Lecture #13: February 4, 2008
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Theorem. *(Tests for uniform convergence)*

(a) (Weierstraß M-test) *Assume that*

1. *for each* n, $f_n : E \to \mathbb{C}$,
2. *for each* n, $|f_n(x)| \le M_n$ *for* $x \in E$
3. $\sum_{n=1}^{\infty} M_n < \infty$.

Then $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly on* E

Proof. For each $x \in E$, $\{\sum_{n=1}^{\infty} f_n(x)\}_{n\in\mathbb{N}}$ converges (absolutely) by comparison to $\sum_{n=1}^{\infty} M_n$. Call the limit $f(x)$. To prove that it converges uniformly, observe that

$$\left|f(x) - \sum_{m=1}^{n} f_m(x)\right| = \left|\lim_{p\to\infty}\sum_{m=1}^{p} f_m(x) - \sum_{m=1}^{n} f_m(x)\right| = \lim_{p\to\infty}\left|\sum_{m=1}^{p} f_m(x) - \sum_{m=1}^{n} f_m(x)\right|$$
$$= \lim_{p\to\infty}\left|\sum_{m=n+1}^{p} f_m(x)\right| \le \sum_{m=n+1}^{p} M_m$$

which means that

$$\left|f(x) - \sum_{m=1}^{n} f_m(x)\right| \le \sum_{m=n+1}^{\infty} M_m = \sum_{m=n+1}^{\infty} M_m - \sum_{m=1}^{n} M_m \to 0 \text{ uniformly in } x \text{ as } n\to\infty$$

Since $\sum_{m=1}^{\infty} M_m$ converges,

$$\left\|f - \sum_{m=1}^{n} f_m\right\|_{\infty} \le \sum_{m=1}^{\infty} M_m - \sum_{m=1}^{n} M_m \to 0 \text{ uniformly in } x \text{ as } n\to\infty.$$ □

(b) *If* $f(x) = \sum_{m=1}^{\infty} a_n x^n$ *has radius of convergence* $R > 0$ *and if* $0 < R' < R$ *(Note the strict inequality* $R' < R$*), then the series converges uniformly on* $|x| \le R'$. (Note: To compute R, recall that

$$R = \limsup_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}. \quad \text{The series } \sum_{n=0}^{\infty} a_n x^n \text{ converges for } |x| < R, \text{ but it diverges for } |x| > R.)$$

Proof. See Problem Set 5 question 1.

(c) *Dini's Theorem (assumes (1) compact domain and (2) monotone convergence.)*

(d) *Dirichlet test (uses summation by parts.)*

Issue #2: Suppose you know

1. $f_n : E \to \mathbb{C}$ is continuous.
2. $\lim_{n\to\infty} f_n(x) = f(x)$.

Does this ensure that $f(x)$ is continuous? No. Not in general. (See Problem Set 5 Question 2(d)) There is a sequence $f_n(x)$ with

(1) Each f_n is continuous,

(2) $\{f_n(x)\}$ converges pointwise and in the mean to $f(x)$, but

(3) $f(x)$ is NOT continuous.

Fortunately, if

(1) Each f_n is continuous, and

(2) $f_n \to f$ uniformly,

then $f(x)$ continuous.

Notation. $p \in E'$ means p is a limit point of E. Recall that if p is a limit point, then by definition

$$\exists \{p_n\} \subset E \setminus \{p\} \quad \text{such that} \quad p = \lim_{n\to\infty} p_n$$

and that p is not necessarily inside E.

Theorem. *Let X be a metric space, and $E \subset X$, and $p \in E'$. Let, for each $n \in \mathbb{N}$, $f_n : E \to \mathbb{C}$ obey*

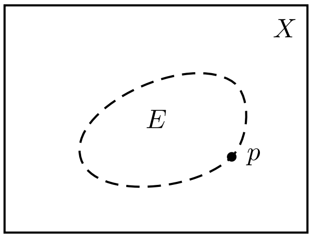
(Hyp 1) *$f_n \to f$ uniformly on E,*

(Hyp 2) *for each $n \in \mathbb{N}$, $\lim_{t\to p} f_n(t) = A_n$ exists,*

then

(a) *$\lim_{n\to\infty} A_n = A$ exists, and*

(b) *$\lim_{t\to p} f(t) = A$.*



Remark. This is an "exchange of limits" problem

$$\left\{ \begin{array}{c} \lim_{t\to p} f(t) \overset{?}{=} A \\ \lim_{t\to p} f(t) = \lim_{t\to p} \left(\lim_{n\to\infty} f_n(t) \right) \end{array} \right\} \overset{?}{=} \left\{ A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \left(\lim_{t\to p} f_n(t) \right) \right\}$$

Proof of Theorem:

(a) We first prove that $\{A_n\}$ is Cauchy. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, $\exists N$ such that

$$\begin{array}{lll} n \geq \mathbb{N} & \Longrightarrow & |f_n(t) - f(t)| < \frac{\varepsilon}{3}, \quad \forall t \in E \\ m \geq \mathbb{N} & \Longrightarrow & |f_m(t) - f(t)| < \frac{\varepsilon}{3}, \quad \forall t \in E \end{array}$$

and, by the triangle inequality, we have

$$m, n \geq \mathbb{N} \quad \Longrightarrow \quad |f_n(t) - f_m(t)| \leq |f_n(t) - f(t)| + |f(t) - f_m(t)| < \tfrac{2\varepsilon}{3}, \quad \forall t \in E$$

As $t \to p$, we have

$$m, n \geq \mathbb{N} \quad \Longrightarrow \quad |A_n - A_m| \leq \tfrac{2\varepsilon}{3} < \varepsilon \quad \Longrightarrow \quad \{A_n\} \text{ is Cauchy.}$$

Hence, $\lim_{n\to\infty} A_n = A$ exists.

Lecture #14: February 6, 2008
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Theorem. *Let X be a metric space, $E \subset X$, p be a limit point of E, and $f_n, f : E \to \mathbb{C}$. Assume that*

(H1) *$f_n \to f$ uniformly on E,*

(H2) *for each $n \in \mathbb{N}$, $\lim_{t\to p} f_n(t) = A_n$ exists. (Not assuming uniformity)*

Then

(a) *$\lim_{n\to\infty} A_n = A$ exists,*

(b) *$\lim_{t\to p} f(t) = A$.*

Proof. (a) Done last class.

(b) Let $\varepsilon > 0$.

$$|f(t) - A| \le |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$$

By uniform convergence of f_n,

$$\text{(H1)} \implies \exists N_1(\varepsilon) \text{ such that } n \ge N_1 \implies |f(t) - f_n(t)| < \frac{\varepsilon}{3}, \quad \forall t \in E$$

$$\text{(a)} \implies \exists N_2(\varepsilon) \text{ such that } n \ge N_2 \implies |A_n - A| < \frac{\varepsilon}{3}$$

Simply choose $n = \max\{N_1, N_2\}$. Note that we have not assume uniform convergence, so

$$\text{(H2)} \implies \exists \delta > 0 \text{ such that } d(t,p) < \delta \implies |f_n(t) - A_n| < \frac{\varepsilon}{3}$$

Thus, $|f(t) - A| < \varepsilon$. Hence, $\lim_{t\to p} f(t) = A$. □

Corollary 1. *If X is a metric space, $p \in X$ and $f_n, f : X \to \mathbb{C}$ obey*

(H1) *$f_n \to f$ uniformly as $n \to \infty$,*

(H2) *for each $n \in \mathbb{N}$, f_n is continuous at p.*

Then f is also continuous at p.

Proof. Apply the above theorem with $E = X$ and $A_n = f_n(p)$. □

Corollary 2. *Let X be a metric space, $C(X) = \{\, f : X \to \mathbb{R} \text{ or } \mathbb{C} \mid f \text{ is continuous and bounded} \,\}$ with the metric $d(f,g) = \sup_{p\in X} |f(p) - g(p)|$, then this is a complete metric space.*

Proof. Let $\{\, f_n \,\}_{n\in\mathbb{N}} \subset C(X)$ be a Cauchy sequence. Let $\varepsilon > 0$. There exists $m, n \ge N$ such that

$$\begin{aligned}
d(f_m, f_n) < \varepsilon \quad &\implies \quad |f_m(p) - f_n(p)| < \varepsilon \quad \text{for each} \quad p \in X \\
&\implies \quad \text{for each} \quad p \in X, \quad \{\, f_j(p) \,\}_{j\in\mathbb{N}} \text{ is Cauchy in } \mathbb{R} \text{ or } \mathbb{C} \\
&\implies \quad \text{for each} \lim_{n\to\infty} f_n(p) \quad \text{exists and we call it } f(p) \\
&\implies \quad |f_m(p) - f_n(p)| \quad \text{whenever } n, m \ge N \implies \lim_{m\to\infty} |f_m(p) - f_n(p)| \le \varepsilon \\
&\implies \quad |f(p) - f_n(p)| \le \varepsilon
\end{aligned}$$

We know $\exists N$ such that $n \ge N \implies |f(p) - f_n(p)| \le \varepsilon, \forall p$, which means $d(f, f_n) \le \varepsilon$; and the fact that $f_n \to f$ uniformly implies f is continuous and bounded, which further implies $f \in C(X)$. Hence, $f \in C(X)$ and $f_n \to f$ in the metric for $C(X)$. Therefore, $(C(X), d)$ is a complete metric space. □

Corollary. (Interchanging the Order of Summation) *If* $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|a_{jk}| < +\infty$, *then* $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}a_{jk}$.

Remark. $\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|a_{jk}|\right) < \infty$ means that for each $j \in \mathbb{N}$, $\sum_{k=1}^{\infty}|a_{jk}| = M_j$ converges and $\sum_{j=1}^{\infty}M_j < \infty$. The meaning of the conclusion

$$\begin{aligned}
\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|a_{jk}|\right) &= \lim_{n\to\infty}\left(\sum_{j=1}^{n}\sum_{k=1}^{\infty}|a_{jk}|\right)\\
&= \lim_{n\to\infty}\sum_{j=1}^{n}\left(\lim_{m\to\infty}\sum_{k=1}^{m}|a_{jk}|\right)\\
&= \lim_{n\to\infty}\lim_{m\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{m}a_{jk} < S_{nm}\\
\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}|a_{jk}|\right) &= \lim_{m\to\infty}\sum_{k=1}^{m}\left(\lim_{n\to\infty}\sum_{j=1}^{n}|a_{jk}|\right)\\
&= \lim_{m\to\infty}\lim_{n\to\infty}\sum_{k=1}^{m}\sum_{j=1}^{n}a_{jk} < S_{nm}
\end{aligned}$$

Proof of Corollary: Apply the theorem with $X = \mathbb{R}$, $E = \{\,1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m} \mid \,\}$. For $p = 0$,

$$f_n(t_m) = \sum_{j=1}^{n}\sum_{k=1}^{m}a_{jk}, \qquad f(t_m) = \sum_{j=1}^{\infty}\sum_{k=1}^{m}a_{jk}$$

where $f(t_m)$ converges by comparison with $\sum_{j=1}^{\infty}M_j$. Apply the theorem using $f_n \to f$ uniformly by the Weierstraß M-test. See Notes on Web. $\square$

Lecture #15: February 8, 2008
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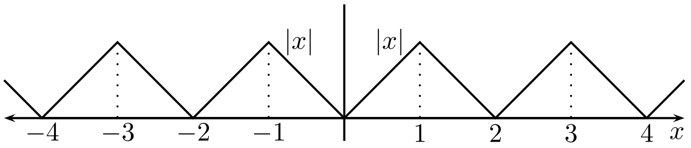
News: The deadline for NSERC-USRA application (MATH) is Wednesday, February 20.

Remark. (A Continuous nowhere differentiable function)
Define ϕ by:



$0 \le \phi(x) \le 1$, period 2

1. ϕ is continuous,
2. $\phi(x)$ is not differentiable at each $x \in \mathbb{Z}$.
3. $f(x) = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x)$ where
    - $\phi(4^n x)$ is continuous, bounded by 1, and nondifferentiable at $\frac{1}{4^n}\mathbb{Z}$
    - $\left(\frac{3}{4}\right)^n \phi(4^n x)$ continuous, bounded by $M_m = \left(\frac{3}{4}\right)^n$
    - $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x)$ converges uniformly by WeierstraßM-test.

    So $f(x)$ is continuous, but not differentiable at any $x \in \mathbb{R}$.

Proof. See the notes on web. ☐

Convergence and Integration

If $f_n \to f$, then does $\int_a^b f_n \, dx \to \int_a^b f \, dx$? No.

Examples.

1. There is an example (Problem Set 5 #2(e)) with $f_n \in \mathcal{R}$ on $[0,1]$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n(x) = f(x)$ exists for each x but $f \notin \mathcal{R}$.
2. There is an example (Problem Set 5 #2(f)) with $f, f_n \in \mathcal{R}$ on $[0,1]$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in [0,1]$, but $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \ne \int_0^1 f(x)\,dx$.

Theorem. *If*

(1) $\alpha : [a,b] \to \mathbb{R}$ *monotone increasing*

(2) *for each* $n \in \mathbb{N}$, $f_n : [a,b] \to \mathbb{R}$ *with* $f_n \in \mathcal{R}(\alpha)$ *on* $[a,b]$

(3) $f_n \to f$ *uniformly on* $[a,b]$,

then $f \in \mathcal{R}$ *on* $[a,b]$ *and* $\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha$ (Isn't this beautiful?)

Proof. Given

(i) $\sup_{x\in[a,b]} |f_n(x) - f(x)| = \varepsilon_n \to 0$ as $n \to \infty$,

(ii) $-\varepsilon_n \le f_n(x) - f(x) \le \varepsilon_n$ for all $x \in [a,b]$.

We have

$$\begin{aligned}
\int_a^b f_n \, d\alpha - \varepsilon_n[\alpha(b) - \alpha(a)] &= \int_a^b [f_n(x) - \varepsilon_n] \, d\alpha \\
&= \underline{\int_a^b} [f_n(x) - \varepsilon_n] \, d\alpha \\
&\leq \underline{\int_a^b} f(x) \, d\alpha(x) \\
&\leq \overline{\int_a^b} f(x) \, d\alpha(x) \\
&\leq \overline{\int_a^b} [f_n(x) + \varepsilon_n] \, d\alpha(x) \\
&= \int_a^b [f_n(x) + \varepsilon_n] \, d\alpha(x) \\
&= \int_a^b f(x) \, d\alpha(x) + \varepsilon_n[\alpha(b) - \alpha(a)]
\end{aligned}$$

such that

$$0 \leq \underbrace{\overline{\int_a^b} f \, d\alpha - \underline{\int_a^b} f \, d\alpha}_{\text{independent of } n} \leq \underbrace{2\varepsilon_n[\alpha(b) - \alpha(a)]}_{\to 0 \text{ as } n \to \infty} \Longrightarrow \underline{\int_a^b} f \, d\alpha = \overline{\int_a^b} f \, d\alpha \Longrightarrow f \in \mathcal{R}(\alpha) \text{ on } [a, b].$$

Thus,

$$\begin{aligned}
\left| \int_a^b f \, d\alpha - \int_a^b f_n \, d\alpha \right| &\leq \varepsilon_n[\alpha(b) - \alpha(a)] \to 0 \text{ as } n \to \infty \\
\Longrightarrow \quad \int_a^b f \, d\alpha &= \lim_{n \to \infty} \int_a^b f_n \, d\alpha
\end{aligned}$$

□

Remark. (Not part of this course)

Theorem. *(Arzelà) If*

1. $f_n, f : [a, b] \to \mathbb{R}$
2. $f_n, f \in \mathcal{R}$ *on* $[a, b]$
3. $|f(x)|, |f(x)| \leq M$ *on* $[a, b]$
4. $\lim_{n \to \infty} f_n(x) = f(x)$ *for each* $x \in [a, b]$

then

$$\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx$$

Theorem. *(Lebesgue Dominated Convergence Theorem)*
This is why analysts (not pyschoanalysts) and probabilists use the Lebesgue integral rather than Riemann-Stieltjes integral. Assume

1. μ *be a measure [assigns lengths to complicated subsets of* $\mathbb{R}$*:* $\int f \, d\mu$ *generalizes* $\int f \, d\alpha$*]*

More on this next week.