Functions of Bounded Variation

Our main theorem concerning the existence of Riemann–Stieltjes integrals assures us that the integral \( \int_a^b f(x) \, d\alpha(x) \) exists when \( f \) is continuous and \( \alpha \) is monotonic. Our linearity theorem then guarantees that the integral \( \int_a^b f(x) \, d\alpha(x) \) exists when \( f \) is continuous and \( \alpha \) is the difference of two monotonic functions. In these notes, we prove that \( \alpha \) is the difference of two monotonic functions if and only if it is of bounded variation, where

**Definition 1**

(a) The function \( \alpha : [a, b] \to \mathbb{R} \) is said to be of bounded variation on \( [a, b] \) if and only if there is a constant \( M > 0 \) such that

\[
\sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| \leq M
\]

for all partitions \( P = \{x_0, x_1, \ldots, x_n\} \) of \( [a, b] \).

(b) If \( \alpha : [a, b] \to \mathbb{R} \) is of bounded variation on \( [a, b] \), then the total variation of \( \alpha \) on \( [a, b] \) is defined to be

\[
V_\alpha(a, b) = \sup \left\{ \sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| \mid P = \{x_0, x_1, \ldots, x_n\} \text{ is a partition of } [a, b]\right\}
\]

**Example 2** If \( \alpha : [a, b] \to \mathbb{R} \) is monotonically increasing, then, for any partition \( P = \{x_0, x_1, \ldots, x_n\} \) of \( [a, b] \)

\[
\sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| = \sum_{i=1}^{n} [\alpha(x_i) - \alpha(x_{i-1})] = \alpha(x_n) - \alpha(x_0) = \alpha(b) - \alpha(a)
\]

Thus \( \alpha \) is of bounded variation and \( V_f(a, b) = \alpha(b) - \alpha(a) \).

**Example 3** If \( \alpha : [a, b] \to \mathbb{R} \) is continuous on \( [a, b] \) and differentiable on \( (a, b) \) with \( \sup_{a < x < b} |\alpha'(x)| \leq M \), then, for any partition \( P = \{x_0, x_1, \ldots, x_n\} \) of \( [a, b] \), we have, by the Mean Value Theorem,

\[
\sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| = \sum_{i=1}^{n} |\alpha'(t_i)[x_i - x_{i-1}]| \leq \sum_{i=1}^{n} M|x_i - x_{i-1}| = M(b - a)
\]

Thus \( \alpha \) is of bounded variation and \( V_f(a, b) \leq M(b - a) \).
Example 4  Define the function $\alpha : [0, 1] \to \mathbb{R}$ by

$$\alpha(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \cos \frac{\pi}{x} & \text{if } x \neq 0 \end{cases}$$

This function is continuous, but is not of bounded variation because it wobbles too much near $x = 0$. To see this, consider, for each $m \in \mathbb{N}$, the partition

$$P_m = \{ x_0 = 0, x_1 = \frac{1}{2m}, x_2 = \frac{1}{2m-1}, \ldots, x_{2m-2} = \frac{1}{3}, x_{2m-1} = \frac{1}{2}, x_{2m} = 1 \}$$

The values of $\alpha$ at the points of this partition are

$$\alpha(P_m) = \{ 0, \frac{1}{2m}, -\frac{1}{2m-1}, \frac{1}{2m-2}, \ldots, -\frac{1}{3}, \frac{1}{2}, -1 \}$$

For this partition,

$$\sum_{i=1}^{2m} |\alpha(x_i) - \alpha(x_{i-1})| = \frac{1}{2m} - 0 + \left| -\frac{1}{2m-1} - \frac{1}{2m} \right| + \left| \frac{1}{2m} + \frac{1}{2m-2} - \frac{1}{2m} \right| + \cdots + \left| -\frac{1}{3} - \frac{1}{4} \right| + \left| \frac{1}{2} + \frac{1}{3} \right| + |1 - \frac{1}{2}|$$

$$= \frac{1}{2m} + 0 + \frac{1}{2m-1} + \frac{1}{2m} + \frac{1}{(2m-2)} + \frac{1}{2m-1} + \cdots + \frac{1}{3} + \frac{1}{4} + \frac{1}{2} + \frac{1}{3} + 1 + \frac{1}{2}$$

$$= 2 \left( \frac{1}{2m} + \frac{1}{2m-1} + \cdots + \frac{1}{2} \right) + 1$$

The harmonic series $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges. So given any $M$, there is a partition $P_m$ for which

$$\sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| > M$$

Theorem 5

(a) If $\alpha, \beta : [a, b] \to \mathbb{R}$ are of bounded variation and $c, d \in \mathbb{R}$, then $c\alpha + d\beta$ is of bounded variation and

$$V_{c\alpha + d\beta}(a, b) \leq |c|V_\alpha(a, b) + |d|V_\beta(a, b)$$
(b) If \( \alpha : [a, b] \to \mathbb{R} \) is of bounded variation on \([a, b]\) and \([c, d] \subset [a, b]\), then \( \alpha \) is of bounded variation on \([c, d]\) and

\[
V_\alpha(c, d) \leq V_\alpha(a, b)
\]

(c) If \( \alpha : [a, b] \to \mathbb{R} \) is of bounded variation and \( c \in (a, b) \), then

\[
V_\alpha(a, b) = V_\alpha(a, c) + V_\alpha(c, b)
\]

(d) If \( \alpha : [a, b] \to \mathbb{R} \) is of bounded variation then the functions \( V(x) = V_\alpha(a, x) \) and \( V(x) - \alpha(x) \) are both increasing on \([a, b]\).

(e) The function \( \alpha : [a, b] \to \mathbb{R} \) is of bounded variation if and only if it is the difference of two increasing functions.

**Proof:** We shall use the shorthand notation

\[
\sum_P |\Delta_i \alpha| \quad \text{for} \quad \sum_{i=1}^n |\alpha(x_i) - \alpha(x_{i-1})|
\]

where the partition \( P = \{x_0, x_1, \ldots, x_n\} \).

(a) follows from the observation that, for any \( P \) partition of \([a, b]\),

\[
\sum_P |\Delta_i (c \alpha + d \beta)| \leq |c| \sum_P |\Delta_i \alpha| + |d| \sum_P |\Delta_i \beta| \leq |c| V_\alpha(a, b) + |d| V_\beta(a, b)
\]

(b) follows from the observation that, for any partition \( P \) of \([c, d]\),

\[
\sum_P |\Delta_i \alpha| \leq \sum_{P \cup \{a, b\}} |\Delta_i \alpha| \leq V_\alpha(a, b)
\]

(c) Since \( |\alpha(x_i) - \alpha(x_{i-1})| \leq |\alpha(x_i) - \alpha(c)| + |\alpha(c) - \alpha(x_{i-1})| \), we have

\[
\sum_P |\Delta_i \alpha| \leq \sum_{P \cup \{c\}} |\Delta_i \alpha| = \sum_{(P \cup \{c\}) \cap [a, c]} |\Delta_i \alpha| + \sum_{(P \cup \{c\}) \cap [c, b]} |\Delta_i \alpha| \leq V_\alpha(a, c) + V_\alpha(c, b)
\]

which implies that \( V_\alpha(a, b) \leq V_\alpha(a, c) + V_\alpha(c, b) \). To prove the other inequality, we let \( \varepsilon > 0 \) and select a partition \( P_1 \) of \([a, c]\) for which \( \sum_{P_1} |\Delta_i \alpha| \geq V_\alpha(a, c) - \varepsilon \) and a partition \( P_2 \) of \([c, b]\) for which \( \sum_{P_2} |\Delta_i \alpha| \geq V_\alpha(c, b) - \varepsilon \). Then

\[
\sum_{P_1 \cup P_2} |\Delta_i \alpha| = \sum_{P_1} |\Delta_i \alpha| + \sum_{P_2} |\Delta_i \alpha| \geq V_\alpha(a, c) + V_\alpha(c, b) - 2\varepsilon
\]
This assures that $V_{\alpha}(a, b) \geq V_{\alpha}(a, c) + V_{\alpha}(c, b) - 2\varepsilon$ for all $\varepsilon > 0$ and hence that $V_{\alpha}(a, b) \geq V_{\alpha}(a, c) + V_{\alpha}(c, b)$.

(d) Let $a \leq x_1 \leq x_2 \leq b$. That $V(x_1) = V_{\alpha}(a, x_1) \leq V_{\alpha}(a, x_2) = V(x_2)$ follows immediately from part (b). By part (c),

$$\{V(x_2) - \alpha(x_2)\} - \{V(x_1) - \alpha(x_1)\} = V_{\alpha}(x_1, x_2) - \{\alpha(x_2) - \alpha(x_1)\}$$

$$\geq V_{\alpha}(x_1, x_2) - |\alpha(x_2) - \alpha(x_1)|$$

So the inequality $[V(x_2) - \alpha(x_2)] \geq [V(x_1) - \alpha(x_1)]$ follows from

$$|\alpha(x_2) - \alpha(x_1)| = \sum_{\{x_1, x_2\}} |\Delta_i \alpha| \leq V_{\alpha}(x_1, x_2)$$

(e) If $\alpha$ is of bounded variation then $\alpha(x) = V_{\alpha}(a, x) - [V_{\alpha}(a, x) - \alpha(x)]$ expresses $\alpha$ as the difference of two increasing functions. On the other hand if $\alpha$ is the difference $\beta - \gamma$ of two increasing functions, then $\beta$ and $\gamma$ are of bounded variation by Example 2 and $\alpha$ is of bounded variation by part (a).

Example 6 We know that if $f$ is continuous and $\alpha$ is of bounded variation on $[a, b]$, then $f \in R(\alpha)$ on $[a, b]$. If $f$ is of bounded variation and $\alpha$ is continuous on $[a, b]$, then we have $f \in R(\alpha)$ on $[a, b]$ with

$$\int_a^b f \, d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha \, df$$

by our integration by parts theorem. It is possible to have $f \in R(\alpha)$ on $[a, b]$ even if neither $f$ nor $\alpha$ are of bounded variation on $[a, b]$. For example, we have seen, in Example 4, that

$$\alpha(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \cos \frac{\pi}{x} & \text{if } x \neq 0 \end{cases}$$

is continuous but not of bounded variation on $[0, 1]$, because of excessive oscillation near $x = 0$. So $f(x) = \alpha(1 - x)$ (still with the $\alpha$ of Example 4) is continuous but not of bounded variation on $[0, 1]$, because of excessive oscillation near $x = 1$. But $f \in R(\alpha)$ on $[0, \frac{1}{2}]$, by integration by parts, because $f$ is of bounded variation on $[0, \frac{1}{2}]$. And $f \in R(\alpha)$ on $[\frac{1}{2}, 1]$, because $\alpha$ is of bounded variation on $[0, \frac{1}{2}]$. So $f \in R(\alpha)$ on $[0, 1]$, by our linearity theorem.