Cardinality

The “cardinality” of a set provides a precise meaning to “the number of elements” in the set, even when the set contains infinitely many elements. We start with the following very reasonable definitions of “the set $S$ contains the same number of elements as the set $T$”, “the set $S$ contains fewer elements than the set $T$” and “the set $S$ contains more elements then the set $T$”.

**Definition 1** Let $S$ and $T$ be nonempty sets. Then

- $\text{card}(S) = \text{card}(T)$ if and only if there exists an $f : S \to T$ which is one-to-one and onto (i.e. bijective).
- $\text{card}(S) \leq \text{card}(T)$ if and only if there exists an $f : S \to T$ which is one-to-one (i.e. injective).
- $\text{card}(S) \geq \text{card}(T)$ if and only if there exists an $f : S \to T$ which is onto (i.e. surjective).
- $\text{card}(S) < \text{card}(T)$ if and only if $\text{card}(S) \leq \text{card}(T)$ but $\text{card}(S) \neq \text{card}(T)$.
- $\text{card}(S) > \text{card}(T)$ if and only if $\text{card}(S) \geq \text{card}(T)$ but $\text{card}(S) \neq \text{card}(T)$.

**Definition 2** Let $S$ be a nonempty set. Then

(a) The set $S$ has $\text{card}(S) = n \in \mathbb{N}$ if $\text{card}(S) = \text{card}\left(\{1, 2, 3, \ldots, n\}\right)$.
(b) The set $S$ is countable if $\text{card}(S) \leq \text{card}(\mathbb{N})$.

The set $S$ is countably infinite if $\text{card}(S) = \text{card}(\mathbb{N})$.
(c) The set $S$ is uncountable if it is not countable.
(d) The set $S$ has the cardinality of the continuum if $\text{card}(S) = \text{card}(\mathbb{R})$.

We now verify that $\leq$ and $\geq$, in the sense of cardinality, work as expected.

**Proposition 3** Let $S$, $T$ and $U$ be nonempty sets. Then

(a) $\text{card}(S) \leq \text{card}(T)$ if and only if $\text{card}(T) \geq \text{card}(S)$.
(b) Either $\text{card}(S) \leq \text{card}(T)$ or $\text{card}(S) \geq \text{card}(T)$.
(c) If $\text{card}(S) \leq \text{card}(T)$ and $\text{card}(S) \geq \text{card}(T)$, then $\text{card}(S) = \text{card}(T)$.
(d) If $\text{card}(S) \leq \text{card}(T)$ and $\text{card}(T) \leq \text{card}(U)$, then $\text{card}(S) \leq \text{card}(U)$.

**Proof:** (a) If $\text{card}(S) \leq \text{card}(T)$, then there is a one-to-one function $f : S \to T$. Fix any $s_0 \in S$. As $f$ is one-to-one

$$g : T \to S$$

$$g(t) = \begin{cases} f^{-1}(t) & \text{if } t \text{ is in the range of } f \\ s_0 & \text{otherwise} \end{cases}$$

is a well defined function. As $f$ is defined on all of $S$ the range of $f^{-1}$ is all of $S$ and $g$ maps $T$ onto $S$, so that $\text{card}(T) \geq \text{card}(S)$.

If $\text{card}(T) \geq \text{card}(S)$, then there is a function $g$ that maps $T$ onto $S$. So the sets

$$g^{-1}(s) = \{ t \in T \mid g(t) = s \}, \quad s \in S$$
are disjoint and nonempty. For each \( s \in S \) select one \( t_s \in g^{-1}(s) \). Then \( f(s) = t_s \) defines a one-to-one map from \( S \) into \( T \). So \( \text{card}(S) \leq \text{card}(T) \).

(b) Denote by \( \mathcal{I} \) the set of all one-to-one maps defined on a subset of \( S \) and taking values in \( T \). Define a partial ordering on \( \mathcal{I} \) by \( f \leq g \) if

- the domain of \( f \) is contained in the domain of \( g \) and
- \( f(s) = g(s) \) for all \( s \) in the domain of \( f \).

Let \( \mathcal{I}' \) be any linearly ordered\(^{(1)}\) subset of \( \mathcal{I} \). Let \( F \) be the function

- whose domain is the union of all of the domains of functions in \( \mathcal{I}' \) and
- whose value at \( s \) is \( f(s) \) for any \( f \in \mathcal{I}' \) whose domain contains \( s \).

Then \( F \) is an upper bound for \( \mathcal{I}' \). So, by Zorn’s lemma (see Folland page 5), \( \mathcal{I} \) has a maximal element, say \( G \). itemo In the event that the domain of \( G \) is \( S \), then \( G : S \to T \) is one-to-one and \( \text{card}(S) \leq \text{card}(T) \).

- In the event that the range of \( G \) is \( T \), then \( G^{-1} : T \to S \) is one-to-one and \( \text{card}(T) \leq \text{card}(S) \).
- The only remaining possibility is that the domain of \( G \) is a proper subset of \( S \) and the range of \( G \) is a proper subset of \( T \). But then there is an \( s_0 \in S \) that is not in the domain of \( G \) and there is a \( t_0 \in T \) that is not in the range of \( G \), and the function

\[
H(s) = \begin{cases} 
G(s) & \text{if } s \text{ is in the domain of } G \\
0 & \text{if } s = s_0 
\end{cases}
\]

is a one-to-one map defined on a subset of \( S \) and taking values in \( T \) that obeys \( H > G \). This violates the assumed maximality of \( G \).

(c) Because \( \text{card}(S) \leq \text{card}(T) \) and \( \text{card}(S) \geq \text{card}(T) \), there exist one-to-one maps \( f : S \to T \) and \( g : T \to S \). For each \( s \in S \), define the points \( x_1(s) \in S, x_2(s) \in T, x_3(s) \in S, x_4(s) \in T, \ldots \) as follows.

- \( x_1(s) = s \in S \)
- if \( x_1(s) \in g(T), \) set \( x_2(s) = g^{-1}(x_1(s)) \in T \)
- if \( x_2(s) \in f(S), \) set \( x_3(s) = f^{-1}(x_2(s)) \in S \)
- and so on.

Either

- this process continues indefinitely, or,
- for some odd number \( j \in \mathbb{N} \), \( x_j(s) \in S \setminus g(T) \) and the process terminates or,
- for some even number \( j \in \mathbb{N} \), \( x_j(s) \in T \setminus f(S) \) and the process terminates.

In these three cases we say that

- \( s \) is in \( S_\infty \) (never terminates), or
- \( s \) is in \( S_S \) (terminates with \( x_j(s) \in S \setminus g(T) \) for some odd \( j \)) or
- \( s \) is in \( S_T \) (terminates with \( x_j(s) \in T \setminus f(S) \) for some \( j \) even).

So \( S \) is the disjoint union of \( S_\infty \), \( S_S \) and \( S_T \). Similarly, define \( y_1(t) \in T, y_2(t) \in S, y_3(t) \in T, y_4(t) \in S, \ldots \) by

- \( y_1(t) = t \in T \)
- if \( y_1(t) \in f(S), \) set \( y_2(t) = f^{-1}(y_1(t)) \in S \)

\(^{(1)}\) This means that if \( f, g \in \mathcal{I}' \), then either \( f \leq g \) or \( g \leq f \).
Thus $f$ maps $S$ onto $T$ and $S$ is a one-to-one map from $S$ into $T$.

Moreover, by construction,

Consequently,

Observ that $y_2(f(s)) = f^{-1}(y_1(f(s))) = f^{-1}(f(s)) = s = x_1(s)$. Similarly, by induction,

for all $j \geq 2$.

Observe that $x_2(g(t)) = g^{-1}(x_1(g(t))) = g^{-1}(g(t)) = t = y_1(t)$. Similarly, by induction,

for all $j \geq 2$.

Consequently,

$s \in S_\infty \implies x_{j-1}(s) \in \begin{cases} g(T) & j \text{ even} \\ f(S) & j \text{ odd} \end{cases} \implies y_j(f(s)) \in \begin{cases} g(T) & j \text{ even} \\ f(S) & j \text{ odd} \end{cases} \implies f(s) \in T_\infty$

That is,

- $f(S_\infty) \subset T_\infty$, and similarly $f(S_S) \subset T_S$, $f(S_T) \subset T_T$, $g(T_\infty) \subset S_\infty$, $g(T_S) \subset S_S$ and $g(T_T) \subset S_T$.

Furthermore, by construction,

- if $s \in S_T$, then $x_1(s)$ is not terminal. That is, $x_1(s) \notin S \setminus g(T)$. So $s = x_1(s) \in g(T)$. That is, $g$ maps onto $S_T$.
- Similarly, if $t \in T_\infty \cup T_S$, then $y_1(t)$ is not terminal and $y_1(t) \notin T \setminus f(S)$. So $t = y_1(t) \in f(S)$. That is, $f$ maps onto $T_\infty \cup T_S$.

Thus $f$ maps $S_\infty \cup S_S$ one-to-one and onto $T_\infty \cup T_S$ and $g$ maps $T_T$ one-to-one and onto $S_T$. Hence

$F(s) = \begin{cases} f(s) & \text{if } s \in S_\infty \cup S_S \\ g^{-1}(s) & \text{if } s \in S_T \end{cases}$

maps $S$ one-to-one and onto $T$.

(d) If $f$ is a one-to-one map from $S$ into $T$ and $g$ is a one-to-one map from $T$ into $U$, then $h = g \circ f$ is a one-to-one map from $S$ into $U$. ■

Proposition 4

(a) If $I$ is countable and, for each $i \in I$, $S_i$ is countable, then $\bigcup_{i \in I} S_i$ is countable.

(b) If $S$ is a nonempty set and $P(S)$ is the collection of all subsets of $S$, then

$$\text{card}(S) < \text{card}(P(S))$$
Proof: (a) Since $I$ is countable, there is a function $f : \mathbb{N} \rightarrow I$ that is onto $I$. For each $i \in I$, $S_i$ is countable so that there is a function $F_i : \mathbb{N} \rightarrow S_i$ that is onto $S_i$. We need to define a function $g : \mathbb{N} \rightarrow \bigcup_{i \in I} S_i$ that is onto $\bigcup_{i \in I} S_i$. Every element of $\bigcup_{i \in I} S_i$ is of the form $F_{f(m)}(n)$ for some $(m, n) \in \mathbb{N} \times \mathbb{N}$. Every element $(m, n)$ of $\mathbb{N} \times \mathbb{N}$ has $m + n \in \mathbb{N}$ and $m + n \geq 2$. We first ensure that the range of $g$ contains all $F_{f(m)}(n)$’s with $m + n = 2$ by setting

$$g(1) = F_{f(1)}(1).$$

We next ensure that the range of $g$ contains all $F_{f(m)}(n)$’s with $m + n = 3$ by setting

$$g(2) = F_{f(1)}(2), \quad g(3) = F_{f(2)}(1).$$

We next ensure that the range of $g$ contains all $F_{f(m)}(n)$’s with $m + n = 4$ by setting

$$g(4) = F_{f(1)}(3), \quad g(5) = F_{f(2)}(2), \quad g(6) = F_{f(3)}(1).$$

We next ensure that the range of $g$ contains all $F_{f(m)}(n)$’s with $m + n = 5$ by setting

$$g(7) = F_{f(1)}(4), \quad g(8) = F_{f(2)}(3), \quad g(9) = F_{f(3)}(2), \quad g(10) = F_{f(4)}(1)$$

and so on.

(b) The function $f(s) = \{s\}$ is a one-to-one map from $S$ into $\mathcal{P}(S)$, so it suffices to prove that there does not exist a function $g : S \rightarrow \mathcal{P}(S)$ that is onto $\mathcal{P}(S)$. We find a contradiction to the hypothesis that there does exist such a function. Suppose that $g : S \rightarrow \mathcal{P}(S)$ is onto $\mathcal{P}(S)$. Set $T = \{s \in S \mid s \notin g(s)\} \in \mathcal{P}(S)$. If $T$ is in the range of $g$, there is an $s_0 \in S$ with $T = g(s_0)$.

- If $s_0 \in T$, then $s_0 \in T = g(s_0)$ and the definition of $T$ gives that $s_0 \notin T$, which is a contradiction.
- If $s_0 \notin T$, then $s_0 \notin T = g(s_0)$ and the definition of $T$ gives that $s_0 \in T$, which is a contradiction.

Example 5
(a) $\mathbb{Z}$ is countable.
(b) $\mathbb{Q}$ is countable.
(c) $\mathbb{R}$ is uncountable.

Proof: (a) The function

$$f(p) = \begin{cases} 1 & \text{if } p = 0 \\ 2p & \text{if } p > 0 \\ -2p + 1 & \text{if } p < 0 \end{cases}$$

is a one-to-one, onto function from $\mathbb{Z}$ to $\mathbb{N}$.

(b) Write $\mathbb{Q} = \bigcup_{n \in \mathbb{N}} \{\frac{p}{n} \mid p \in \mathbb{Z}\}$ and apply part (a) of Proposition 4.
(c) If \( \mathbb{R} \) were countable, there would be an \( f : \mathbb{N} \rightarrow \mathbb{R} \) that maps \( \mathbb{N} \) onto \( \mathbb{R} \). We shall show that this is impossible by constructing a real number that is not in the range of \( f \). Write out a decimal representation of \( f(n) \) for each \( n \in \mathbb{N} \).

\[
\begin{align*}
f(1) &= d_0^{(1)} \cdot d_1^{(1)} d_2^{(1)} d_3^{(1)} d_4^{(1)} \ldots \\
f(2) &= d_0^{(2)} \cdot d_1^{(2)} d_2^{(2)} d_3^{(2)} d_4^{(2)} \ldots \\
f(3) &= d_0^{(3)} \cdot d_1^{(3)} d_2^{(3)} d_3^{(3)} d_4^{(3)} \ldots \\
&\vdots
\end{align*}
\]

with \( d_0^{(n)} \in \mathbb{Z} \) for each \( n \in \mathbb{N} \) and \( d_m^{(n)} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \) for each \( (n, m) \in \mathbb{N} \times \mathbb{N} \). Define, for each \( m \in \mathbb{N} \),

\[
d_m = \begin{cases} 
7 & \text{if } d_m^{(n)} \in \{0, 1, 2, 3, 4\} \\
2 & \text{if } d_m^{(n)} \in \{5, 6, 7, 8, 9\}
\end{cases}
\]

Then the decimal number

\[
d = 0 \cdot d_1 d_2 d_3 d_4 \ldots
\]

cannot be \( f(1) \) because its first decimal place is wrong. It cannot be \( f(2) \) because its second decimal place is wrong. And so on. So \( d \) is not in the range of \( f \).