Interchanging the Order of Summation

Corollary (Interchanging the Order of Summation)

$$\textit{If} \qquad \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\big|a_{jk}\big| < \infty \qquad \textit{then} \qquad \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{jk}$$

Remark. The hypothesis $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\big|a_{jk}\big| < \infty$ really means that

$$\text{for each } j \in \mathbb{N},\ \sum_{k=1}^{\infty}\big|a_{jk}\big| = M_j < \infty \qquad \text{and} \qquad \sum_{j=1}^{\infty} M_j < \infty$$

The two double sums in the conclusion really mean

$$\begin{aligned}
\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} &= \lim_{n\to\infty}\sum_{j=1}^{n}\Big[\sum_{k=1}^{\infty} a_{jk}\Big] = \lim_{n\to\infty}\sum_{j=1}^{n}\Big[\lim_{m\to\infty}\sum_{k=1}^{m} a_{jk}\Big] \\
&= \lim_{n\to\infty}\lim_{m\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{m} a_{jk} \\
\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{jk} &= \lim_{m\to\infty}\sum_{k=1}^{m}\Big[\sum_{j=1}^{\infty} a_{jk}\Big] = \lim_{m\to\infty}\sum_{k=1}^{m}\Big[\lim_{n\to\infty}\sum_{j=1}^{n} a_{jk}\Big] \\
&= \lim_{m\to\infty}\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{m} a_{jk}
\end{aligned}$$

That all of these limits exist is part of the conclusion of the corollary.

This result is a corollary of the following theorem, which has already been proven in class.

Theorem. *Let X be a metric space, $E \subset X$ and $p \in E'$, the set of limit points of E. Let $f : E \to \mathbb{C}$ and, for each $n \in \mathbb{N}$, $f_n : E \to \mathbb{C}$ and assume that*

(H1) $\lim_{n\to\infty} f_n(t) = f(t)$ *uniformly on E and*

(H2) for each $n \in \mathbb{N}$, $\lim_{t\to p} f_n(t) = A_n$ *exists*

Then

(a) $\lim_{n\to\infty} A_n = A$ *exists and*

(b) $\lim_{t\to p} f(t) = A$. *That is,* $\lim_{t\to p}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to p} f_n(t)$.

$$
\begin{array}{c} t \\ \downarrow \\ p \end{array}
\left\{
\overbrace{
\begin{array}{cccccc}
f_1(t) & f_2(t) & f_3(t) & \cdots & \xrightarrow{\text{unif}} & f(t) \\
\downarrow & \downarrow & \downarrow & \cdots & & \downarrow\text{(b)} \\
A_1 & A_2 & A_3 & \cdots & \xrightarrow{\text{(a)}} & A
\end{array}
}^{n\to\infty}
\right.
$$

Proof of the Corollary: Set $X = \mathbb{R}$, $E = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{m}, \cdots\right\}$ and $p = 0$. Write $\frac{1}{m} = t_m$ and define

$$f_n(t_m) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_{jk} \qquad f(t_m) = \sum_{j=1}^{\infty}\sum_{k=1}^{m} a_{jk}$$

The infinite sum in the definition of $f(t_m)$ converges by comparison with $\sum_{j=1}^{\infty} M_j$.

Verification of (H1): For each $j \in \mathbb{N}$, $\left|\sum\limits_{k=1}^{m} a_{jk}\right| \le M_j$ for all $m \in \mathbb{N}$. So the Weierstrass M–test implies that f_n converges to f, uniformly on E.

Verification of (H2): For each $j \in \mathbb{N}$, $\sum\limits_{k=1}^{\infty} a_{jk}$ converges absolutely by the hypothesis that $\sum\limits_{k=1}^{\infty} \left|a_{jk}\right| = M_j < \infty$. So

$$\lim_{m\to\infty} f_n(t_m) = \lim_{m\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{m} a_{jk} = \sum_{j=1}^{n}\lim_{m\to\infty}\sum_{k=1}^{m} a_{jk} = A_n$$

exists.

So the theorem now tells is that

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\lim_{m\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{m} a_{jk}$$

and

$$\lim_{m\to\infty} f(t_m) = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{m} a_{jk}$$

exist and are equal.

■

Example. Here is an example which illustrates the need for the hypothesis that the double sum converges absolutely. We choose

$$a_{jk} = \begin{cases} 1 & \text{if } j = k = 1 \\ 1 & \text{if } k = j + 1 \\ -1 & \text{if } k = j - 1 \\ 0 & \text{otherwise} \end{cases}$$

This example is rigged to give the partial sums

$$S_{mn} = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} = \begin{cases} 1 & \text{if } m = n \\ 2 & \text{if } n > m \\ 0 & \text{if } n < m \end{cases}$$

Pictorially

a_{jk}	k	$\rightarrow$				
j	1	1	0	0	0	$\cdots$
$\downarrow$	-1	0	1	0	0	$\cdots$
	0	-1	0	1	0	$\cdots$
	0	0	-1	0	1	
	$\vdots$	$\vdots$		$\ddots$	$\ddots$	$\ddots$

S_{mn}	n	$\rightarrow$					
m	1	2	2	2	$\cdots$	$\rightarrow$	2
$\downarrow$	0	1	2	2	$\cdots$	$\rightarrow$	2
	0	0	1	2	$\cdots$	$\rightarrow$	2
	0	0	0	1			
	$\vdots$	$\vdots$	$\vdots$	0			$\downarrow$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$\searrow$	2
	0	0	0	0	$\rightarrow$	0	1

For any fixed n, $S_{m,n} = 0$ for all $m > n$ and so converges to 0 as $m \to \infty$. Hence

$$\lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} = \lim_{n\to\infty} \left(\lim_{m\to\infty} S_{m,n} \right) = \lim_{n\to\infty} 0 = 0$$

Similarly, for each fixed m, $S_{m,n} = 2$ for all $n > m$ and so converges to 2 as $n \to \infty$. Hence

$$\lim_{m\to\infty} \lim_{n\to\infty} \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} = \lim_{m\to\infty} \left(\lim_{n\to\infty} S_{m,n} \right) = \lim_{m\to\infty} 2 = 2$$

And the sequence $S_{m,m} = 1$ converges to 1 as $m \to \infty$. So

$$\lim_{m\to\infty} \sum_{j=1}^{m} \sum_{k=1}^{m} a_{jk} = \lim_{m\to\infty} S_{m,m} = \lim_{m\to\infty} 1 = 1$$