MATHEMATICS 420/507 December 2019 Final Exam Solutions

1. Define
   (a) an outer measure
   (b) $f = \lim_{n\to\infty} f_n$ a.e.
   (c) a positive set

   Solution. (a) An **outer measure** on X is a function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ such that
   (i) $\mu^*(\emptyset) = 0$
   (ii) If $E \subset F$, then $\mu^*(E) \le \mu^*(F)$.
   (iii) If $\{A_j\}$ is a countable collection of subsets of X, then

$$\mu^*\Big(\bigcup_j A_j\Big) \le \sum_j \mu^*(A_j)$$

   (b) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f, f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$. Then $f = \lim_{n\to\infty} f_n$ **a.e.** if there is a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \notin E$.
   (c) Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. A set $E \in \mathcal{M}$ is said to be **positive** for ν if

$$F \in \mathcal{M},\ F \subset E \implies \nu(F) \ge 0$$

2. Give an example of each of the following, together with a brief explanation of your example. If an example does not exist, explain why not.
   (a) A measure space, which does not involve Lebesgue measure or counting measure.
   (b) A sequence of functions $f_n(x)$ obeying $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in [0,1]$ but with $\int_{[0,1]} f(x)\, dm(x) \ne \lim_{n\to\infty} \int_{[0,1]} f_n(x)\, dm(x)$.
   (c) A Borel measure on $\mathbb{R}^2$ that is not of the form $\mu \times \nu$ for any Borel measures μ, ν on $\mathbb{R}$.

   Solution. (a) $X = \mathbb{R}$, $\mathcal{M} = \mathcal{P}(\mathbb{R})$, $\mu(E) = 1$ if $0 \in E$ and $\mu(E) = 0$ if $0 \notin E$.
   (b) $f = 0$, $f_n = n\chi_{(0,1/n]}$. Then $\int_{[0,1]} f(x)\ dm(x) = 0$ but $\int_{[0,1]} f_n(x)\ dm(x) = 1$ for all $n \in \mathbb{N}$.
   (c) Let m be the usual Lebesgue measure on $\mathbb{R}$. Set, for any $B \in \mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$

$$\lambda(B) = m\big(\{\ x \in \mathbb{R} \mid (x,x) \in B\ \}\big)$$

   It is easy to check that this is well–defined (let $\mathcal{D}$ be the collection of all $E \subset \mathbb{R}^2$ such that the diagonal $\{\ x \in \mathbb{R} \mid (x,x) \in E\ \} \in \mathcal{B}_{\mathbb{R}}$ and verify that $\mathcal{D}$ is closed under complements and under countable unions and contains rectangles) and is a measure. If there existed Borel measures μ and ν on $\mathbb{R}$ with $\lambda = \mu \times \nu$ we would have

$$1 = \lambda\big([0,1] \times [0,1]\big) = \mu\big([0,1]\big)\nu\big([0,1]\big) \quad 1 = \lambda\big([2,3] \times [2,3]\big) = \mu\big([2,3]\big)\nu\big([2,3]\big)$$

   This forces all of $\mu\big([0,1]\big)$, $\nu\big([0,1]\big)$, $\mu\big([2,3]\big)$ and $\nu\big([2,3]\big)$ all to be nonzero. On the other hand

$$0 = \lambda\big([0,1] \times [2,3]\big) = \mu\big([0,1]\big)\nu\big([2,3]\big)$$

   forces at least one of $\mu\big([0,1]\big)$ and $\nu\big([2,3]\big)$ to be zero, which is a contradiction.

3) Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$. Suppose that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Prove that μ-a.e. points of X belong only to a finite number of E_n's.

Solution. Set $B = \{\ x \in X \mid x \text{ is in infinitely many } E_n\text{'s}\ \}$. Let, for each $n \in \mathbb{N}$, $F_n = \bigcup_{m\geq n} E_m$. Then $B \subset F_n$ for all $n \in \mathbb{N}$ and hence $B \subset \bigcap_{n=1}^{\infty} F_n$. As $F_{n+1} \subset F_n$ for all $n \in \mathbb{N}$ and

$$\mu(F_1) \leq \sum_{m=1}^{\infty} \mu(E_m) < \infty$$

continuity from above implies that

$$\mu\Big(\bigcap_{n=1}^{\infty} F_n\Big) = \lim_{n\to\infty} \mu(F_n) \leq \limsup_{n\to\infty} \sum_{m=n}^{\infty} \mu(E_m) = 0$$

Since $\sum_{m=n}^{\infty} \mu(E_m)$ is the tail of the convergent series $\sum_{m=1}^{\infty} \mu(E_m)$. As B is a subset of $\bigcap_{n=1}^{\infty} F_n$, which has measure zero, μ-a.e. points of X belong only to a finite number of E_n's (even if $B \notin \mathcal{M}$).

4) Let $\mathcal{M}$ be a σ–algebra of subsets of a nonempty set X. Define $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and

$$\mathcal{B}_{\overline{\mathbb{R}}} = \{\ E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\ \}$$

(a) Prove that $\mathcal{B}_{\overline{\mathbb{R}}}$ is the σ–algebra generated by $\{\ (a, \infty] \mid a \in \mathbb{R}\ \}$.

(b) Let, for each $n \in \mathbb{N}$, $f_n : X \to [0, \infty]$ be $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$–measurable. Prove that $\liminf_{n\to\infty} f_n(x)$ and $\limsup_{n\to\infty} f_n(x)$ are $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$–measurable.

Solution. (a) Let $\mathcal{B}'$ be the σ–algebra of subsets of $\overline{\mathbb{R}}$ generated by $\{\ (a, \infty] \mid a \in \mathbb{R}\ \}$. We have already proven in a problem set that $\mathcal{B}_{\overline{\mathbb{R}}}$ is a σ–algebra. Furthermore, directly from the definition of $\mathcal{B}_{\overline{\mathbb{R}}}$, we have that $\{\ (a, \infty] \mid a \in \mathbb{R}\ \} \subset \mathcal{B}_{\overline{\mathbb{R}}}$ so that $\mathcal{B}' \subset \mathcal{B}_{\overline{\mathbb{R}}}$. Hence it suffices to prove that $\mathcal{B}_{\overline{\mathbb{R}}} \subset \mathcal{B}'$. Now

$$\begin{aligned}
\{\infty\} &= \bigcap_{n\in\mathbb{N}} (n, \infty] && \implies \{\infty\} \in \mathcal{B}' \\
\{-\infty\} &= \overline{\mathbb{R}} \cap \Big(\bigcup_{n\in\mathbb{Z}} (n, \infty]\Big)^c && \implies \{-\infty\} \in \mathcal{B}' \\
\mathbb{R} &= \overline{\mathbb{R}} \cap \{-\infty, \infty\}^c && \implies \mathbb{R} \in \mathcal{B}'
\end{aligned}$$

Also $\{\ E' \cap \mathbb{R} \mid E' \in \mathcal{B}'\ \}$ is a σ–algebra of subsets of $\mathbb{R}$ that contains $\{\ (a, \infty) \mid a \in \mathbb{R}\ \}$ and hence contains $\mathcal{B}_{\mathbb{R}}$. So if $E \in \mathcal{B}_{\overline{\mathbb{R}}}$, then

- $E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ so that
- there is an $E' \in \mathcal{B}'$ with $E \cap \mathbb{R} = E' \cap \mathbb{R} \in \mathcal{B}'$ so that
- E, which is the union of $E' \cap \mathbb{R}$ with one of $\emptyset$, $\{-\infty\}$, $\{\infty\}$, $\{-\infty, \infty\}$, is in $\mathcal{B}'$

(b) Let, for each $n \in \mathbb{N}$, $g_n : X \to [0, \infty]$ be $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$–measurable and set $G(x) = \sup_{n\in\mathbb{N}} g_n(x)$ and $g(x) = \inf_{n\in\mathbb{N}} g_n(x)$. Then, for all $a \in \mathbb{R}$,

$$\begin{aligned}
G^{-1}\big((a, \infty]\big) &= \bigcup_{n\in\mathbb{N}} g_n^{-1}\big((a, \infty]\big) \in \mathcal{M} \\
g^{-1}\big((a, \infty]\big) &= \bigcup_{m\in\mathbb{N}} \Big(\bigcap_{n\in\mathbb{N}} g_n^{-1}\big((a + \tfrac{1}{m}, \infty]\big)\Big) \in \mathcal{M}
\end{aligned}$$

so that G and g are $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$–measurable. Now it suffices to observe that

$$\liminf_{n\to\infty} f_n(x) = \sup_{n\in\mathbb{N}} \Big(\inf_{m>n} f_n(x)\Big) \qquad \limsup_{n\to\infty} f_n(x) = \inf_{n\in\mathbb{N}} \Big(\sup_{m>n} f_n(x)\Big)$$

are $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$–measurable.

5) Let $(X, \mathcal{M}, \mu)$ be a measure space and f be a non–negative measurable function on X.
(a) Prove that if $\int_X f\ d\mu < \infty$ then $\lim\limits_{a\to\infty} a\ \mu\big(\{\ x \in X \mid f(x) > a\ \}\big) = 0$.
(b) Assume that $\lim\limits_{a\to\infty} a\ \mu\big(\{\ x \in X \mid f(x) > a\ \}\big) = 0$. Either prove that $\int_X f\ d\mu < \infty$ or provide a counterxample.

Solution. (a) Set, for each $a > 0$, $E_a = f^{-1}\big((a, \infty)\big)$ and $f_a(x) = a\chi_{E_a}$. Note that

$$\int_{\mathcal{M}} f_a\ d\mu = a\mu\big(\{\ x \in X \mid f(x) > a\ \}\big)$$

and that $0 \le f_a(x) \le f(x)$ for all $x \in \mathbb{R}$ and that for all (or almost all, depending on whether or not we allow $f(x) = \infty$) $x \in \mathbb{R}$, $f(x) < \infty$ so that $\lim_{a\to\infty} f_a(x) = 0$ since $f(x) < a$ for all sufficiently large a. Hence, by the Lebesgue Dominated Convergence Theorem

$$\lim_{a\to\infty} a\ \mu\big(\{\ x \in X \mid f(x) > a\ \}\big) = \lim_{a\to\infty} \int_{\mathcal{M}} f_a\ d\mu = \int_{\mathcal{M}} \lim_{a\to\infty} f_a\ d\mu = \int_{\mathcal{M}} 0\ d\mu = 0$$

(b) The converse is false. For a counterexample, take $X = \mathbb{R}$ with μ being Lebesgue measure and $f(x) \equiv 1$. Then $\{\ x \in X \mid f(x) > a\ \} = \emptyset$ for all $a > 1$ so that $\lim_{a\to\infty} a\ \mu\big(\{\ x \in X \mid f(x) > a\ \}\big) = 0$. But $\int_X f\ d\mu = \infty$.

6) Denote by $\mathcal{L}$ the σ–algebra of Lebesgue measurable sets on $\mathbb{R}$ and by m the Lebesgue measure. Let $f : \mathbb{R}^2 \to [0, \infty]$ be $\mathcal{L} \otimes \mathcal{L}$–measurable. Suppose that for m–a.e. $x \in \mathbb{R}$, $f_x(y) = f(x, y)$ is m–a.e. finite. Prove that for m–almost all $y \in \mathbb{R}$ $f^y(x) = f(x, y)$ is m–a.e. finite.

Solution. Let $E = f^{-1}(\{\infty\})$. Since f is $\mathcal{L} \otimes \mathcal{L}$–measurable, $E \in \mathcal{L} \otimes \mathcal{L}$. Set $g(x, y) = \chi_E(x, y)$. Then $f(x, y) = \infty \iff g(x, y) = \infty$, so that for m–a.e. $x \in \mathbb{R}$, $g_x(y) = g(x, y)$ is m–a.e. finite and hence is m–a.e. zero. Hence for m–a.e. $x \in \mathbb{R}$, $\int g(x, y)\ dm(y) = 0$. By the Tonelli theorem,

$$\begin{aligned} 0 = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(x, y)\ dm(y) \right) dm(x) &= \int_{\mathbb{R}\times\mathbb{R}} g(x, y)\ dm \times m(x, y) \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(x, y)\ dm(x) \right) dm(y) \end{aligned}$$

This implies that $\int_{\mathbb{R}} g(x, y)\ dm(x) = 0$ for almost every y and hence that, for m–almost all $y \in \mathbb{R}$, $g^y(x) = g(x, y)$ is m–a.e. zero and hence that, for m–almost all $y \in \mathbb{R}$, $f^y(x) = f(x, y)$ is m–a.e. finite.

7) Let $\nu = \nu_+ - \nu_-$ be the Jordan decomposition of a signed measure on the measurable space $(X, \mathcal{M})$. Prove that, for all $E \in \mathcal{M}$,

$$\begin{aligned} \nu_+(E) &= \sup\{\ \nu(F) \mid F \in \mathcal{M},\ F \subset E\ \} \\ \nu_-(E) &= -\inf\{\ \nu(F) \mid F \in \mathcal{M},\ F \subset E\ \} \end{aligned}$$

Solution. Let $X = P \cup N$ be a Hahn decomposition for ν. That is, $P \cup N = X$, $P \cap N = \emptyset$, $P, N \in \mathcal{M}$, P is a positive set for ν and N is a negative set for ν. Then $\nu_+(E) = \nu(E \cap P)$ and $\nu_-(E) = -\nu(E \cap N)$. Since

$$\begin{aligned} F \subset E &\implies \nu(F) = \nu_+(F) - \nu_-(F) \le \nu_+(F) \le \nu_+(E) \\ &\implies \nu_+(E) \ge \sup\{\ \nu(F) \mid F \in \mathcal{M},\ F \subset E\ \} \end{aligned}$$

On the other hand, as $E \cap P$ is one allowed choice of F in the sup

$$\sup\{\ \nu(F) \mid F \in \mathcal{M},\ F \subset E\ \} \ge \nu(E \cap P) = \nu_+(E)$$

Similarly

$$\begin{aligned} F \subset E &\implies \nu(F) = \nu_+(F) - \nu_-(F) \ge -\nu_-(F) \ge -\nu_-(E) \\ &\implies \nu_-(E) \ge -\nu(F) \\ &\implies \nu_-(E) \ge \sup\{\ -\nu(F) \mid F \in \mathcal{M},\ F \subset E\ \} \\ &\implies \nu_-(E) \ge -\inf\{\ \nu(F) \mid F \in \mathcal{M},\ F \subset E\ \} \end{aligned}$$

On the other hand, as $E \cap N$ is one allowed choice of F in the inf

$$\begin{aligned} &\inf\{\ \nu(F) \mid F \in \mathcal{M},\ F \subset E\ \} \le \nu(E \cap N) = -\nu_-(E) \\ &\implies -\inf\{\ \nu(F) \mid F \in \mathcal{M},\ F \subset E\ \} \ge \nu_-(E) \end{aligned}$$