example heavily uses measure theory. In particular, it uses the following theorem, in which $dm(x)$ refers to the Lebesgue measure, $\mathbb{R}\!\!\!\!\!\int_b^a$ refers to the Riemann integral and $\int_{[a,b]}$ refers to the Lebesgue integral.

Theorem. *Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be bounded.*

(a) If f is Riemann integrable, then f is Lebesgue measurable (and hence L^1, since f is bounded and the interval $[a, b]$ is finite) and

$$\mathbb{R}\!\!\!\!\!\int_b^a f(x)\ dx = \int_{[a,b]} f(x)\ dm(x)$$

(b) f is Riemann integrable if and only if $\big\{\ x \in [a, b]\ \big|\ f \text{ is not continuous at } x\ \big\}$ has Lebesgue measure zero.

Example 2 Let q_k, $k \in \mathbb{N}$, be an enumeration of the rational numbers in $[0, 1]$. For each $n \in \mathbb{N}$, we construct the subset $F_n \subset [0, 1]$ by "fattening out" the first n rationals.

$$F_n = \bigcup_{k=1}^{n} \big(q_k - \tfrac{1}{2^{k+2}}\ ,\ q_k + \tfrac{1}{2^{k+2}}\big) \cap [0, 1]$$

Set

$$F = \bigcup_{n=1}^{\infty} F_n = \bigcup_{k=1}^{\infty} \big(q_k - \tfrac{1}{2^{k+2}}\ ,\ q_k + \tfrac{1}{2^{k+2}}\big) \cap [0, 1]$$

The length of the interval $\big(q_k - \frac{1}{2^{k+2}}\ ,\ q_k + \frac{1}{2^{k+2}}\big)$ is $\frac{1}{2^{k+1}}$. As F is a union of intervals whose lengths add up to $\sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2}$, F is a proper subset of $[0, 1]$, and indeed contains less than half of $[0, 1]$. Define the corresponding functions

$$f_n = \chi_{F_n} \qquad f = \chi_F$$

- We first show that, if the sequence $\big\{f_n\big\}_{n=1}^{\infty}$ converges, with respect to the norm $\|\cdot\|$, to a Riemann integrable function g, then $g = f$ (except possibly on a set of measure zero). Assume that $\big\{f_n\big\}_{n=1}^{\infty}$ converges to a Riemann integrable function g. Then g is also Lebesgue integrable and $g = \lim_{n\to\infty} f_n$ in $L^1\big([0, 1]\big)$. As $\big\{f_n\big\}_{n=1}^{\infty}$ is uniformly bounded and converges pointwise to f, we also have that $f = \lim_{n\to\infty} f_n$ in $L^1\big([0, 1]\big)$, by the Lebesgue dominated convergence theorem. Consequently $f = g$, except possibly on a set of measure zero.

- We next show that f is not Riemann integrable. Note that
  - if $q \in [0,1] \cap \mathbb{Q}$, then $f(q) = 1$ and
  - if $x \in [0,1] \setminus F$, then $f(x) = 0$ and f is not continuous at x, as there is a sequence $\{q_n\}_{n=1}^{\infty} \subset [0,1]$ of rational numbers that converge to x and every $f(q_n) = 1$.
  - So $\{\ x \in [0,1] \mid f \text{ is not continuous at } x\ \}$ has measure at least $m\big([0,1] \setminus F\big) \geq \frac{1}{2}$ and f is not Riemann integrable.
- Next, here is a second, independent, proof that f is not Riemann integrable. If you are happy with the last bullet, you can skip this one. This proof does not use any measure theory. In the definition of the Riemann integral $\int_0^1 f(x)\ dx$, we pick any natural number n and partition the interval $[0,1]$ into n subintervals $[x_{i-1}, x_i]$, $1 \leq i \leq n$. The corresponding upper and lower Riemann sums are

$$U_n = \sum_{i=1}^{n} \max_{x_{i-1} \leq t \leq x_i} f(t)\ \big[x_i - x_{i-1}\big] \qquad L_n = \sum_{i=1}^{n} \min_{x_{i-1} \leq t \leq x_i} f(t)\ \big[x_i - x_{i-1}\big]$$

  - The function f is a characteristic function and so only takes the values 0 and 1.
  - Every subinterval contains a rational number, where f takes the value 1. So the upper sum $U_n = \sum_{i=1}^{n} \big[x_i - x_{i-1}\big] = 1$.
  - If the ith subinterval contains at least one point in $[0,1] \setminus F$, where f takes the value 0, then $\min_{x_{i-1} \leq t \leq x_i} f(t) = 0$. So the lower sum $L_n = \sum_{i \notin I_n} \big[x_i - x_{i-1}\big] = 1 - \sum_{i \in I_n} \big[x_i - x_{i-1}\big]$, where

$$I_n = \{\ 1 \leq i \leq n \mid \big[x_{i-1}, x_i\big] \cap [0,1] \setminus F \neq \emptyset\ \}$$

  - The union of $\bigcup_{i \in I_n} \big[x_{i-1}, x_i\big]$ with $F = \bigcup_{k \in \mathbb{N}} \big(q_k - \frac{1}{2^{k+2}}\ ,\ q_k + \frac{1}{2^{k+2}}\big)$ covers all of $[0,1]$. So the sum of the lengths of the intervals $\big(q_k - \frac{1}{2^{k+2}}\ ,\ q_k + \frac{1}{2^{k+2}}\big)$, $k \in \mathbb{N}$, which we already know is $\frac{1}{2}$, added to the sum of the lengths of the subintervals $\big[x_i - x_{i-1}\big]$, $i \in I_n$ is greater than 1. So $\sum_{i \in I_n} \big[x_i - x_{i-1}\big] \geq \frac{1}{2}$ and the lower sum $L_n \leq \frac{1}{2}$.

  Taking the limit over finer and finer partitions, we have that the upper Riemann integral $\displaystyle\overline{\int_a^b} f(x)\ dx = 1$ and the lower Riemann integral $\displaystyle\underline{\int_a^b} f(x)\ dx \leq \frac{1}{2}$. So f is not Riemann integrable.
- Finally we show that if $g = f$, except possibly on a set of measure zero, then g is also not Riemann integrable. Write $F' = [0,1] \setminus F$ and $G = \{\ x \in [0,1] \mid f(x) \neq g(x)\ \}$. Then
  - $g(x) = 1$ for all $x \in F \setminus G$
  - $g(x) = 0$ for all $x \in F' \setminus G$
  - $m(F' \setminus G) = m(F') \geq \frac{1}{2}$

It remains only to show that g is not continuous at every point of $F' \setminus G$, which we now do. Let $x \in F' \setminus G$. Then

- $g(x) = f(x) = 0$.
- Let $\varepsilon > 0$. Then $\big(x - \varepsilon, x + \varepsilon\big) \cap [0, 1]$ contains a rational number and so also contains an interval I_ε of nonzero length that is contained in F.
- Then $I_\varepsilon \setminus G$ has nonzero measure and in particular is not empty.
- As $g(y) = f(y) = 1$ for every $y \in I_\varepsilon \setminus G$ with $\varepsilon > 0$, g is not continuous at x.