Theorem. (Pointwise Convergence of Fourier Series)
Assume: \( f: \mathbb{R} \to \mathbb{C} \) be 2\( \pi \)-periodic and Riemann integrable on \([-\pi, \pi] \); \( x \in \mathbb{R} \)

\[
|f(x + t) - f(x^+)| \leq M|t| \quad \text{for all } 0 < x < \delta
\]

\[
|f(x + t) - f(x^-)| \leq M|t| \quad \text{for all } -\delta < x < 0
\]

for some \( \delta > 0 \) and \( M < \infty \), then \( \lim_{n \to \infty} S_n = \frac{1}{2}[f(x^+) + f(x^-)] \)

Proof. Last class we defined

\[
g(t) = \begin{cases} 
\frac{f(x-t) - f(x^+)}{\sin(t/2)}, & \text{if } t \in [-\pi, 0), \\
\frac{f(x-t) - f(x^-)}{\sin(t/2)}, & \text{if } t \in (0, \pi], \\
0, & \text{if } t = 0.
\end{cases}
\]

and showed that

\[
S_n(x) - \frac{1}{2}[f(x^+) - f(x^-)]
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)e^{\frac{i}{2}t}e^{int} dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)e^{-\frac{i}{2}t}e^{-int} dt
\]

\[
= \frac{1}{2\pi}(\text{nth Fourier coefficient of } g(t)e^{(i/2)t}) - \frac{1}{2\pi}(\text{-nth Fourier coefficient of } g(t)e^{-(i/2)t})
\]

so the theorem follows from

Lemma. \( g(t) \) is Riemann integrable on \([-\pi, \pi] \).

Proof. Let \( \varepsilon > 0 \), \( \varepsilon < \delta \). Note that

(1) On \([-\pi, -\varepsilon] \cup [\varepsilon, \pi] \), \( g(t) \) is Riemann-integrable since \( f(x - t), f(x^{\pm}), \frac{1}{\sin(t/2)} \) are integrable.

(2) There is a constant \( M' > 0 \) such that \( |g(t)| \leq M' \) for all \( t \in [-\varepsilon, \varepsilon] \) because

\[
|g(t)| \leq \left| \frac{2|\frac{t}{\sin(t/2)}|}{\sin(t/2)} \right| \leq M'
\]

since \( \lim_{t \to 0} \frac{t}{\sin(t/2)} = 2 \). Thus,

\[
\int_{-\pi}^{\pi} g(t) dt - \int_{-\pi}^{\pi} g(t) dt = \int_{-\pi}^{-\varepsilon} g(t) dt + \int_{-\varepsilon}^{\varepsilon} g(t) dt + \int_{\varepsilon}^{\pi} g(t) dt - \int_{-\varepsilon}^{-\pi} g(t) dt - \int_{-\pi}^{\varepsilon} g(t) dt - \int_{\varepsilon}^{\pi} g(t) dt
\]

\[
= \int_{-\varepsilon}^{\varepsilon} g(t) dt - \int_{-\varepsilon}^{\varepsilon} g(t) dt
\]

Since \( \int_{-\varepsilon}^{\varepsilon} g(t) dt < M'(2\varepsilon) \) and \( \int_{-\varepsilon}^{\varepsilon} g(t) dt < M'(2\varepsilon) \),

\[
\int_{-\pi}^{\pi} g(t) dt - \int_{-\pi}^{\pi} g(t) dt \leq 4M'\varepsilon
\]

for all \( 0 < \varepsilon < \delta \). This means \( \int = \int \). Hence, \( g \in \mathcal{R} \).
Theorem. (Uniform Convergence of Fourier Series)
If \( f : \mathbb{R} \to \mathbb{C} \) is \( 2\pi \)-periodic and is piecewise \( C^1 \), then \( S_n(f, x) \) converges uniformly to \( f \) on any closed interval that contains no point of discontinuity of \( f \).

Definition. \( f \) is piecewise \( C^1 \) on \([-\pi, \pi]\) if there exist \( n \in \mathbb{N} \) and \(-\pi = x_0 < x_1 < x_2 < \cdots < x_n = \pi\) such that

1. for each \( 1 \leq j \leq n \), \( f \) has at least one continuous derivative on \((x_{j-1}, x_j)\)
2. \( \lim_{t \to 0^+} f(x_j + t) \) and \( \lim_{t \to 0^+} f'(x_j + t) \) exist. That is,

\[
f(x_j^+), \quad f(x_j^-), \quad f'(x_j^+), \quad f'(x_j^-)
\]

all exist.

Proof of Theorem:

Step 1: \( f \) is continuous and piecewise \( C^1 \). If \( f \) has Fourier series expansion \( \sum_{m=-\infty}^{\infty} c_m e^{imx} \), then \( f' \) has \( \sum_{m=-\infty}^{\infty} im c_m e^{imx} \)
Theorem. (Uniform Convergence of Fourier Series)
If \( f : \mathbb{R} \to \mathbb{C} \) is 2\( \pi \)-periodic and is piecewise \( C^1 \), then the Fourier series of \( f \) converges uniformly on any closed interval that does not contain a point of discontinuity of \( f \).

**Proof.** Step 1: If \( f \) is continuous and piecewise \( C^1 \), then the Fourier series of \( f \) converges uniformly to \( f \).

**Step (1a):** If \( f \) has Fourier series \( \sum_{n=-\infty}^{\infty} a_n e^{inx} \), then \( f' \) has Fourier series \( \sum_{n=-\infty}^{\infty} in a_n e^{inx} \).

Proof. The \( n \)-th Fourier coefficient of \( f' \) is

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x)e^{-inx} \, dx = \frac{1}{2\pi} \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f'(x)e^{-inx} \, dx
\]

(where \( f' \) is continuous on \([-\pi, \pi]\) except at \( -\pi = x_0 < x_1 < x_2 < \cdots < x_n = \pi \))

\[
= \frac{1}{2\pi} \sum_{k=1}^{n} \lim_{\varepsilon \to 0} \int_{x_{k-1}+\varepsilon}^{x_k-\varepsilon} f'(x)e^{-inx} \, dx
\]

\[
= \frac{1}{2\pi} \sum_{k=1}^{n} \lim_{\varepsilon \to 0} \left\{ f(x)e^{-inx} \bigg|_{x_{k-1}+\varepsilon}^{x_k-\varepsilon} + \int_{x_{k-1}}^{x_k-\varepsilon} f(x)e^{-inx} \, dx \right\}
\]

\[
= \frac{1}{2\pi} \sum_{k=1}^{n} f(x)e^{-inx} \bigg|_{x_{k-1}=x_{k-1}}^{x_k=x_k} + (in) \frac{1}{2\pi} \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)e^{-inx} \, dx
\]

the \( x_k \) with \( k = 1, \ldots, n-1 \) contributions cancel; and the \( x_0 = -\pi, x_n = \pi \) terms cancel by periodicity.

\[
a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx
\]

\[
= 0 + in a_n = in a_n.
\]

**Step (1b):** If \( f \) is piecewise \( C^1 \) and continuous, the Fourier series converges uniformly. We know

\[
\sum_{n=-\infty}^{\infty} a_n e^{inx}
\]

converges pointwise to \( f(x) \) since \( |f(x + t) - f(x)| \leq \sup |f'| |t| \leq M |t| \) and we know

\[
\sum_{n=-\infty}^{\infty} in a_n e^{inx}
\]
converges in the mean to \(f'(x)\). So for each \(x\),

\[
\left| f(x) - \sum_{n=-N}^{N} a_n e^{inx} \right| = \left| \sum_{n=-\infty}^{\infty} a_n e^{inx} - \sum_{n=-N}^{N} a_n e^{inx} \right| \leq \sum_{|n|>N} |a_n| = \sum_{|n|>N} \frac{1}{n} |a_n| \leq \sqrt{\sum_{|n|>N} \frac{1}{n^2}} \sqrt{\sum_{|n|>N} |a_n|^2}
\]

But \(\sum_{n=-\infty}^{\infty} \frac{1}{n^2}\) converges and \(\sum_{n=-\infty}^{\infty} |a_n|^2\) converges to

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(x)|^2 \, dx < \infty.
\]

So,

\[
\lim_{N \to \infty} \sqrt{\sum_{|n|>N} \frac{1}{n^2}} = 0 \quad \text{and} \quad \lim_{N \to \infty} \sqrt{\sum_{|n|>N} |a_n|^2} = 0.
\]

**Step 2:** Let \(F(x) = \begin{cases} \frac{\pi-x}{2\pi}, & 0 < x < 2\pi \\ 0, & x = 0,2\pi \end{cases}\)

and \(F(x + 2\pi) = F(x)\) for all \(x \in \mathbb{R}\). By Problem Set 9, #3, \(F(x)\) has Fourier series

\[
\frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \frac{1}{n} e^{inx}
\]

which converges uniformly on \([\varepsilon, 2\pi - \varepsilon]\) for any \(\varepsilon > 0\).

**Step 3:** (Proof of theorem) Suppose \(f\) has jump discontinuities on \([-\pi, \pi]\) at \(p_1, \ldots, p_m\). Let, for \(1 \leq j \leq m\), let \(\delta_j = f(p_j^+)-f(p_j^-)\). Write

\[
f(x) = \sum_{j=1}^{n} \delta_j F(x - p_j) + f(x) - \sum_{j=1}^{n} \delta_j F(x - p_j)
\]

FS conv. uniformly except at \(p_j\)  continuous FS converges uniformly