Examples. \((SO(3), O(3))\)

\[
O(3) = \{3 \times 3 \text{ real matrices } R \mid R^T R = I\}
\]

\[
SO(3) = \{3 \times 3 \text{ real matrices } R \mid R^T R = I, \det R = 1\}
\]

Claim. \(SO(3)\) and \(O(3)\) are 3-dimensional surfaces in \(\mathbb{R}^9\).

\[
R = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}
\]

\(\mathbb{R}^9 = \{(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3)\}\)

\[
I = R^T R = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}
\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}
\]

\[
(R^T R)_{1,1} = a_1^2 + a_2^2 + a_3^2 = 1, \quad |\vec{a}| = 1
\]

\[
(R^T R)_{2,2} = b_1^2 + b_2^2 + b_3^2 = 1, \quad |\vec{b}| = 1
\]

\[
(R^T R)_{3,3} = c_1^2 + c_2^2 + c_3^2 = 1, \quad |\vec{c}| = 1
\]

\[
(R^T R)_{1,2} = (R^T R)_{2,1} = a_1 b_1 + a_2 b_2 + a_3 b_3 = 0, \quad (i.e. \vec{a} \perp \vec{b})
\]

\[
(R^T R)_{1,3} = (R^T R)_{3,1} = a_1 c_1 + a_2 c_2 + a_3 c_3 = 0, \quad (i.e. \vec{a} \perp \vec{c})
\]

\[
(R^T R)_{2,3} = (R^T R)_{3,2} = b_1 c_1 + b_2 c_2 + b_3 c_3 = 0, \quad (i.e. \vec{b} \perp \vec{c})
\]

There are 6 equations with 9 unknowns. Thus we expect 3 free parameters. Apply the implicit function theorem to solve the equations near

\[
\vec{R} = \begin{bmatrix} \tilde{a}_1 & \tilde{b}_1 & \tilde{c}_1 \\ \tilde{a}_2 & \tilde{b}_2 & \tilde{c}_2 \\ \tilde{a}_3 & \tilde{b}_3 & \tilde{c}_3 \end{bmatrix}
\]

This allows us to conclude that we have a 3-d manifold provided we show that for each \(\vec{R}\), the 6 gradients

\[
\vec{\nabla} g_1(\vec{R}) = (2\tilde{a}_1, 2\tilde{a}_2, 2\tilde{a}_3, 0, 0, 0, 0, 0, 0)
\]

\[
\vdots
\]

\[
\vec{\nabla} g_6(\vec{R}) = (0, 0, 0, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{b}_1, \tilde{b}_2, \tilde{b}_3)
\]

are linearly independent. The details are given as a homework problem.

Some intuition instead:

Find all allowed \(\vec{a}, \vec{b}, \vec{c}\)’s near \(\vec{R}\), for example, \(\vec{R} = I\).

**Step 1:** Find all \(\vec{a}\)’s near \((1, 0, 0)\) obeying \(|\vec{a}| = 1\), i.e., \(a_1^2 + a_2^2 + a_3^2 = 1\). We can assign any small values to \(a_2\) and \(a_3\) such that \(a_1\) must be \(\pm \sqrt{1 - a_2^2 - a_3^2}\). Choose + so that \(a_1 \approx 1\).
STEP 2: We have picked \( \vec{a} \). Find all \( \vec{b} \)'s obeying \( |\vec{b}| = 1, \vec{a} \perp \vec{b} \).

\[
\{ \vec{b} \in \mathbb{R}^3 \mid \vec{b} \perp \vec{a} \} \text{ is a plane through the origin}
\]
\[
\{ \vec{b} \in \mathbb{R}^3 \mid |\vec{b}| = 1 \} \text{ is the unit sphere centered on} \vec{0}
\]
\[
\{ \vec{b} \in \mathbb{R}^3 \mid \vec{b} \perp \vec{a}, |\vec{b}| = 1 \} \text{ is a great circle}
\]

There is a 1-parameter family of allowed \( \vec{b} \)'s. That’s the third parameter.

STEP 3: We have picked \( \vec{a} \) and \( \vec{b} \). Final allowed \( \vec{c} \)'s obeying

\[
\vec{c} \perp \vec{a}, \vec{c} \perp \vec{b}, |\vec{c}| = 1
\]

so that \( \{ \vec{c} \in \mathbb{R}^3 \mid \vec{c} \perp \vec{a}, \vec{c} \perp \vec{b} \} \) is a line through the origin. Since \( |\vec{c}| = 1 \), only two \( \vec{c} \)'s are obeying

\[
\vec{c} \perp \vec{a}, \vec{c} \perp \vec{b}, |\vec{c}| = 1
\]

Pick the \( \vec{c} \) near \((0,0,1)\) such that \( O(3) \) is a 3-d surface in \( \mathbb{R}^3 \).

What about \( SO(3) \), i.e., what effect does \( \det R = \pm 1 \)?

\[
R \in O(3) \iff R^\dagger R = I \\
\implies \det(R^\dagger R) = I \\
\implies (\det R^\dagger)(\det R) = I \\
\implies (\det R)^2 = 1 \implies \det R = \pm 1.
\]

Since \( \det R \) depends continuously on \( R \), \( O(3) \) consists of disconnected components

\[
SO(3) = \{ R \in O(3) \mid \det R = 1 \} \text{ and } \{ R \in O(3) \mid \det R = -1 \}
\]

For practical coordinates, use Euler angles. (See Web Notes)

**Example.** Consider a set \( \mathcal{M} = [0,1) \times (-1,1) \) with two atlas

\[
\mathcal{A}_1 = \{(U_1, \psi_1), (U_2, \psi_2)\} \text{ and } \mathcal{A}_1 = \{(U_1, \varphi_1), (U_2, \varphi_2)\}
\]

where

\[
U_1 = \left(\frac{1}{4}, \frac{3}{4}\right) \times (-1,1) \text{ and } U_2 = \left[0, \frac{1}{4}\right) \times (-1,1) \cup \left(\frac{3}{4}, 1\right) \times (-1,1)
\]

Intuition for \( \mathcal{A}_1 \):

Identify

\[
\psi_1(x, y) = (x, y) \\
\psi_2(x, y) = \begin{cases} (x, y), & \text{if } 0 \leq x \leq \frac{1}{4} \\ (-x, y), & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}
\]
Example. Given a set \( M = [0, 1) \times (-1, 1) \) and two atlases

\[
\mathcal{A} = \{ (U_1, \psi_1), (U_2, \psi_2) \} \quad \text{and} \quad \mathcal{B} = \{ (U_1, \varphi_1), (U_2, \varphi_2) \}
\]

where

\[
U_1 = \left( \left[ \frac{1}{4}, \frac{7}{8} \right) \times (-1, 1) \right) \quad \text{and} \quad U_2 = \left[ 0, \frac{1}{4} \right) \times (-1, 1) \cup \left( \frac{3}{4}, 1 \right) \times (-1, 1)
\]

Atlas \( \mathcal{A} \):

Intuition: 

Identify \( \psi_1(x, y) = (x, y) \)

\[
\psi_2(x, y) = \begin{cases} 
(x, y), & \text{for } 0 \leq x < \frac{1}{4} \\
(x - 1, y), & \text{for } \frac{3}{4} < x < 1
\end{cases}
\]

Challenge: Find a metric for \( M \) so that \( \psi_1, \psi_2 \) are homeomorphisms.

Range of \( \psi_2 = \psi_2 \left( \left[ 0, \frac{1}{4} \right) \times (-1, 1) \right) \cup \psi_2 \left( \left( \frac{3}{4}, 1 \right) \times (-1, 1) \right) = [0, \frac{1}{4}) \times (-1, 1) \cup (-\frac{1}{4}, 0) \times (-1, 1) = (-\frac{1}{4}, \frac{1}{4}) \times (-1, 1) \]

\[
\psi_2^{-1}(x, y) = \begin{cases} 
(x, y), & \text{if } 0 \leq x < \frac{1}{4} \\
(x + 1, y), & \text{if } -\frac{1}{4} < x < 0
\end{cases}
\]

\[
\psi_2^{-1} \left( B_{1/4} \left( 0, \frac{1}{2} \right) \right) = \psi_2^{-1} \left( B_{1/4} \left( 0, \frac{1}{2} \right) \cap \{ x \geq 0 \} \right) \cup \psi_2^{-1} \left( B_{1/4} \left( 0, \frac{1}{2} \right) \cap \{ x > 0 \} \right) = B_{1/4} \left( 0, \frac{1}{2} \right) \cap \{ x \geq 0 \} \cup B_{1/2} \left( 0, \frac{1}{2} \right) \cap \{ x < 1 \}
\]

Atlas \( \mathcal{B} \):

Intuition: 

Identify \( \varphi_1(x, y) = (x, y) \)

\[
\varphi_2(x, y) = \begin{cases} 
(x, y), & \text{for } 0 \leq x < \frac{1}{4} \\
(x - 1, -y), & \text{for } \frac{3}{4} < x < 1
\end{cases}
\]

Challenges:

(1) Find a metric for \( M \) so that \( \psi_1, \psi_2 \) are homeomorphisms.

(2) Show that \( (U_2, \psi_2) \) and \( (U_2, \varphi_2) \) are not compatible.
Integration on Manifolds

Integrals over zero-dimensional domains:

- **0-domain**
  - called a 0-chain
  - is a finite number of points with multiplicity and signs
  - denoted $n_1p_1 + n_2p_2 + \cdots + n_\ell p_\ell$ with $p_1, \ldots, p_\ell \in \mathcal{M}$ and $n_1, \ldots, n_\ell \in \mathbb{Z}$. 
0-dimensional Integral case:

- domain of integration
  - called a 0-chain
  - a finite number of points in $\mathcal{M}$ with multiplicity and sign
  - denoted $n_1 p_1 + \cdots + n_k p_k$ with $p_1, \ldots, p_k$ being distinct points of $\mathcal{M}$ and $n_1, \ldots, n_k \in \mathbb{Z}$
  - Formal definition: A function $\sigma : \mathcal{M} \to \mathbb{Z}$ which vanishes except at finitely many points. The $\sigma$ corresponding to $n_1 p_1 + \cdots + n_k p_k$ is
    $$\sigma(p) = \begin{cases} n_j, & \text{if } p = p_j \text{ for some } 1 \leq j \leq k, \\ 0, & \text{otherwise} \end{cases}$$

- the integrand is called a 0-form, which is just a function $F : \mathcal{M} \to \mathbb{C}$
- the integral is
  $$\int_{n_1 p_1 + \cdots + n_k p_k} F = \sum_{j=1}^{k} n_j F(p_j)$$

1-dimensional Integral case:

Brief review of work integrals

- have a particle which is at $\vec{r}(t)$ at time $t$
- feels a force $\vec{F}(\vec{r})$
- by Newton
  $$m \frac{d^2 \vec{r}}{dt^2}(t) = \vec{F}(\vec{r}(t)) \Rightarrow m \frac{d^2 \vec{r}}{dt^2}(t) \cdot \frac{d \vec{r}}{dt}(t) = \vec{F}(\vec{r}(t)) \cdot \frac{d \vec{r}}{dt}(t)$$
  $$\Rightarrow \frac{d}{dt} \left( \frac{1}{2} m \frac{d \vec{r}}{dt} \cdot \frac{d \vec{r}}{dt} \right) = \vec{F}(\vec{r}(t)) \cdot \frac{d \vec{r}}{dt}$$
  $$\Rightarrow \int_{t_1}^{t_2} \frac{1}{2} m \vec{v}^2(t_2) - \frac{1}{2} m \vec{v}^2(t_1) = \int_{t_1}^{t_2} \vec{F}(\vec{r}(t)) \cdot \frac{d \vec{r}}{dt}(t) \, dt$$

Usual notation for the work integral $\int_{t_1}^{t_2} \vec{F}(\vec{r}(t)) \cdot \frac{d \vec{r}}{dt}(t) \, dt$ is

$$\int_C \vec{F} \cdot d\vec{r} = \int_C F_1 \, dx + F_2 \, dy + F_3 \, dz.$$
On a manifold of dimension $n$

- **domain of integration**
  - called a 1-chain
  - denoted $n_1C_1 + \cdots + n_kC_k$ with $C_1, \ldots, C_k$ begin paths, and $n_1, \ldots, n_k \in \mathbb{Z}$
  - a path is a function $r : [0,1] \to M$ which is $C^\infty$; a function $r : [0,1] \to M$ is $C^\infty$ at $t_0$ if there is a coordinate patch (chart) $\{U, \varphi\}$ with $r(t_0) \in U$ and $\varphi(r(t))$ is $C^\infty$ at $t_0$. If so, $\psi(r(t))$ is automatically $C^\infty$ at $t_0$ for any coordinate patch $\{V, \psi\}$ with $r(t_0) \in V$.

- **integrand**
  - called a 1-form
  - a 1-form is a rule which assigns to each coordinate patch $\{U, \varphi = (x^1, \ldots, x^n)\}$, $n$ functions $f_1, \ldots, f_n : \varphi(U) \to \mathbb{C}$
  - denoted $\omega|_{\{U, \varphi = (x^1, \ldots, x^n)\}} = f_1dx^1 + \cdots + f_ndx^n$
  - The functions must obey the following change of coordinates rule:
    * Let $\{U, \varphi = (x^1, \ldots, x^n)\}$ and $\{V, \psi = (y^1, \ldots, y^n)\}$ with $U \cap V = \emptyset$. Write $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V) \subset \mathbb{R}^n$.
      
    $\left. \omega \right|_{\{U, \varphi\}} = f_1dx^1 + \cdots + f_ndx^n = \sum_{k=1}^n f_kdx^k = \sum_{j,k} f_k \frac{\partial x^k}{\partial y^j} dy^j$
    $\left. \omega \right|_{\{V, \psi\}} = g_1dy^1 + \cdots + g_n dy^n$

Then $g_j(\vec{y}) = \sum_{k=1}^n f_k(\vec{x}(\vec{y})) \frac{\partial x^k}{\partial y^j}$. Recall $dx^k = \sum_{j=1}^n \frac{\partial x^k}{\partial y^j} dy^j$

- **integral**

\[
\int_{n_1C_1 + \cdots + n_kC_k} \omega = n_1 \int_{C_1} \omega + \cdots + n_k \int_{C_k} \omega
\]

If $C$ is the path $r : [0,1] \to M$ and if $\{U, \varphi = (x^1, \ldots, x^n)\}$ is a chart with $r(t) \in U$ for all $0 \leq t \leq 1$, then

\[
\int_C \omega = \int_0^1 \sum_{j=1}^n f_j(\varphi(r(t))) \frac{dx^j(r(t))}{dt} dt
\]

if $\omega|_{\{U, \varphi = (x^1, \ldots, x^n)\}} = f_1dx^1 + \cdots + f_ndx^n$. Note that this integral gives the same answer for all charts.

- **One last note:**

\[
\int_{\partial R} \omega = \int_R d\omega
\]

This identity is, in fact, the Fundamental Theorem of Calculus, Green’s Theorem, Stokes’ Theorem, Divergence Theorem, and half a dozen more...

That’s all, folks!