1. Define
   (a) an outer measure
   (b) \( f = \lim_{n \to \infty} f_n \) a.e.
   (c) a positive set

**Solution.** (a) An **outer measure** on \( X \) is a function \( \mu^* : \mathcal{P}(X) \to [0, \infty] \) such that
   (i) \( \mu^*(\emptyset) = 0 \)
   (ii) If \( E \subset F \), then \( \mu^*(E) \leq \mu^*(F) \).
   (iii) If \( \{A_j\} \) is a countable collection of subsets of \( X \), then
   \[
   \mu^*(\bigcup_j A_j) \leq \sum_j \mu^*(A_j)
   \]

(b) Let \((X, \mathcal{M}, \mu)\) be a measure space and \( f, f_n : X \to \mathbb{R} \) for all \( n \in \mathbb{N} \). Then \( f = \lim_{n \to \infty} f_n \) a.e. if there is a set \( E \in \mathcal{M} \) such that \( \mu(E) = 0 \) and \( f(x) = \lim_{n \to \infty} f_n(x) \) for all \( x \notin E \).

(c) Let \( \nu \) be a signed measure on the measurable space \((X, \mathcal{M})\). A set \( E \in \mathcal{M} \) is said to be **positive** for \( \nu \) if
   \[
   F \in \mathcal{M}, \; F \subset E \implies \nu(F) \geq 0
   \]

2. Give an example of each of the following, together with a brief explanation of your example. If an example does not exist, explain why not.
   (a) A measure space, which does not involve Lebesgue measure or counting measure.
   (b) A sequence of functions \( f_n(x) \) obeying \( f(x) = \lim_{n \to \infty} f_n(x) \) for all \( x \in [0, 1] \) but with \( \int_{[0,1]} f(x) dm(x) \neq \lim_{n \to \infty} \int_{[0,1]} f_n(x) dm(x) \).
   (c) A Borel measure on \( \mathbb{R}^2 \) that is not of the form \( \mu \times \nu \) for any Borel measures \( \mu, \nu \) on \( \mathbb{R} \).

**Solution.** (a) \( X = \mathbb{R}, \mathcal{M} = \mathcal{P}(\mathbb{R}), \mu(E) = 1 \) if \( 0 \in E \) and \( \mu(E) = 0 \) if \( 0 \notin E \).

(b) \( f = 0, \; f_n = n\chi_{(0,1/n]} \). Then \( \int_{[0,1]} f(x) dm(x) = 0 \) but \( \int_{[0,1]} f_n(x) dm(x) = 1 \) for all \( n \in \mathbb{N} \).

(c) Let \( m \) be the usual Lebesgue measure on \( \mathbb{R} \). Set, for any \( B \in \mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \)
   \[
   \lambda(B) = m(\{x \in \mathbb{R} \mid (x, x) \in B \})
   \]

It is easy to check that this is well-defined (let \( \mathcal{D} \) be the collection of all \( E \subset \mathbb{R}^2 \) such that the diagonal \( \{x \in \mathbb{R} \mid (x, x) \in E \} \in \mathcal{B}_{\mathbb{R}} \) and verify that \( \mathcal{D} \) is closed under complements and under countable unions and contains rectangles) and is a measure. If there existed Borel measures \( \mu \) and \( \nu \) on \( \mathbb{R} \) with \( \lambda = \mu \times \nu \) we would have

\[
1 = \lambda([0, 1] \times [0, 1]) = \mu([0, 1])\nu([0, 1]) \quad 1 = \lambda([2, 3] \times [2, 3]) = \mu([2, 3])\nu([2, 3])
\]

This forces all of \( \mu([0, 1]), \nu([0, 1]), \mu([2, 3]) \) and \( \nu([2, 3]) \) all to be nonzero. On the other hand

\[
0 = \lambda([0, 1] \times [2, 3]) = \mu([0, 1])\nu([2, 3])
\]

forces at least one of \( \mu([0, 1]) \) and \( \nu([2, 3]) \) to be zero, which is a contradiction.
3) Let \((X, \mathcal{M}, \mu)\) be a measure space and \(\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M}\). Suppose that \(\sum_{n=1}^{\infty} \mu(E_n) < \infty\). Prove that \(\mu\)-a.e. points of \(X\) belong only to a finite number of \(E_n's\).

**Solution.** Set \(B = \{ \ x \in X \mid x \text{ is in infinitely many } E_n's \} \). Let, for each \(n \in \mathbb{N}\), \(F_n = \bigcup_{m \geq n} E_m\). Then \(B \subset F_n\) for all \(n \in \mathbb{N}\) and hence \(B \subset \bigcap_{n=1}^{\infty} F_n\). As \(F_{n+1} \subset F_n\) for all \(n \in \mathbb{N}\) and

\[
\mu(F_1) \leq \sum_{m=1}^{\infty} \mu(E_m) < \infty
\]

continuity from above implies that

\[
\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim_{n \to \infty} \mu(F_n) \leq \limsup_{n \to \infty} \sum_{m=n}^{\infty} \mu(E_m) = 0
\]

Since \(\sum_{m=n}^{\infty} \mu(E_m)\) is the tail of the convergent series \(\sum_{m=1}^{\infty} \mu(E_m)\). As \(B\) is a subset of \(\bigcap_{n=1}^{\infty} F_n\), which has measure zero, \(\mu\)-a.e. points of \(X\) belong only to a finite number of \(E_n's\) (even if \(B \notin \mathcal{M}\)).

4) Let \(\mathcal{M}\) be a \(\sigma\)-algebra of subsets of a nonempty set \(X\). Define \(\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}\) and

\[
\mathcal{B}_{\overline{\mathbb{R}}} = \{ \ E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}} \ \}
\]

(a) Prove that \(\mathcal{B}_{\overline{\mathbb{R}}}\) is the \(\sigma\)-algebra generated by \(\{(a, \infty) \mid a \in \mathbb{R}\}\).

(b) Let, for each \(n \in \mathbb{N}\), \(f_n : X \to [0, \infty] \) be \((\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}}\))-measurable. Prove that \(\liminf_{n \to \infty} f_n(x)\) and 

\[
\limsup_{n \to \infty} f_n(x) \text{ are } (\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}}\))-measurable.
\]

**Solution.** (a) Let \(\mathcal{B}'\) be the \(\sigma\)-algebra of subsets of \(\overline{\mathbb{R}}\) generated by \(\{(a, \infty) \mid a \in \mathbb{R}\}\). We have already proven in a problem set that \(\mathcal{B}_{\overline{\mathbb{R}}}\) is a \(\sigma\)-algebra. Furthermore, directly from the definition of \(\mathcal{B}_{\overline{\mathbb{R}}}\), we have that \(\{(a, \infty) \mid a \in \mathbb{R}\} \subset \mathcal{B}_{\overline{\mathbb{R}}}\) so that \(\mathcal{B}' \subset \mathcal{B}_{\overline{\mathbb{R}}}\). Hence it suffices to prove that \(\mathcal{B}_{\overline{\mathbb{R}}} \subset \mathcal{B}'\). Now

\[
\{\infty\} = \bigcap_{n \in \mathbb{N}} (n, \infty) \quad \implies \{\infty\} \in \mathcal{B}'
\]

\[
\{-\infty\} = \overline{\mathbb{R}} \cap \bigcup_{n \in \mathbb{Z}} (n, \infty)^c \quad \implies \{-\infty\} \in \mathcal{B}'
\]

\[
\mathbb{R} = \overline{\mathbb{R}} \cap (-\infty, \infty)^c \quad \implies \mathbb{R} \in \mathcal{B}'
\]

Also \(\{E' \cap \mathbb{R} \mid E' \in \mathcal{B}'\}\) is a \(\sigma\)-algebra of subsets of \(\mathbb{R}\) that contains \(\{(a, \infty) \mid a \in \mathbb{R}\}\) and hence contains \(\mathcal{B}_{\overline{\mathbb{R}}}\). So if \(E \in \mathcal{B}_{\overline{\mathbb{R}}}\) then

\[
\circ E \cap \mathbb{R} \in \mathcal{B}_{\overline{\mathbb{R}}} \text{ so that}
\]

\[
\circ \text{ there is an } E' \in \mathcal{B}' \text{ with } E \cap \mathbb{R} = E' \cap \mathbb{R} \in \mathcal{B}' \text{ so that}
\]

\[
\circ E, \text{ which is the union of } E' \cap \mathbb{R} \text{ with one of } 0, \{-\infty\}, \{\infty\}, \{-\infty, \infty\}, \text{ is in } \mathcal{B}'
\]

(b) Let, for each \(n \in \mathbb{N}\), \(g_n : X \to [0, \infty] \) be \((\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}}\))-measurable and set \(G(x) = \sup_{n \in \mathbb{N}} g_n(x)\) and \(g(x) = \inf_{n \in \mathbb{N}} g_n(x)\). Then, for all \(a \in \mathbb{R}\),

\[
G^{-1}((a, \infty]) = \bigcup_{n \in \mathbb{N}} g_n^{-1}((a, \infty]) \in \mathcal{M}
\]

\[
g^{-1}((a, \infty]) = \bigcup_{m \in \mathbb{N}} \left( \bigcap_{n \in \mathbb{N}} g_n^{-1}\left(\left[a + \frac{1}{m}, \infty]\right)\right) \in \mathcal{M}
\]

so that \(G\) and \(g\) are \((\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}}\))-measurable. Now it suffices to observe that

\[
\liminf_{n \to \infty} f_n(x) = \sup_{n \in \mathbb{N}} \left( \inf_{m > n} f_n(x) \right) \quad \limsup_{n \to \infty} f_n(x) = \inf_{n \in \mathbb{N}} \left( \sup_{m > n} f_n(x) \right)
\]

are \((\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}}\))-measurable.
5) Let \((X, \mathcal{M}, \mu)\) be a measure space and \(f\) be a non-negative measurable function on \(X\).

(a) Prove that if \(\int_X f \, d\mu < \infty\) then \(\lim_{a \to \infty} \mu(\{ x \in X \mid f(x) > a \}) = 0\).

(b) Assume that \(\lim_{a \to \infty} \mu(\{ x \in X \mid f(x) > a \}) = 0\). Either prove that \(\int_X f \, d\mu < \infty\) or provide a counterexample.

**Solution.** (a) Set, for each \(a > 0\), \(E_a = f^{-1}((a, \infty))\) and \(f_a(x) = a \chi_{E_a}\). Note that
\[
\int_{\mathbb{R}} f_a \, d\mu = a \mu(\{ x \in X \mid f(x) > a \})
\]
and that \(0 \leq f_a(x) \leq f(x)\) for all \(x \in \mathbb{R}\) and that for all (or almost all, depending on whether or not we allow \(f(x) = \infty\)) \(x \in \mathbb{R}\), \(f(x) < \infty\) so that \(\lim_{a \to \infty} f_a(x) = 0\) since \(f(x) < a\) for all sufficiently large \(a\). Hence, by the Lebesgue Dominated Convergence Theorem
\[
\lim_{a \to \infty} a \mu(\{ x \in X \mid f(x) > a \}) = \lim_{a \to \infty} \int_{\mathbb{R}} f_a \, d\mu = \int_{\mathbb{R}} \lim_{a \to \infty} f_a \, d\mu = \int_{\mathbb{R}} 0 \, d\mu = 0
\]
(b) The converse is false. For a counterexample, take \(X = \mathbb{R}\) with \(\mu\) being Lebesgue measure and \(f(x) \equiv 1\). Then \(\{ x \in X \mid f(x) > a \} = \emptyset\) for all \(a > 1\) so that \(\lim_{a \to \infty} a \mu(\{ x \in X \mid f(x) > a \}) = 0\). But \(\int_{\mathbb{R}} f \, d\mu = \infty\).

6) Denote by \(\mathcal{L}\) the \(\sigma\)-algebra of Lebesgue measurable sets on \(\mathbb{R}\) and by \(m\) the Lebesgue measure. Let \(f : \mathbb{R}^2 \to [0, \infty]\) be \(\mathcal{L} \otimes \mathcal{L}\)-measurable. Suppose that for \(m\)-a.e. \(x \in \mathbb{R}\), \(f(x, y) = f(y, x)\) is \(m\)-a.e. finite. Prove that for \(m\)-almost all \(y \in \mathbb{R}\) \(f(y, x) = f(x, y)\) is \(m\)-a.e. finite.

**Solution.** Let \(E = f^{-1}(\{\infty\})\). Since \(f\) is \(\mathcal{L} \otimes \mathcal{L}\)-measurable, \(E \in \mathcal{L} \otimes \mathcal{L}\). Set \(g(x, y) = \chi_{E(x, y)}\). Then \(f(x, y) = \infty \iff g(x, y) = \infty\), so that for \(m\)-a.e. \(x \in \mathbb{R}\), \(g_x(y) = g(x, y)\) is \(m\)-a.e. finite and hence is \(m\)-a.e. zero. Hence for \(m\)-a.e. \(x \in \mathbb{R}\), \(\int g(x, y) \, d\mu(y) = 0\). By the Tonelli theorem,
\[
0 = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} g(x, y) \, d\mu(y) \right) \, d\mu(x) = \int_{\mathbb{R} \times \mathbb{R}} g(x, y) \, d\mu(x, y)
\]
This implies that \(\int_{\mathbb{R}} g(x, y) \, d\mu(x) = 0\) for almost every \(y\) and hence that, for \(m\)-almost all \(y \in \mathbb{R}\), \(g(y, x) = g(x, y)\) is \(m\)-a.e. zero and hence that, for \(m\)-almost all \(y \in \mathbb{R}\), \(f(y, x) = f(x, y)\) is \(m\)-a.e. finite.

7) Let \(\nu = \nu_+ - \nu_-\) be the Jordan decomposition of a signed measure on the measurable space \((X, \mathcal{M})\). Prove that, for all \(E \in \mathcal{M}\),
\[
\nu_+(E) = \sup \{ \nu(F) \mid F \in \mathcal{M}, F \subset E \}
\]
\[
\nu_-(E) = -\inf \{ \nu(F) \mid F \in \mathcal{M}, F \subset E \}
\]

**Solution.** Let \(X = P \cup N\) be a Hahn decomposition for \(\nu\). That is, \(P \cup N = X, P \cap N = \emptyset\), \(P, N \in \mathcal{M}\), \(P\) is a positive set for \(\nu\) and \(N\) is a negative set for \(\nu\). Then \(\nu_+(E) = \nu(E \cap P)\) and \(\nu_-(E) = -\nu(E \cap N)\). Since
\[
F \subset E \implies \nu(F) = \nu_+(F) - \nu_-(F) \leq \nu_+(F) \leq \nu_+(E)
\]
\[
\implies \nu_+(E) \geq \sup \{ \nu(F) \mid F \in \mathcal{M}, F \subset E \}
\]
On the other hand, as \(E \cap P\) is one allowed choice of \(F\) in the sup
\[
\sup \{ \nu(F) \mid F \in \mathcal{M}, F \subset E \} \geq \nu(E \cap P) = \nu_+(E)
\]
Similarly
\[
F \subset E \implies \nu(F) = \nu_+(F) - \nu_-(F) \geq -\nu_-(F) \geq -\nu_-(E)
\]
\[
\implies -\nu_-(E) \geq \sup \{ -\nu(F) \mid F \in \mathcal{M}, F \subset E \}
\]
\[
\implies -\nu_-(E) \geq -\inf \{ \nu(F) \mid F \in \mathcal{M}, F \subset E \}
\]
On the other hand, as \(E \cap N\) is one allowed choice of \(F\) in the inf
\[
\inf \{ \nu(F) \mid F \in \mathcal{M}, F \subset E \} \leq \nu(E \cap N) = -\nu_-(E)
\]
\[
\implies -\inf \{ \nu(F) \mid F \in \mathcal{M}, F \subset E \} \geq \nu_-(E)
\]