Lecture #35: April 7, 2008
Instructor: Dr. Joel Feldman
Scribe: Peter Wong

Examples. **($SO(3)$, $O(3)$)**

$$O(3) = \{\, 3\times 3 \text{ real matrices } \mathbf{R} \mid \mathbf{R}^\dagger\mathbf{R} = \mathbf{I} \,\}$$
$$SO(3) = \{\, 3\times 3 \text{ real matrices } \mathbf{R} \mid \mathbf{R}^\dagger\mathbf{R} = \mathbf{I},\ \det\mathbf{R} = 1 \,\}$$

Claim. *$SO(3)$ and $O(3)$ are 3-dimensional surfaces in $\mathbb{R}^9$.*

$$\mathbf{R} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$
$$\mathbb{R}^9 = \{\, (a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) \,\}$$
$$\mathbf{I} = \mathbf{R}^\dagger\mathbf{R} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$
$$(\mathbf{R}^\dagger\mathbf{R})_{1,1} = a_1^2 + a_2^2 + a_3^2 = 1, \quad |\vec{a}| = 1$$
$$(\mathbf{R}^\dagger\mathbf{R})_{2,2} = b_1^2 + b_2^2 + b_3^2 = 1, \quad |\vec{a}| = 1$$
$$(\mathbf{R}^\dagger\mathbf{R})_{3,3} = c_1^2 + c_2^2 + c_3^2 = 1, \quad |\vec{a}| = 1$$
$$(\mathbf{R}^\dagger\mathbf{R})_{2,1} = (\mathbf{R}^\dagger\mathbf{R})_{1,2} = a_1b_1 + a_2b_2 + a_3b_3 = 0, \quad (\text{i.e. } \vec{a} \perp \vec{b})$$
$$(\mathbf{R}^\dagger\mathbf{R})_{3,1} = (\mathbf{R}^\dagger\mathbf{R})_{1,3} = a_1c_1 + a_2c_2 + a_3c_3 = 0, \quad (\text{i.e. } \vec{a} \perp \vec{c})$$
$$(\mathbf{R}^\dagger\mathbf{R})_{3,2} = (\mathbf{R}^\dagger\mathbf{R})_{2,3} = b_1c_1 + b_2c_2 + b_3c_3 = 0, \quad (\text{i.e. } \vec{b} \perp \vec{c})$$

There are 6 equations with 9 unknowns. Thus we expect 3 free parameters. Apply the implicit function theorem to solve the equations near

$$\tilde{\mathbf{R}} = \begin{bmatrix} \tilde{a}_1 & \tilde{b}_1 & \tilde{c}_1 \\ \tilde{a}_2 & \tilde{b}_2 & \tilde{c}_2 \\ \tilde{a}_3 & \tilde{b}_3 & \tilde{c}_3 \end{bmatrix}$$

This allows us to conclude that we have a 3-d manifold provided we show that for each $\tilde{\mathbf{R}}$,, the 6 gradients

$$\vec{\nabla} g_1(\tilde{\mathbf{R}}) = (2\tilde{a}_1, 2\tilde{a}_2, 2\tilde{a}_3, 0, 0, 0, 0, 0, 0)$$
$$\vdots$$
$$\vec{\nabla} g_6(\tilde{\mathbf{R}}) = (0, 0, 0, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{b}_1, \tilde{b}_2, \tilde{b}_3)$$

are linearly independent. The details are given as a homework problem.

Some intuition instead:

Find all allowed $\vec{a}$, $\vec{b}$, $\vec{c}$'s near $\tilde{\mathbf{R}}$, for example, $\tilde{\mathbf{R}} = \mathbf{I}$.

Step 1: Find all $\vec{a}$'s near $(1, 0, 0)$ obeying $|\vec{a}| = 1$, i. e., $a_1^2 + a_2^2 + a_3^2 = 1$. We can assign any small values to a_2 and a_3 such that a_1 must be $\pm\sqrt{1 - a_2^2 - a_3^2}$. Choose $+$ so that $a_1 \approx 1$.

STEP 2: We have picked $\vec{a}$. Find all $\vec{b}$'s obeying $|\vec{b}| = 1$, $\vec{a} \perp \vec{b}$.

$$\begin{aligned} \{\, \vec{b} \in \mathbb{R}^3 \mid \vec{b} \perp \vec{a} \,\} &= \text{a plane through the origin} \\ \{\, \vec{b} \in \mathbb{R}^3 \mid |\vec{b}| = 1 \,\} &= \text{the unit sphere centered on } \vec{0} \\ \{\, \vec{b} \in \mathbb{R}^3 \mid \vec{b} \perp \vec{a},\, |\vec{b}| = 1 \,\} &= \text{a great circle} \end{aligned}$$

There is a 1-parameter family of allowed $\vec{b}$'s. That's the third parameter.

STEP 3: We have picked $\vec{a}$ and $\vec{b}$. Final allowed $\vec{c}$'s obeying

$$\vec{c} \perp \vec{a}, \qquad \vec{c} \perp \vec{b}, \qquad |\vec{c}| = 1$$

so that $\{\, \vec{c} \in \mathbb{R}^3 \mid \vec{c} \perp \vec{a}, \vec{c} \perp \vec{b} \,\}$ is a line through the origin. Since $|\vec{c}| = 1$, only two $\vec{c}$'s are obeying

$$\vec{c} \perp \vec{a}, \qquad \vec{c} \perp \vec{b}, \qquad |\vec{c}| = 1$$

Pick the $\vec{c}$ near $(0, 0, 1)$ such that $O(3)$ is a 3-d surface in $\mathbb{R}^9$.

What about $SO(3)$, i. e., what effect does $\det \mathbf{R} = \pm 1$?

$$\begin{aligned} \mathbf{R} \in O(3) \iff & \mathbf{R}^\dagger \mathbf{R} = \mathbf{I} \\ & \Longrightarrow \det(\mathbf{R}^\dagger \mathbf{R}) = \mathbf{I} \\ & \Longrightarrow (\det \mathbf{R}^\dagger)(\det \mathbf{R}) = \mathbf{I} \\ & \Longrightarrow (\det \mathbf{R})^2 = 1 \implies \det R = \pm 1. \end{aligned}$$

Since $\det R$ depends continuously on $\mathbf{R}$, $O(3)$ consists of disconnected components

$$SO(3) = \{\, R \in O(3) \mid \det R = 1 \,\} \quad \text{and} \quad \{\, R \in O(3) \mid \det R = -1 \,\}$$

For practical coordinates, use Euler angles. (See Web Notes)

Example. Consider a set $\mathcal{M} = [0, 1) \times (-1, 1)$ with two atlas

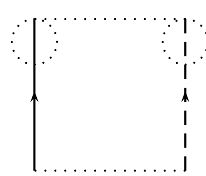
$$\mathcal{A}_1 = \{\, (U_1, \psi_1), (U_2, \psi_2) \,\} \quad \text{and} \quad \mathcal{A}_1 = \{\, (U_1, \varphi_1), (U_2, \varphi_2) \,\}$$

where

$$U_1 = \left(\tfrac{1}{8}, \tfrac{7}{8}\right) \times (-1, 1) \quad \text{and} \quad U_2 = \left[0, \tfrac{1}{4}\right) \times (-1, 1) \cup \left(\tfrac{3}{4}, 1\right) \times (-1, 1)$$

Intuition for $\mathcal{A}_1$:

Identify



$$\psi_1(x, y) = (x, y)$$

$$\psi_2(x, y) = \begin{cases} (x, y), & \text{if} \quad 0 \le x \le \frac{1}{4} \\ (-x, y), & \text{if} \quad \frac{3}{4} \le x \le 1 \end{cases}$$

Lecture #36: April 9, 2008
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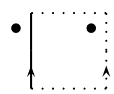
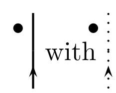
Example. Given a set $\mathcal{M} = [0,1) \times (-1,1)$ and two atlas

$$\mathcal{A} = \{\, (U_1, \psi_1), (U_2, \psi_2) \,\} \quad \text{and} \quad \mathcal{B} = \{\, (U_1, \varphi_1), (U_2, \varphi_2) \,\}$$

where

$$U_1 = \left(\tfrac{1}{8}, \tfrac{7}{8}\right) \times (-1,1) \quad \text{and} \quad U_2 = \left[0, \tfrac{1}{4}\right) \times (-1,1) \cup \left(\tfrac{3}{4}, 1\right) \times (-1,1)$$

Atlas $\mathcal{A}$:

Intuition:  Identify 

$$\psi_1(x,y) = (x,y)$$

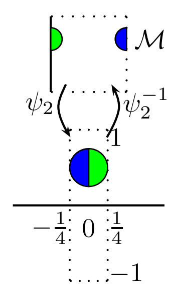
$$\psi_2(x,y) = \begin{cases} (x,y), & \text{for } 0 \le x < \frac{1}{4} \\ (x-1,y), & \text{for } \frac{3}{4} < x < 1 \end{cases}$$

Challenge: Find a metric for $\mathcal{M}$ so that ψ_1, ψ_2 are homeomorphisms.

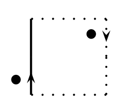
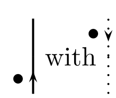
$$\begin{aligned} \text{Range of } \psi_2 &= \psi_2\left(\left[0,\tfrac{1}{4}\right) \times (-1,1)\right) \cup \psi_2\left(\left(\tfrac{3}{4},1\right) \times (-1,1)\right) \\ &= \left[0,\tfrac{1}{4}\right) \times (-1,1) \cup \left(-\tfrac{1}{4},0\right) \times (-1,1) \\ &= \left(-\tfrac{1}{4},\tfrac{1}{4}\right) \times (-1,1) \end{aligned}$$

$$\psi_2^{-1}(x,y) = \begin{cases} (x,y), & \text{if } 0 \le x < \frac{1}{4} \\ (x+1,y), & \text{if } -\frac{1}{4} < x < 0 \end{cases}$$

$$\psi_2^{-1}\left(B_{1/4}\left(0,\tfrac{1}{2}\right)\right) = \psi_2^{-1}\left(B_{1/4}\left(0,\tfrac{1}{2}\right) \cap \{\, x \ge 0 \,\}\right) \cup \psi_2^{-1}\left(B_{1/4}\left(0,\tfrac{1}{2}\right) \cap \{\, x > 0 \,\}\right)$$



Atlas $\mathcal{B}$:

Intuition:  Identify 

$$\varphi_1(x,y) = (x,y)$$

$$\varphi_2(x,y) = \begin{cases} (x,y), & \text{for } 0 \le x < \frac{1}{4} \\ (x-1,-y), & \text{for } \frac{3}{4} < x < 1 \end{cases}$$

Challenges:

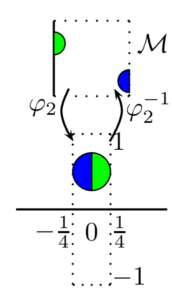
(1) Find a metric for $\mathcal{M}$ so that ψ_1, ψ_2 are homeomorphisms.

(2) Show that (U_2, ψ_2) and (U_2, φ_2) are not compatible.

$$\begin{aligned} \text{Range of } \varphi_2 &= \varphi_2\left(\left[0,\tfrac{1}{4}\right) \times (-1,1)\right) \cup \varphi_2\left(\left(\tfrac{3}{4},1\right) \times (-1,1)\right) \\ &= \left[0,\tfrac{1}{4}\right) \times (-1,1) \cup \left(-\tfrac{1}{4},0\right) \times (-1,1) \\ &= \left(-\tfrac{1}{4},\tfrac{1}{4}\right) \times (-1,1) \end{aligned}$$

$$\varphi_2^{-1}(x,y) = \begin{cases} (x,y), & \text{if } 0 \le x < \frac{1}{4} \\ (x+1,-y), & \text{if } -\frac{1}{4} < x < 0 \end{cases}$$

$$\begin{aligned} \varphi_2^{-1}\left(B_{1/4}\left(0,\tfrac{1}{2}\right)\right) &= \varphi_2^{-1}\left(B_{1/4}\left(0,\tfrac{1}{2}\right) \cap \{\, x \ge 0 \,\}\right) \cup \varphi_2^{-1}\left(B_{1/4}\left(0,\tfrac{1}{2}\right) \cap \{\, x > 0 \,\}\right) \\ &= B_{1/2}\left(0,\tfrac{1}{2}\right) \cap \{\, x \ge 0 \,\} \cup B_{1/2}\left(0,\tfrac{1}{2}\right) \cap \{\, x < 1 \,\} \end{aligned}$$



Integration on Manifolds

Integrals over zero-dimensional domains:

- 0-domain
  - called a 0-chain
  - is a finite number of points with multiplicity and signs
  - denoted $n_1 p_1 + n_2 p_2 + \cdots + n_\ell p_\ell$ with $p_1, \ldots, p_\ell \in \mathcal{M}$ and $n_1, \ldots, n_\ell \in \mathbb{Z}$.

Lecture #37: April 11, 2008
Instructor: Dr. Joel Feldman
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Review Session: Wednesday, April 16, 3:00 pm in room MATH 103.
Exam: Thursday, April 17, 3:30 pm in room MATH 104.

0-dimensional Integral case:

- domain of integration
  - called a 0-chain
  - a finite number of points in $\mathcal{M}$ with multiplicity and sign
  - denoted $n_1p_1 + \cdots + n_kp_k$ with $p_1, \ldots, p_k$ being distinct points of $\mathcal{M}$ and $n_1, \ldots, n_k \in \mathbb{Z}$
  - Formal definition: A function $\sigma : \mathcal{M} \to \mathbb{Z}$ which vanishes except at finitely many points. The σ corresponding to $n_1p_1 + \cdots + n_kp_k$ is

$$\sigma(p) = \begin{cases} n_j, & \text{if } p = p_j \text{ for some } 1 \le j \le k, \\ 0, & \text{otherwise} \end{cases}$$

- the integrand is called a 0-form, which is just a function $F : \mathcal{M} \to \mathbb{C}$
- the integral is

$$\int_{n_1p_1+\cdots+n_kp_k} F = \sum_{j=1}^{k} n_j\, F(p_j)$$

1-dimensional Integral case:

Brief review of work integrals

- have a particle which is at $\vec{r}(t)$ at time t
- feels a force $\vec{F}(\vec{r})$
- by Newton

$$\begin{aligned} m\frac{d^2\vec{r}}{dt^2}(t) = \vec{F}(\vec{r}(t)) &\Longrightarrow m\frac{d^2\vec{r}}{dt^2}(t)\cdot\frac{d\vec{r}}{dt}(t) = \vec{F}(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}(t) \\ &\Longrightarrow \frac{d}{dt}\left(\frac{1}{2}m\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\right) = \vec{F}(\vec{r}(t))\cdot\frac{d\vec{r}}{dt} \\ &\Longrightarrow \int_{t_1}^{t_2} \underbrace{\frac{1}{2}m\vec{v}^{\,2}(t_2) - \frac{1}{2}m\vec{v}^{\,2}(t_1)}_{\text{change in kinetic energy}} = \int_{t_1}^{t_2} \underbrace{\vec{F}(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}(t)\,dt}_{\text{work}} \end{aligned}$$

Usual notation for the work integral $\int_{t_1}^{t_2} \vec{F}(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}(t)\,dt$ is

$$\int_C \vec{F}\cdot d\vec{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz.$$

On a manifold of dimension n

- domain of integration
  - called a 1-chain
  - denoted $n_1C_1 + \cdots + n_kC_k$ with $C_1, \ldots, C_k$ begin paths, and $n_1, \ldots, n_k \in \mathbb{Z}$
  - a path is a function $r : [0,1] \mapsto \mathcal{M}$ which is C^∞; a function $r : [0,1] \mapsto \mathcal{M}$ is C^∞ at t_0 if there is a coordinate patch (chart) $\{\mathcal{U}, \varphi\}$ with $r(t_0) \in \mathcal{U}$ and $\varphi(r(t))$ is C^∞ at t_0. If so, $\psi(r(t))$ is automatically C^∞ at t_0 for any coordinate patch $\{\mathcal{V}, \psi\}$ with $r(t_0) \in \mathcal{V}$.
- integrand
  - called a 1-form
  - a 1-form is a rule which assigns to each coordinate patch $\left\{\mathcal{U}, \varphi = (x^1, \ldots, x^n)\right\}$, n functions $f_1, \ldots, f_n : \varphi(\mathcal{U}) \to \mathbb{C}$
  - denoted $\omega\big|_{\{\mathcal{U},\varphi=(x^1,\ldots,x^n)\}} = f_1 d_{x^1} + \cdots + f_n d_{x^n}$
  - The functions must obey the following change of coordinates rule:
    - Let $\left\{\mathcal{U}, \varphi = (x^1, \ldots, x^n)\right\}$ and $\left\{\mathcal{V}, \psi = (y^1, \ldots, y^n)\right\}$ with $\mathcal{U} \cap \mathcal{V} = \emptyset$. Write $\varphi \circ \psi^{-1} : \psi(\mathcal{U} \cap \mathcal{V}) \to \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n$.

$$\omega\big|_{\{\mathcal{U},\varphi\}} = f_1 dx^1 + \cdots + f_n\, dx^n = \sum_{k=1}^n f_k dx^k = \sum_{j,k} f_k \frac{\partial x^k}{\partial y^j} dy^j$$
$$\omega\big|_{\{\mathcal{V},\psi\}} = g_1 dy^1 + \cdots + g_n\, dy^n$$

Then $g_j(\vec{y}) = \sum_{k=1}^n f_k(\vec{x}(\vec{y}))\frac{\partial x^k}{\partial y^j}$. Recall $dx^k = \sum_{j=1}^n \frac{\partial x^k}{\partial y^j} dy^j$

- integral

$$\int_{n_1C_1+\cdots+n_kC_k} \omega = n_1 \int_{C_1} \omega + \cdots + n_k \int_{C_k} \omega$$

If C is the path $r : [0,1] \to \mathcal{M}$ and if $\left\{\mathcal{U}, \varphi = (x^1, \ldots, x^n)\right\}$ is a chart with $r(t) \in \mathcal{U}$ for all $0 \le t \le 1$, then

$$\int_C \omega = \int_0^1 \sum_{j=1}^n f_j(\varphi(r(t))) \frac{dx^j\,(r(t))}{dt} dt$$

if $\omega\big|_{\{\mathcal{U},\varphi=(x^1,\ldots,x^n)\}} = f_1 d_{x^1} + \cdots + f_n d_{x^n}$. Note that this integral gives the same answer for all charts.

- **One last note:**

$$\int_{\partial R} \omega = \int_R d\omega$$

This identity is, in fact, the Fundamental Theorem of Calculus, Green's Theorem, Stokes' Theorem, Divergence Theorem, and half a dozen more...

That's all, folks!