Functions of Bounded Variation

Our main theorem concerning the existence of Riemann–Stietjes integrals assures us that the integral $\int_a^b f(x)\ d\alpha(x)$ exists when f is continuous and α is monotonic. Our linearity theorem then guarantees that the integral $\int_a^b f(x)\ d\alpha(x)$ exists when f is continuous and α is the difference of two monotonic functions. In these notes, we prove that α is the difference of two monotonic functions if and only if it is of bounded variation, where

Definition 1

(a) The function $\alpha : [a,b] \to \mathbb{R}$ is said to be of bounded variation on $[a,b]$ if and only if there is a constant $M > 0$ such that

$$\sum_{i=1}^{n} \big|\alpha(x_i) - \alpha(x_{i-1})\big| \le M$$

for all partitions $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$.

(b) If $\alpha : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$, then the total variation of α on $[a,b]$ is defined to be

$$V_\alpha(a,b) = \sup\Big\{ \sum_{i=1}^{n} \big|\alpha(x_i) - \alpha(x_{i-1})\big| \ \Big|\ P = \{x_0, x_1, \cdots, x_n\} \text{ is a partition of } [a,b] \Big\}$$

Example 2 If $\alpha : [a,b] \to \mathbb{R}$ is monotonically increasing, then, for any partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$

$$\sum_{i=1}^{n} \big|\alpha(x_i) - \alpha(x_{i-1})\big| = \sum_{i=1}^{n} \big[\alpha(x_i) - \alpha(x_{i-1})\big] = \alpha(x_n) - \alpha(x_0) = \alpha(b) - \alpha(a)$$

Thus α is of bounded variation and $V_f(a,b) = \alpha(b) - \alpha(a)$.

Example 3 If $\alpha : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with $\sup_{a<x<b} |\alpha'(x)| \le M$, then, for any partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$, we have, by the Mean Value Theorem,

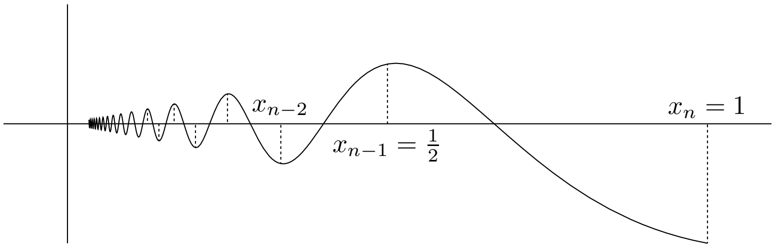
$$\sum_{i=1}^{n} \big|\alpha(x_i) - \alpha(x_{i-1})\big| = \sum_{i=1}^{n} \big|\alpha'(t_i)[x_i - x_{i-1}]\big| \le \sum_{i=1}^{n} M[x_i - x_{i-1}] = M(b-a)$$

Thus α is of bounded variation and $V_f(a,b) \le M(b-a)$.

Example 4 Define the function $\alpha : [0,1] \to \mathbb{R}$ by

$$\alpha(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \cos\frac{\pi}{x} & \text{if } x \ne 0 \end{cases}$$

This function is continuous, but is not of bounded variation because it wobbles too much near $x = 0$. To see this, consider, for each $m \in \mathbb{N}$, the partition $P_m = \{0, \frac{1}{2m}, \frac{1}{2m-1}, \cdots, \frac{1}{3}, \frac{1}{2}, 1\}$. The values of α at the points of this partition are $\alpha(P_m) = \{0, \frac{1}{2m}, -\frac{1}{2m-1}, \frac{1}{2m-2}, \cdots, -\frac{1}{3}, \frac{1}{2}, -1\}$.



For this partition,

$$\begin{aligned}\sum_{i=1}^{n} \big|\alpha(x_i) - \alpha(x_{i-1})\big| &= \big|\tfrac{1}{2m} - 0\big| + \big|-\tfrac{1}{2m-1} - \tfrac{1}{2m}\big| + \big|\tfrac{1}{(2m-2)} + \tfrac{1}{2m-1}\big| + \cdots + \big|-\tfrac{1}{3} - \tfrac{1}{4}\big| + \big|\tfrac{1}{2} + \tfrac{1}{3}\big| + \big|-1 - \tfrac{1}{2}\big| \\ &= \tfrac{1}{2m} + \tfrac{1}{2m-1} + \tfrac{1}{2m} + \tfrac{1}{(2m-2)} + \tfrac{1}{2m-1} + \cdots + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{3} + 1 + \tfrac{1}{2} \\ &= 2\big(\tfrac{1}{2m} + \tfrac{1}{2m-1} + \cdots + \tfrac{1}{2}\big) + 1\end{aligned}$$

The series $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges. So given any M, there is a partition P_m for which

$$\sum_{i=1}^{n} \big|\alpha(x_i) - \alpha(x_{i-1})\big| > M$$

Theorem 5

(a) If $\alpha, \beta : [a,b] \to \mathbb{R}$ are of bounded variation and $c, d \in \mathbb{R}$, then $c\alpha + d\beta$ is of bounded variation and

$$V_{c\alpha+d\beta}(a,b) \le |c| V_\alpha(a,b) + |d| V_\beta(a,b)$$

(b) If $\alpha : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$ and $[c,d] \subset [a,b]$, then α is of bounded variation on $[c,d]$ and

$$V_\alpha(c,d) \le V_\alpha(a,b)$$

(c) If $\alpha : [a,b] \to \mathbb{R}$ is of bounded variation and $c \in (a,b)$, then

$$V_\alpha(a,b) = V_\alpha(a,c) + V_\alpha(c,b)$$

(d) If $\alpha : [a,b] \to \mathbb{R}$ is of bounded variation then the functions $V(x) = V_\alpha(a,x)$ and $V(x) - \alpha(x)$ are both increasing on $[a,b]$.

(e) The function $\alpha : [a,b] \to \mathbb{R}$ is of bounded variation if and only if it is the difference of two increasing functions.

Proof: We shall use the shorthand notation

$$\sum_P \big|\Delta_i \alpha\big| \quad \text{for} \quad \sum_{i=1}^{n} \big|\alpha(x_i) - \alpha(x_{i-1})\big|$$

where the partition $P = \{x_0, x_1, \cdots, x_n\}$.

(a) follows from the observation that, for any P partition of $[a,b]$,

$$\sum_P \big|\Delta_i(c\alpha + d\beta)\big| \le |c| \sum_P \big|\Delta_i \alpha\big| + |d| \sum_P \big|\Delta_i \beta\big| \le |c| V_\alpha(a,b) + |d| V_\beta(a,b)$$

(b) follows from the observation that, for any partition P of $[c,d]$,

$$\sum_P \big|\Delta_i\alpha\big| \le \sum_{P\cup\{a,b\}} \big|\Delta_i\alpha\big| \le V_\alpha(a,b)$$

(c) Since $\big|\alpha(x_i)-\alpha(x_{i-1})\big| \le \big|\alpha(x_i)-\alpha(c)\big| + \big|\alpha(c)-\alpha(x_{i-1})\big|$, we have

$$\sum_P \big|\Delta_i\alpha\big| \le \sum_{P\cup\{c\}} \big|\Delta_i\alpha\big| = \sum_{(P\cup\{c\})\cap[a,c]} \big|\Delta_i\alpha\big| + \sum_{(P\cup\{c\})\cap[c,b]} \big|\Delta_i\alpha\big| \le V_\alpha(a,c)+V_\alpha(c,b)$$

which implies that $V_\alpha(a,b) \le V_\alpha(a,c)+V_\alpha(c,b)$. To prove the other inequality, we let $\varepsilon>0$ and select a partition P_1 of $[a,c]$ for which $\sum_{P_1}\big|\Delta_i\alpha\big| \ge V_\alpha(a,c)-\varepsilon$ and a partition P_2 of $[c,b]$ for which $\sum_{P_2}\big|\Delta_i\alpha\big| \ge V_\alpha(c,b)-\varepsilon$. Then

$$\sum_{P_1\cup P_2} \big|\Delta_i\alpha\big| = \sum_{P_1} \big|\Delta_i\alpha\big| + \sum_{P_2} \big|\Delta_i\alpha\big| \ge V_\alpha(a,c)+V_\alpha(c,b)-2\varepsilon$$

This assures that $V_\alpha(a,b) \ge V_\alpha(a,c)+V_\alpha(c,b)-2\varepsilon$ for all $\varepsilon>0$ and hence that $V_\alpha(a,b) \ge V_\alpha(a,c)+V_\alpha(c,b)$.

(d) Let $a\le x_1\le x_2\le b$. That $V(x_1)=V_\alpha(a,x_1)\le V_\alpha(a,x_2)=V(x_2)$ follows immediately from part (b). By part (c),

$$[V(x_2)-\alpha(x_2)]-[V(x_1)-\alpha(x_1)] = V_\alpha(x_1,x_2)-[\alpha(x_2)-\alpha(x_1)] \ge V_\alpha(x_1,x_2)-\big|\alpha(x_2)-\alpha(x_1)\big|$$

So the inequality $[V(x_2)-\alpha(x_2)] \ge [V(x_1)-\alpha(x_1)]$ follows from

$$\big|\alpha(x_2)-\alpha(x_1)\big| = \sum_{\{x_1,x_2\}} \big|\Delta_i\alpha\big| \le V_\alpha(x_1,x_2)$$

(e) If α is of bounded variation then $\alpha(x)=V_\alpha(a,x)-\big[V_\alpha(a,x)-\alpha(x)\big]$ expresses α as the difference of two increasing functions. On the other hand if α is the difference $\beta-\gamma$ of two increasing functions, then β and γ are of bounded variation by Example 2 and α is of bounded variation by part (a). ■

Example 6 We know that if f is continuous and α is of bounded variation on $[a,b]$, then $f\in\mathcal{R}(\alpha)$ on $[a,b]$. If f is of bounded variation and α is continuous on $[a,b]$, then we have $f\in\mathcal{R}(\alpha)$ on $[a,b]$ with

$$\int_a^b f\,d\alpha = f(b)\alpha(b)-f(a)\alpha(a)-\int_a^b \alpha\,df$$

by our integration by parts theorem. It is possible to have $f\in\mathcal{R}(\alpha)$ on $[a,b]$ even if neither f nor α are of bounded variation on $[a,b]$. For example, we have seen, in Example 4, that

$$\alpha(x)=\begin{cases} 0 & \text{if } x=0 \\ x\cos\frac{\pi}{x} & \text{if } x\ne 0 \end{cases}$$

is continuous but not of bounded variation on $[0,1]$, because of excessive oscillation near $x=0$. So $f(x)=\alpha(1-x)$ (still with the α of Example 4) is continuous but not of bounded variation on $[0,1]$, because of excessive oscillation near $x=1$. But $f\in\mathcal{R}(\alpha)$ on $[0,\frac12]$, by integration by parts, because f is of bounded variation on $[0,\frac12]$. And $f\in\mathcal{R}(\alpha)$ on $[\frac12,1]$, because α is of bounded variation on $[0,\frac12]$. So $f\in\mathcal{R}(\alpha)$ on $[0,1]$, by our linearity theorem.