Theorem. (Uniform Convergence of Fourier Series)
If \( f : \mathbb{R} \to \mathbb{C} \) is \( 2\pi \)-periodic and is \textbf{piecewise} \( C^1 \), then the Fourier series for \( f \) converges uniformly to \( f \) on any closed interval that does not contain a point of discontinuity of \( f \).

Proof. \textbf{Step 1:} If, in addition, \( f \) is continuous, then the Fourier series of \( f \) converges uniformly to \( f \).

Proof. Done.

\textbf{Step 2:} Define

\[
\sum_{k=1}^{n} \delta_j F(x - p_j)
\]

The Fourier series converges uniformly on any closed interval that does not contain a point in \( 2\pi \mathbb{Z} = \{ 2\pi k \mid k \in \mathbb{Z} \} \).

Proof. Done.

\textbf{Step 3:} The general case:
Let the points of discontinuity of \( f \) on \( [-\pi, \pi] \) be \( p_1, \ldots, p_m \). Let the jump heights be \( \delta_j = f(p_j^+) - f(p_j^-) \). Write \( f(x) = g(x) + \sum_{j=1}^{m} \delta_j F(x - p_j) \) with \( g(x) = f(x) - \sum_{j=1}^{m} \delta_j F(x - p_j) \). The Fourier series for \( \delta_j F(x - p_j) \) converges uniformly on any closed interval not containing a point of \( p_j + 2\pi \mathbb{Z} \). It suffices to show that \( g(x) \) is continuous. \( g(x) \) is obviously continuous except possibly at the \( p_k \)'s, which are in \( [-\pi, \pi] \). To check continuity at \( p_k \)'s

\[
g(p_k^+) - g(p_k^-) = f(p_k^+) - f(p_k^-) - \sum_{j=1}^{n} \delta_j [F((p_k - p_j)^+) - F((p_k - p_j)^-)]
\]

\[
= \delta_k - \sum_{j=1}^{n} \delta_j \begin{cases} 0, & j \neq k; \\ 1, & j = k \end{cases}
\]

\[
= 0
\]

\[
\square
\]

\textbf{Manifolds (See web notes and references therein)}

(1) Definition

(2) Examples

(3) Integration on Manifolds & Stoke’s theorem
Rough Definition

- an \( n \)-dimensional manifold is something that looks locally like \( \mathbb{R}^n \)
- it is a union of subsets with each subset having coordinates that run over any open subset of \( \mathbb{R}^n \)

Definition. (An \( n \)-dimensional manifold)

Let \( \mathcal{M} \) be a metric space.

(a) A **chart** on \( \mathcal{M} \) is a pair \( \{U, \varphi\} \) with \( U \subset \mathcal{M} \) open and \( \varphi: U \to \mathbb{R}^n \) a homeomorphism between \( U \) and \( \varphi(U) \). Think of \( \varphi \) as assigning coordinates in \( \mathbb{R}^n \) to each \( m \in U \).

(b) Two charts \( \{U, \varphi\} \) and \( \{V, \psi\} \) are **compatible** if

\[
\psi \circ \varphi^{-1} : \varphi(U \cap V) \subset \mathbb{R}^n \to \psi(U \cap V) \subset \mathbb{R}^n \\
\varphi \circ \psi^{-1} : \psi(U \cap V) \subset \mathbb{R}^n \to \varphi(U \cap V) \subset \mathbb{R}^n
\]

are \( C^\infty \) (i.e., all partial derivatives are defined and continuous.)
Definition. (Manifold of dimension $n$) Let $\mathcal{M}$ be a metric space.

(a) A chart is a pair $\{ \mathcal{U}, \varphi \}$ with
   - $\mathcal{U}$ an open subset of $\mathcal{M}$
   - $\varphi$ a homeomorphism from $\mathcal{U}$ to an open subset of $\mathbb{R}^n$

(b) $\{ \mathcal{U}, \varphi \}$ and $\{ \mathcal{V}, \psi \}$ are compatible if $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are $C^\infty$

(c) An atlas $\mathcal{A}$ for the $\mathcal{M}$ is a set $\mathcal{A} = \{ \{ \mathcal{U}_i, \varphi_i \} \mid i \in I \}$, with $I$ being a completely arbitrary index set, of charts obeying
   - (a) $\{ \mathcal{U}_i \}$ covers $\mathcal{M}$ (i.e., $\bigcup_{i \in I} \mathcal{U}_i = \mathcal{M}$)
   - (b) Every pair of charts in $\mathcal{A}$ is compatible.

A maximal atlas is an atlas $\mathcal{A}$ with the property that $\{ \mathcal{U}, \varphi \}$ being any chart that is compatible with every chart in $\mathcal{A}$ implies $\{ \mathcal{U}, \varphi \} \in \mathcal{A}$.

(d) A manifold of dimension $n$ is a metric space $\mathcal{M}$ together with a maximal atlas $\mathcal{A}$.

Lemma. Let $\{ \mathcal{U}, \varphi \}$ and $\{ \mathcal{V}, \psi \}$ be

- $\mathcal{U}, \mathcal{V} \in \mathcal{M}$ with $\mathcal{U} \cup \mathcal{V} \neq \emptyset$
- $\varphi : \mathcal{U} \to \mathbb{R}^n$ be a homeomorphism from $\mathcal{U}$ to $\varphi(\mathcal{U}) \subseteq \mathbb{R}^n$ open
- $\psi : \mathcal{V} \to \mathbb{R}^m$ be a homeomorphism from $\mathcal{V}$ to $\psi(\mathcal{V}) \subseteq \mathbb{R}^m$ open
- $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are $C^\infty$

Proof. Homework problem.

Lemma. If $\mathcal{A}$ is any atlas, then there exists a unique maximal atlas containing $\mathcal{A}$

Proof. Homework problem. (Hint: Try to guess what the maximal atlas is.)

Example 1. Let $\mathcal{M}$ be an open subset of $\mathbb{R}^n$ (e.g., $\mathcal{M} = \mathbb{R}^n$) and $\mathcal{A} = \{ \{ \mathcal{U}, \varphi \} \mid \mathcal{U} = \mathcal{M}, \varphi = \text{identity map} \}$. Then $(\mathcal{U}, \varphi)$ is an identity map.

Example 2. Let $\mathcal{M} = S^1 = \{ (x, y) \in \mathbb{R}^n \mid x^2 + y^2 = 1 \}$ and

- $\mathcal{U}_1 = S^1 \setminus \{ (-1, 0) \}$, $\varphi_1(x, y) = \arctan \frac{y}{x} \in (-\pi, \pi)$ is a unique angle with $(x, y) = (\cos \varphi_1, \sin \varphi_1)$
- $\mathcal{U}_2 = S^1 \setminus \{ (1, 0) \}$, $\varphi_2(x, y) = \arctan \frac{y}{x} \in (0, 2\pi)$ is a unique angle with $(x, y) = (\cos \varphi_2, \sin \varphi_2)$
Then \( \{ \{ U_1, \varphi_1 \}, \{ U_2, \varphi_2 \} \} \) is an atlas for \( S^1 \) verification of compatibility:

\[
\begin{align*}
\varphi_1(U_1 \cap U_2) & = (-\pi, 0) \cup (0, \pi) \\
\varphi_2(U_1 \cap U_2) & = (0, \pi) \cup (\pi, 2\pi) \\
\varphi_2 \circ \varphi_1^{-1}(\theta) & = \begin{cases} 
\varphi_2 \left( \begin{pmatrix} \cos \theta, \sin \theta \end{pmatrix} \right), & \text{if } -\pi < \theta < 0, \\
\varphi_2 \left( \begin{pmatrix} \cos \theta, \sin \theta \end{pmatrix} \right), & \text{if } 0 < \theta < \pi,
\end{cases} \\
& = \begin{cases} 
\theta + 2\pi, & \text{if } -\pi < \theta < 0, \\
\theta, & \text{if } 0 < \theta < \pi,
\end{cases} \\
& \in C^\infty
\end{align*}
\]

Similarly for \( \varphi_1 \circ \varphi_2^{-1} \).

**Example. (Any \( n \)-dimensional surface in \( \mathbb{R}^{n+m} \))** Handwavy definition of an \( n \)-dimensional surface in \( \mathbb{R}^{n+m} \)

- a subset of \( \mathbb{R}^{n+m} \) such that locally
  - \( m \) coordinates of points on the surface are determined by the other \( n \) coordinates in a \( C^\infty \) way.

\[
x = -\sqrt{1-y^2} \\
y = -\sqrt{1-x^2}
\]
Example. \((n\text{-dimensional surface in } \mathbb{R}^{m+n})\)

Rough definition: A subset of \(\mathbb{R}^{m+n}\) is an \(n\)-dimensional surface if locally, \(m\) coordinates of points on the surface are determined by the \(n\) other coordinates (i.e. are functions of the other \(n\) coordinates) in a \(C^\infty\) way.

Example.

\[
S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}
\]

For \(y > 0\), \(y = \sqrt{1 - x^2}\)

For \(x < 0\), \(x = -\sqrt{1 - y^2}\)

Formal definition: \(M \subset \mathbb{R}^{m+n}\) is an \(n\)-dimensional surface if, for each \(\vec{z} \in M\), there are

- a neighborhood \(U_{\vec{z}}\) of \(\vec{z}\) in \(\mathbb{R}^{m+n}\)
- \(n\) natural numbers \(1 \leq j_1 < j_2 < \cdots < j_n \leq m+n\) (indices of the independent coordinates)
- \(m\) \(C^\infty\) functions \(f_k(x_{j_1}, \ldots, x_{j_n})\) with \(k \in \{1, \ldots, m+n\} \setminus \{1 \leq j_1 < \cdots < j_n\}\) such that the point \(\vec{x} \in U_{\vec{z}}\) is in \(M\) if and only if \(x_k = f_k(x_{j_1}, \ldots, x_{j_n})\) for all \(k \in \{1, \ldots, m+n\} \setminus \{1 \leq j_1 < \cdots < j_n\}\)

This is a manifold with the atlas

\[
A = \{ \{ M \cap U_{\vec{z}}, \varphi_{\vec{z}} \} \mid \vec{z} \in M \}
\]

where \(\varphi_{\vec{z}}(x_1, \ldots, x_{n+m}) = (x_{j_1}, \ldots, x_{j_n}).\)

Equivalent definition: \(M \subset \mathbb{R}^{m+n}\) is an \(n\)-dimensional surface if, for each \(\vec{z} \in M\), there are

- a neighborhood \(U_{\vec{z}}\) of \(\vec{z}\) in \(\mathbb{R}^{m+n}\)
- \(m\) \(C^\infty\) functions

\[
g_k : U_{\vec{z}} \to \mathbb{R}, \ k = 1, \ldots, m
\]

such that \(\big\{ \nabla g_k(\vec{z}) \mid 1 \leq k \leq m \big\}\) are linearly independent such that the point \(\vec{x} \in U_{\vec{z}}\) is in \(M\) if and only if \(g_k(\vec{x}) = 0\) for all \(1 \leq k \leq m\).

The two definitions are equivalent because of the \ldots
Implicit Function Theorem

Let

- \( m, n \in \mathbb{N} \)
- \( U \subset \mathbb{R}^{m+n} \) be open
- \( \vec{g} : U \rightarrow \mathbb{R}^m \) be \( C^\infty \)

\( \vec{y} \in \mathbb{R}^m \)
\( \vec{x} \in \mathbb{R}^n \)
\( \vec{x}_0 \in \mathbb{R}^n \)
\( \vec{y}_0 \in \mathbb{R}^m \)
\( U, V, W \subset \mathbb{R}^{m+n} \) be open

- \( \vec{g}(\vec{x}_0, \vec{y}_0) = 0 \) for some \( \vec{x}_0 \in \mathbb{R}^n \), \( \vec{y}_0 \in \mathbb{R}^m \) with \( (\vec{x}_0, \vec{y}_0) \in U \)
- \( \det \left[ \frac{\partial g_i}{\partial y_j}(\vec{x}_0, \vec{y}_0) \right]_{1 \leq i \leq m, 1 \leq j \leq m} \neq 0 \)

Then there exist open sets \( V \subset \mathbb{R}^{m+n} \) and \( W \subset \mathbb{R}^n \) with \( \vec{x}_0 \in W \) and \( (\vec{x}_0, \vec{y}_0) \in V \) such that for each \( \vec{x} \in W \), there is a unique \( (\vec{x}, \vec{y}) \in V \) with \( \vec{g}(\vec{x}, \vec{y}) = \vec{0} \). Call \( \vec{y} = f(\vec{x}) \). Furthermore, \( f : W \rightarrow \mathbb{R}^m \) is \( C^\infty \) and \( f(\vec{x}_0) = \vec{y}_0 \) and \( \vec{g}(\vec{x}, f(\vec{x})) = \vec{0} \) for all \( \vec{x} \in W \).

Example. (Simple – \( S^1 \))

\[
g(x, y) = x^2 + y^2 - 1, \quad (x_0, y_0) = (0, 1) \\
U = \mathbb{R}^2, \quad V = \{ (x, y) \in \mathbb{R}^2 \mid y > 0 \}, \quad W = (-1, 1), \quad f(x) = \sqrt{1 - x^2}
\]

Examples. (\( SO(3), O(3) \))

\[
O(3) = \{ 3 \times 3 \text{ real matrices } R \mid R^\dagger R = I \} \\
SO(3) = \{ 3 \times 3 \text{ real matrices } R \mid R^\dagger R = I, \quad \det R = 1 \}
\]

Think of the matrix

\[
\begin{bmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3
\end{bmatrix}
\]

as a vector in \( \mathbb{R}^9 \). Write \( R^\dagger R = I \) as a linear system of equation.