From Last Time: Surely we had a merry Christmas and a happy New Year.

**Announcements**

- **Office** = MATH 221 (Directly above the Mathematics Department Office.)
- **Email** = feldman@math.ubc.ca
- **Text** = “Baby Rudin” – *Principles of Mathematical Analysis 3/e* by Walter Rudin
- **Course Website** = [http://www.math.ubc.ca/~feldman/m321/](http://www.math.ubc.ca/~feldman/m321/)
- **Topics:**
  - Riemann-Stieltjes integral (§6)
  - Sequences & Series of functions (§7)
  - Power Series, Special Functions, Fourier Series (§8)
  - Another topic to be determined.

- **Grading:**
  1. A midterm on Wed. Feb 27 (25%)
  2. Weekly problem sets due each Wednesday (25%)
  3. Exam (50%)
  4. Grades will probably be scaled (up, usually)

**Integration (Rudin §6)**

Recall the definition of \( \int_a^b f(x) \, dx \) from first-year calculus.

**Step 1:** Slice up \([a, b]\).

**Definition.** A **partition** of \([a, b]\) is a finite set of points \( P = \{x_0, x_1, \ldots, x_n\} \) such that \( a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \).

**Step 2:** Pick a representation value of \( f \) for each subinterval.

**Definition.** A **choice** \( T \) for the partition \( P \) is a finite set of points \( t_1, \ldots, t_n \) obeying \( x_{i-1} \leq t_i \leq x_i \) for each \( 1 \leq i \leq n \). We shall use \( f(t_i) \) as an approximate value of \( f \) on \([x_{i-1}, x_i]\).

**Step 3:** Compute an approximate value of \( \int_a^b f(x) \, dx \).

**Definition.** A **Riemann partial sum** is a sum of the form

\[
\sum_{i=1}^{n} f(t_i)[x_i - x_{i-1}] = S(P, T, f, x).
\]

**Step 4:** Make the partition finer.

**Definition.** A partition \( P' \) is **finer** than the partition \( P \) if \( P' \supset P \).
Step 5: Hope the sums converge as the partition gets finer and finer.

Definition. A function \( f : [a, b] \to \mathbb{R} \) is said to be \textit{Riemann-integrable} on \([a, b]\) (denoted \( f \in \mathcal{R} \) on \([a, b]\)) if there is a number \( I \in \mathbb{R} \) such that

\[
\forall \varepsilon > 0, \ \exists P_\varepsilon \text{ such that for each partition } P \supset P_\varepsilon : |S(P, T, f, x) - I| < \varepsilon
\]

for all choices \( T \) compatible with \( P \). If so, we write

\[
\lim_{P} S(P, T, f, x) = I = \int_{a}^{b} f(x) dx.
\]

Generalization of this definition

Let \( \alpha : [a, b] \to \mathbb{R} \) and replace \( x_i - x_{i-1} \) by \( \alpha(x_i) - \alpha(x_{i-1}) \). The \textit{Riemann-Stieltjes partial sum} is

\[
S(P, T, f, \alpha) = \sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})]
\]

Definition. A function \( f : [a, b] \to \mathbb{R} \) is said to be \textit{Riemann-Stieltjes integrable} with respect to \( \alpha \) on \([a, b]\) (denoted by \( f \in \mathcal{R} (\alpha) \) on \([a, b]\)) if there is a number \( I \in \mathbb{R} \) such that

\[
\forall \varepsilon > 0, \ \exists P_\varepsilon \text{ such that for each partition } P \supset P_\varepsilon : |S(P, T, f, \alpha) - I| < \varepsilon
\]

for all choices \( T \) compatible with \( P \). If so, we write

\[
\lim_{P} S(P, T, f, \alpha) = I = \int_{a}^{b} f(x) d\alpha(x) = \int_{a}^{b} f d\alpha.
\]

[Examples and application shortly].

Remark.

1. The Riemann-Stieltjes integral reduces to the Riemann integral when \( \alpha(x) = x \).

2. We shall eventually prove that

   (a) \underline{\alpha \text{ monotonic, } f \text{ continuous } \Rightarrow f \in \mathcal{R}(\alpha)}
   
   (b) \underline{\alpha \text{ continuous, } f \text{ monotonic } \Rightarrow f \in \mathcal{R}(\alpha)} \text{. (From (a) and Integration by Parts)}
   
   (c) \underline{\alpha \text{ strictly monotone, } f \text{ unbounded on } [a, b] \Rightarrow f \notin \mathcal{R}(\alpha)} \text{ (Use improper integrals)}
From last time: We defined
\[ \int_a^b f(x) \, d\alpha(x) = \lim_{P} S(P, T, f, \alpha). \]
Here, \( P = \{x_0, x_1, \cdots, x_n\} \) with \( a = x_0 < x_1 < x_2 < \cdots < x_n = b, \)
and \( T = \{t_1, \cdots, t_n\} \) with \( t_i \in [x_{i-1}, x_i]. \)
\[ S(P, T, f, \alpha) = \sum_{i=1}^{n} f(t_i) \left[ \alpha(x_i) - \alpha(x_{i-1}) \right] \]
and
\[ \lim_{P} S(P, T, f, \alpha) = I \iff \forall \varepsilon > 0, \exists P_\varepsilon \text{ such that } P \supset P_\varepsilon \implies |S(P, T, f, \alpha) - I| < \varepsilon \]
And this time:

**Remark.** (Connection between \( \int_a^b f \, d\alpha \) and Riemann Integral) Observe that
\[ S(P, T, f, \alpha) \approx \sum_{i=1}^{n} f(t_i) \left[ \alpha(x_i) - \alpha(x_{i-1}) \right] \]
We shall prove that if \( \alpha \) has a continuous derivative, then
\[ \int_a^b f \, d\alpha = \int_a^b f(x) \alpha'(x) \, dx \]

**Definition.** A **step function** on \([a, b]\) is a function \( \alpha : [a, b] \to \mathbb{R} \) such that
(i) \( \alpha \) has finitely many points of discontinuity on \([a, b]\). Call \( s_i \) for \( 1 \leq i \leq n \) where \( a \leq s_1 < s_2 < \cdots < s_n \leq b. \)
(ii) \( \alpha \) is constant on each subinterval.

\[ [a, b), \quad (s_j, s_{j+1}], \quad \text{for } 1 \leq j \leq n, \quad (s_n, b]. \]

**Theorem.** Let \( a < b \). Let \( \alpha : [a, b] \to \mathbb{R} \) be a step function with discontinuities at \( s_1 < \cdots < s_n \). Then \( f \in \mathcal{R}(\alpha) \) on \([a, b]\) and
\[ \int_a^b f \, d\alpha = \sum_{j=1}^{n} f(s_j) \left[ \alpha(s_j^+) - \alpha(s_j^-) \right] \]
where
\[ \alpha(s_j^+) = \lim_{t \to s_j} \alpha(t), \quad \alpha(s_j^-) = \lim_{t \to s_j} \alpha(t) \]
and by convention

\[ s_1 = a \Rightarrow \alpha(s^-) = \alpha(a) \]
\[ s_n = b \Rightarrow \alpha(s^+) = \alpha(b) \]

**Proof.** Write \( I = \sum_{j=1}^{n} f(s_j)[\alpha(s^+) - \alpha(s^-)] \). We prove that

\[ \forall \varepsilon > 0, \exists P_\varepsilon \text{ such that } P \supset P_\varepsilon \implies |S(P, f, \alpha) - I| < \varepsilon \]

Let \( \varepsilon > 0 \). Choose \( P_\varepsilon \) obeying

1. \( \{s_0, \ldots, s_n\} \subset P_\varepsilon = \{a = \tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_m = b\} \)
2. The mesh (or norm) of \( P_\varepsilon \) is

\[ \|P_\varepsilon\| = \max_{1 \leq i \leq m} |\tilde{x}_i - \tilde{x}_{i-1}| < \delta \]

where \( \delta = \min\{s_2 - s_1, s_3 - s_2, \ldots, s_n - s_{n-1}, \delta_0\} \), and \( \delta_0 \) is given by

Insert (*) here.

Let \( P = \{x_0, x_1, \ldots, x_p\} \supset P_\varepsilon \) and \( T = \{t_1, \ldots, t_p\} \) be a choice for \( P \) and consider each term in

\[ S(P, T, f, \alpha) = \sum_{i=1}^{p} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})] \]

Either

1. neither \( x_{i-1} \) nor \( x_i \) is on \( s_j \). In this case, \( \alpha(x_i) = \alpha(x_{i-1}) \), so \( \alpha(x_i) - \alpha(x_{i-1}) = 0 \), or
2. \( x_i = s_j \) for some \( j \). In this case, \( \alpha(x_i) - \alpha(x_{i-1}) = \alpha(s_j) - \alpha(s^-_j) \), or
3. \( x_{i-1} = s_j \) for some \( j \). In this case, \( \alpha(x_i) - \alpha(x_{i-1}) = \alpha(s^+_j) - \alpha(s_j) \).
Proof. (Continued from last time)

\[ S(P, T, f, \alpha) = \sum_{i=1}^{p} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})] \]

Each of \( t_{ij} \) and \( t_{ij}' \) lie in an interval of \( P \) with one endpoint: \( s_j \)

Since

\[ ||P|| < \delta \implies \begin{cases} |s_j - t_{ij}| < \delta \\ |s_j - t_{ij}'| < \delta. \end{cases} \]

We are aiming for the integral to be

\[ \sum_{j=1}^{n} f(s_j)[\alpha(s_j^+) - \alpha(s_j^-)] = \sum_{j=1}^{n} \left\{ f(s_j)[\alpha(s_j) - \alpha(s_j^-)] + f(s_j)[\alpha(s_j^+) - \alpha(s_j)] \right\} \]

But

\[
\left| S(P, T, f, \alpha) - \sum_{j=1}^{n} f(s_j)[\alpha(s_j^+) - \alpha(s_j^-)] \right|
\leq \sum_{j=1}^{n} \left\{ |f(t_{ij}) - f(s_j)|[\alpha(s_j) - \alpha(s_j^-)] + |f(t_{ij}') - f(s_j)|[\alpha(s_j^+) - \alpha(s_j)] \right\}
\]

(*) for each \( 1 \leq j \leq n, \ f \) is continuous at \( s_j \) implies \( \exists \delta_j > 0 \) such that

\[ |f(s_j) - f(t)| < \frac{\epsilon}{\sum_{k=1}^{n} \{|\alpha(s_k) - \alpha(s_k^-)| + |\alpha(s_k^+) - \alpha(s_k)|\}} \]

for all \( |t - s_j| < \delta_j \). Choose \( \delta_0 = \min\{\delta_1, \ldots, \delta_n\} \).

Since \( ||P|| \leq ||P_\epsilon|| < \delta_0 \), we have

\[ \left| S(P, T, f, \alpha) - \sum_{j=1}^{n} f(s_j)[\alpha(s_j^+) - \alpha(s_j^-)] \right| < \epsilon. \]

Remarks. (Applications of the Riemann-Stieltjes integral)

1. We now have a Dirac \( \delta \)-function on a hand-waving level, \( \delta(x) \) is defined by

(a) \( \delta(x) < 0 \) for all \( x \neq 0 \)

(b) \( \delta(0) = +\infty \)
(c) \( \int_{a}^{b} \delta(x) \, dx = 1 \) for all \( a < 0, b > 0 \).

such that

\[
\int_{a}^{b} f(x) \delta(x) \, dx = \int_{a}^{b} f(0) \delta(x) \, dx, \quad \forall a < 0, b > 0 \quad (\text{Not rigorous})
\]

Notice that for \( x \neq 0 \), both integrals equal 0.

We can also define the **Heaviside function**

\[
H(x) = \begin{cases} 
0, & x < 0 \\
1, & x \geq 0
\end{cases}
\]

such that \( \delta(x) = H'(x) \). (Not rigorous)

A more rigorous definition follows from the concept of Riemann-Stieltjes Integral:

\[
\int_{a}^{b} f(x) \delta(x) \, dx = f(0)
\]

\[
\int_{a}^{b} f(x) H'(x) \, dx = f(0)
\]

\[
\int_{a}^{b} f \, dH = f(0)
\]

2. Application to Probability. ([Next Lecture])