Cardinality

The “cardinality” of a set provides a precise meaning to “the number of elements” in the set, even when the set contains infinitely many elements. We start with the following very reasonable definitions of “the set $S$ contains the same number of elements as the set $T$”, “the set $S$ contains fewer elements than the set $T$” and “the set $S$ contains more elements then the set $T$”.

**Definition 1** Let $S$ and $T$ be nonempty sets. Then

- $\text{card} (S) = \text{card} (T) \iff$ there exists an $f : S \to T$ which is one–to–one and onto
- $\text{card} (S) \leq \text{card} (T) \iff$ there exists an $f : S \to T$ which is one–to–one
- $\text{card} (S) \geq \text{card} (T) \iff$ there exists an $f : S \to T$ which is onto
- $\text{card} (S) < \text{card} (T) \iff$ $\text{card} (S) \leq \text{card} (T)$ but $\text{card} (S) \neq \text{card} (T)$
- $\text{card} (S) > \text{card} (T) \iff$ $\text{card} (S) \geq \text{card} (T)$ but $\text{card} (S) \neq \text{card} (T)$

**Definition 2** Let $S$ be a nonempty set. Then

(a) The set $S$ has $\text{card} (S) = n \in \mathbb{N}$ if $\text{card} (S) = \text{card} \lbrace 1, 2, 3, \cdots, n \rbrace$.

(b) The set $S$ is countable if $\text{card} (S) \leq \text{card} (\mathbb{N})$. The set $S$ is countably infinite if $\text{card} (S) = \text{card} (\mathbb{N})$.

(c) The set $S$ is uncountable if it is not countable.

(d) The set $S$ has the cardinality of the continuum if $\text{card} (S) = \text{card} (\mathbb{R})$.

We now verify that $\leq$ and $\geq$, in the sense of cardinality, work as expected.

**Proposition 3** Let $S$, $T$ and $U$ be nonempty sets. Then

(a) $\text{card} (S) \leq \text{card} (T)$ if and only if $\text{card} (T) \geq \text{card} (S)$.

(b) Either $\text{card} (S) \leq \text{card} (T)$ or $\text{card} (S) \geq \text{card} (T)$.

(c) If $\text{card} (S) \leq \text{card} (T)$ and $\text{card} (S) \geq \text{card} (T)$, then $\text{card} (S) = \text{card} (T)$.

(d) If $\text{card} (S) \leq \text{card} (T)$ and $\text{card} (T) \leq \text{card} (U)$, then $\text{card} (S) \leq \text{card} (U)$.

**Proof:** (a) If $\text{card} (S) \leq \text{card} (T)$, then there is a one–to–one function $f : S \to T$. Fix any $s_0 \in S$. Then

$$g(t) = \begin{cases} f^{-1}(t) & \text{if } t \text{ is in the range of } f \\ s_0 & \text{otherwise} \end{cases}$$

maps $T$ onto $S$, so that $\text{card} (T) \geq \text{card} (S)$.

If $\text{card} (T) \geq \text{card} (S)$, then there is a function $g$ that maps $T$ onto $S$. So the sets

$$g^{-1}(s) = \lbrace t \in T \mid g(t) = s \rbrace, \quad s \in S$$

are disjoint and nonempty. For each $s \in S$ select one $t_s \in g^{-1}(s)$. Then $f(s) = t_s$ defines a one–to–one map from $S$ into $T$. So $\text{card} (S) \leq \text{card} (T)$. 

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(b) Denote by $\mathcal{I}$ the set of all one-to-one maps defined on a subset of $S$ and taking values in $T$. Define a partial ordering on $\mathcal{I}$ by $f \leq g$ if
- the domain of $f$ is contained in the domain of $g$ and
- $f(s) = g(s)$ for all $s$ in the domain of $f$.

Let $\mathcal{I}'$ be any linearly ordered\(^{(1)}\) subset of $\mathcal{I}$. Let $F$ be the function
- whose domain is the union of all of the domains of functions in $\mathcal{I}'$ and
- whose value at $s$ is $f(s)$ for any $f \in \mathcal{I}'$ whose domain contains $s$.

Then $F$ is an upper bound for $\mathcal{I}'$. So, by Zorn’s lemma (see Folland page 5), $\mathcal{I}$ has a maximal element, say $G$. In the event that the domain of $G$ is $S$, then $G : S \rightarrow T$ is one-to-one and $\text{card}(S) \leq \text{card}(T)$. In the event that the range of $G$ is $T$, then $G^{-1} : T \rightarrow S$ is one-to-one and $\text{card}(T) \leq \text{card}(S)$. If the domain of $G$ is a proper subset of $S$ and the range of $G$ is a proper subset of $T$, then there is an $s_0 \in S$ that is not in the domain of $G$ and there is a $t_0 \in T$ that is not in the range of $G$. But then the function

$$H(s) = \begin{cases} G(s) & \text{if } s \text{ is in the domain of } G \\ t_0 & \text{if } s = s_0 \end{cases}$$

is a one-to-one map defined on a subset of $S$ and taking values in $T$ that obeys $H > G$. This violates the assumed maximality of $G$.

(c) Let $f : S \rightarrow T$ and $g : T \rightarrow S$ be one-to-one. For each $s \in S$, define the points $x_1(s), x_2(s), x_3(s), \cdots$ as follows.
- $x_1(s) = s \in S$
- if $x_1(s) \in g(T)$, set $x_2(s) = g^{-1}(x_1(s)) \in T$
- if $x_2(s) \in f(S)$, set $x_3(s) = f^{-1}(x_2(s)) \in S$
- and so on.

Either this process continues indefinitely, or, for some odd number $j \in \mathbb{N}$, $x_j(s) \in S \setminus g(T)$ and the process terminates or, for some even number $j \in \mathbb{N}$, $x_j(s) \in T \setminus f(S)$ and the process terminates. In these three cases we say that $s$ is in $S_\infty$, $S_S$ or $S_T$ respectively. So $S$ is the disjoint union of $S_\infty$, $S_S$ and $S_T$. Similarly, define
- $y_1(t) = t \in T$
- if $y_1(t) \in f(S)$, set $y_2(t) = f^{-1}(y_1(t)) \in S$
- if $y_2(t) \in g(T)$, set $y_3(t) = g^{-1}(y_2(t)) \in T$
- and so on.

and decompose $T$ into the disjoint union of $T_\infty$, $T_S$ and $T_T$. Observe that
- $y_2(f(s)) = s = x_1(s)$ so that $y_j(f(s)) = x_{j-1}(s)$ for all $j \geq 2$.
- $x_2(g(t)) = t = y_1(t)$ so that $x_j(g(t)) = y_{j-1}(t)$ for all $j \geq 2$.

Consequently,
- $f(S_\infty) \subset T_\infty$, $f(S_S) \subset T_S$, $f(S_T) \subset T_T$, $g(T_\infty) \subset S_\infty$, $g(T_S) \subset S_S$ and $g(T_T) \subset S_T$.

Furthermore, by construction, if $x_1(s) = s \in S_\infty \cup S_T$, then $x_2(s)$ is defined and $s \in g(T)$. That is, $g$ maps onto $S_\infty \cup S_T$. Similarly, if $y_1(t) = t \in T_\infty \cup T_S$, then $y_2(t)$ is defined and $t \in f(S)$ so that

\(^{(1)}\) This means that if $f, g \in \mathcal{I}'$, then either $f \leq g$ or $g \leq f$. 

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Proof:  (a) Since Proposition 4 maps $S_{\infty}$ onto $T_{\infty}$ and $S_{S}$ onto $T_{S}$ and $g$ maps $T_{T}$ onto $S_{T}$. Hence

$$F(s) = \begin{cases} f(s) & \text{if } s \in S_{\infty} \cup S_{S} \\ g^{-1}(s) & \text{if } s \in S_{T} \end{cases}$$

maps $S$ one–to–one and onto $T$.

(d) If $f$ is a one–to–one map from $S$ into $T$ and $g$ is a one–to–one map from $T$ into $U$, then $h = g \circ f$ is a one–to–one map from $S$ into $U$.

Proposition 4

(a) If $\mathcal{I}$ is countable and, for each $i \in \mathcal{I}$, $S_{i}$ is countable, then $\bigcup_{i \in \mathcal{I}} S_{i}$ is countable.

(b) If $S$ is a nonempty set and $\mathcal{P}(S)$ is the collection of all subsets of $S$, then

$$\text{card} (S) < \text{card} (\mathcal{P}(S))$$

Proof:  (a) Since $\mathcal{I}$ is countable, there is a function $f : \mathbb{N} \to \mathcal{I}$ that is onto $\mathcal{I}$. For each $i \in \mathcal{I}$, $S_{i}$ is countable so that there is a function $F_{i} : \mathbb{N} \to S_{i}$ that is onto $S_{i}$. We need to define a function $g : \mathbb{N} \to \bigcup_{i \in \mathcal{I}} S_{i}$ that is onto $\bigcup_{i \in \mathcal{I}} S_{i}$. Every element of $\bigcup_{i \in \mathcal{I}} S_{i}$ is of the form $F_{j(m)}(n)$ for some $(m, n) \in \mathbb{N} \times \mathbb{N}$. Every element $(m, n)$ of $\mathbb{N} \times \mathbb{N}$ has $m + n \in \mathbb{N}$ and $m + n \geq 2$. We first ensure that the range of $g$ contains all $F_{j(m)}(n)$'s with $m + n = 2$ by setting

$$g(1) = F_{j(1)}(1)$$

We next ensure that the range of $g$ contains all $F_{j(m)}(n)$'s with $m + n = 3$ by setting

$$g(2) = F_{j(1)}(2) \quad g(3) = F_{j(2)}(1)$$

We next ensure that the range of $g$ contains all $F_{j(m)}(n)$'s with $m + n = 4$ by setting

$$g(4) = F_{j(1)}(3) \quad g(5) = F_{j(2)}(2) \quad g(6) = F_{j(3)}(1)$$

We next ensure that the range of $g$ contains all $F_{j(m)}(n)$'s with $m + n = 5$ by setting

$$g(7) = F_{j(1)}(4) \quad g(8) = F_{j(2)}(3) \quad g(9) = F_{j(3)}(2) \quad g(10) = F_{j(4)}(1)$$

and so on.

(b) The function $f(s) = \{s\}$ is a one–to–one map from $S$ into $\mathcal{P}(S)$, so it suffices to prove that there does not exist a function $g : S \to \mathcal{P}(S)$ that is onto $\mathcal{P}(S)$. We find a contradiction to the hypothesis that there does exist such a function. Suppose that $g : S \to \mathcal{P}(S)$ is onto $\mathcal{P}(S)$. Set $T = \{ s \in S \mid s \notin g(s) \}$. If $T$ is in the range of $g$, there is an $s_{0} \in S$ with $T = g(s_{0})$.

- If $s_{0} \in T$, then $s_{0} \in T = g(s_{0})$ and the definition of $T$ gives that $s_{0} \notin T$, which is a contradiction.
- If $s_{0} \notin T$, then $s_{0} \notin T = g(s_{0})$ and the definition of $T$ gives that $s_{0} \in T$, which is a contradiction.
Example 5
(a) $\mathbb{Z}$ is countable.
(b) $\mathbb{Q}$ is countable.
(c) $\mathbb{R}$ is uncountable.

Proof: (a) The function
\[
f(p) = \begin{cases} 
1 & \text{if } p = 0 \\
2p & \text{if } p > 0 \\
-2p + 1 & \text{if } p < 0 
\end{cases}
\]
is a one-to-one, onto function from $\mathbb{Z}$ to $\mathbb{N}$.

(b) Write $\mathbb{Q} = \bigcup_{n \in \mathbb{N}} \{ \frac{p}{n} \mid p \in \mathbb{Z} \}$ and apply part (a) of Proposition 4.

(c) If $\mathbb{R}$ were countable, there would be an $f : \mathbb{N} \to \mathbb{R}$ that maps $\mathbb{N}$ onto $\mathbb{R}$. We shall show that this is impossible by constructing a real number that is not in the range of $f$. Write out a decimal representation of $f(n)$ for each $n \in \mathbb{N}$.

\[
f(1) = d_1^{(1)} \cdot d_2^{(1)} d_3^{(1)} d_4^{(1)} \cdots \\
f(2) = d_1^{(2)} \cdot d_2^{(2)} d_3^{(2)} d_4^{(2)} \cdots \\
f(3) = d_1^{(3)} \cdot d_2^{(3)} d_3^{(3)} d_4^{(3)} \cdots \\
\vdots
\]

with $d_0^{(n)} \in \mathbb{Z}$ for each $n \in \mathbb{N}$ and $d_m^{(n)} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for each $(n, m) \in \mathbb{N} \times \mathbb{N}$. Define, for each $m \in \mathbb{N}$,
\[
d_m = \begin{cases} 
7 & \text{if } d_m^{(m)} \in \{0, 1, 2, 3, 4\} \\
2 & \text{if } d_m^{(m)} \in \{5, 6, 7, 8, 9\} 
\end{cases}
\]

Then the decimal number
\[
d = 0 \cdot d_1 d_2 d_3 d_4 \cdots
\]
cannot be $f(1)$ because its first decimal place is wrong. It cannot be $f(2)$ because its second decimal place is wrong. And so on. So $d$ is not in the range of $f$. 