Taylor’s Theorem

Theorem 1 (Taylor’s Theorem) Let \( a < b, \ n \in \mathbb{N} \cup \{0\}, \) and \( f : [a, b] \to \mathbb{R} \). Assume that \( f^{(n)} \) exists and is continuous on \([a, b]\) and \( f^{(n+1)} \) exists on \((a, b)\). Let \( \alpha \in [a, b] \) and define the Taylor polynomial of degree \( n \) with expansion point \( \alpha \) to be

\[
P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(\alpha) (x-\alpha)^k
\]

Then, for all \( x \in [a, b], \)

\[
f(x) = P_n(x) + R_n(\alpha, x)
\]

where the error term \( R_n(\alpha, x) \) is given by

(a) (integral form) \( R_n(\alpha, x) = \int_{\alpha}^{x} \frac{1}{(n+1)!} f^{(n+1)}(t) (x-t)^n \, dt, \) if \( f^{(n+1)} \) is integrable.

(b) (Lagrange form) \( R_n(\alpha, x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-\alpha)^{n+1} \) for some \( c \) strictly between \( \alpha \) and \( x \).

(c) (Cauchy form) \( R_n(\alpha, x) = \frac{1}{n!} f^{(n)}(c) (x-c)^n (x-\alpha) \) for some \( c \) strictly between \( \alpha \) and \( x \).

(d) If \( r \in \mathbb{N}, \) then \( R_n(\alpha, x) = \frac{1}{r!} f^{(n+1)}(c) (x-c)^{n+r+1} (x-\alpha)^r \) for some \( c \) strictly between \( \alpha \) and \( x \).

(e) \( R_n(\alpha, x) = \frac{1}{n!} Q^{(n)}(\alpha) (x-\alpha)^{n+1} \) where

\[
Q(t) = \begin{cases} 
\frac{f(t)-f(x)}{t-x} & \text{if } t \neq x \\
\frac{f'(x)}{1} & \text{if } t = x
\end{cases}
\]

Proof: Fix any evaluation and expansion points \( x, \alpha \in [a, b] \). Define the function \( S(t) \) by

\[
f(x) = f(t) + (x-t)f'(t) + \frac{1}{2}(x-t)^2 f''(t) + \cdots + \frac{1}{n!}(x-t)^n f^{(n)}(t) + S(t) \tag{1}
\]

Observe that substituting \( t = \alpha \) into (1) gives \( f(x) = P_n(x) + S(\alpha) \). So we wish to find

\[
R_n(\alpha, x) = S(\alpha)
\]

The function \( S(t) \) is determined by its derivative and its value at one point. Finding a value of \( S(t) \) for one value of \( t \) is easy. Substitute \( t = x \) into (1) to yield \( S(x) = 0 \). To find \( S'(t) \), apply \( \frac{d}{dt} \) to both sides of (1). Recalling that \( x \) is just a constant parameter,

\[
0 = f'(t) + \left[ -f'(t) + (x-t)f''(t) \right] + \left[ - (x-t)f''(t) + \frac{1}{2} (x-t)^2 f^{(3)}(t) \right] + \cdots + \left[ - \frac{1}{(n-1)!} (x-t)^{n-1} f^{(n)}(t) + \frac{1}{n!} (x-t)^n f^{(n+1)}(t) \right] + S'(t)
\]

\[
= \frac{1}{n!} (x-t)^n f^{(n+1)}(t) + S'(t)
\]
so that
\[ S'(t) = -\frac{1}{n!}f^{(n+1)}(t) (x-t)^n \]

(a) By the fundamental theorem of calculus
\[ S(\alpha) = -[S(x) - S(\alpha)] = - \int_\alpha^x S'(t) \, dt = \int_\alpha^x \frac{1}{n!}f^{(n+1)}(t) (x-t)^n \, dt \]

(c) By the mean value theorem, there is a \( c \) strictly between \( \alpha \) and \( x \) such that
\[
S(\alpha) = S(\alpha) - S(x) = S'(c) (\alpha - x) = -\frac{1}{n!}f^{(n+1)}(c) (x-c)^n(\alpha - x)
\]
\[
= \frac{1}{n!}f^{(n+1)}(c) (x-c)^n(\alpha - \alpha)
\]

(b) By the generalized mean value theorem (see the notes entitled “The Mean Value Theorem”) with \( F(t) = S(t) \) and \( G(t) = (x-t)^n+1 \), there is a \( c \) strictly between \( \alpha \) and \( x \) such that
\[
S(\alpha) = S(\alpha) - S(x) = \frac{S'(c)}{G'(c)} (G(\alpha) - G(x))
\]
\[
= -\frac{1}{n!}f^{(n+1)}(c) (x-c)^n \frac{1}{(n+1)(x-c)^n} (x-\alpha)^n
\]
\[
= \frac{1}{(n+1)!}f^{(n+1)}(c)(x-\alpha)^n
\]

Don’t forget, when computing \( G'(c) \), that \( G \) is a function of \( t \) with \( x \) just a fixed parameter.

(d) By the generalized mean value theorem with \( G(t) = (x-t)^r \), there is a \( c \) strictly between \( \alpha \) and \( x \) such that
\[
S(\alpha) = S(\alpha) - S(x) = \frac{S'(c)}{G'(c)} (G(\alpha) - G(x))
\]
\[
= -\frac{1}{n!}f^{(n+1)}(c) (x-c)^n \frac{1}{r(x-c)^{-r-1}} (x-\alpha)^r
\]
\[
= \frac{1}{r n}f^{(n+1)}(c)(x-c)^{n-r+1}(x-\alpha)^r
\]

(e) We’ll only consider the case that \( x \neq \alpha \). For \( x = \alpha \), the error \( R_n(\alpha, x) \) is obviously zero and we’ll just take it as a convention that \( \frac{1}{n!}Q^{(n)}(\alpha)(x-\alpha)^{n+1} = 0 \) even if \( Q^{(n)}(\alpha) \) is not defined. Since \( f \) is \( n \) times differentiable, so is \( Q(t) \), at least for all \( t \neq x \). In particular \( Q(t) \) is \( n \) times differentiable at \( t = \alpha \). From the definition of \( Q \) we have that
\[
f(t) = f(x) + (t-x)Q(t) \quad \Rightarrow \quad f(\alpha) = f(x) - (x-\alpha)Q(\alpha)
\]
\[
f'(t) = Q(t) + (t-x)Q'(t) \quad \Rightarrow \quad f'(\alpha) = Q(\alpha) - (x-\alpha)Q'(\alpha)
\]
\[
f^{(2)}(t) = 2Q'(t) + (t-x)Q^{(2)}(t) \quad \Rightarrow \quad f^{(2)}(\alpha) = 2Q'(\alpha) - (x-\alpha)Q^{(2)}(\alpha)
\]
\[
\vdots
\]
\[
f^{(k)}(t) = kQ^{(k-1)}(t) + (t-x)Q^{(k)}(t) \quad \Rightarrow \quad f^{(k)}(\alpha) = kQ^{(k-1)}(\alpha) - (x-\alpha)Q^{(k)}(\alpha)
\]
for \( k \leq n \). So

\[
\begin{align*}
  f(x) - P(x) &= f(x) - f(\alpha) - \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(\alpha) (x-\alpha)^k \\
  &= (x-\alpha)Q(\alpha) - \sum_{k=1}^{n} \left\{ \frac{1}{(k-1)!} Q^{(k-1)}(\alpha) (x-\alpha)^k - \frac{1}{k!} Q^{(k)}(\alpha) (x-\alpha)^{k+1} \right\}
\end{align*}
\]

The sum telescopes leaving

\[
  f(x) - P(x) = \frac{1}{n!} Q^{(n)}(\alpha) (x-\alpha)^{n+1}
\]

as desired.

\[\blacksquare\]

\textbf{Remark 2} It is rarely necessary, or even possible, to evaluate \( R_n(\alpha, x) \) exactly. It is usually sufficient to find a number \( M \) such that \( |f^{(n+1)}(c)| \leq M \) for all \( c \) between the expansion point \( \alpha \) and the \( x \) of interest. Both parts (a) and (b) of Taylor’s Theorem then imply that

\[
|R_n(\alpha, x)| \leq \frac{1}{(n+1)!} M |x-\alpha|^{n+1}
\]

\textbf{Example 3 (Sine and Cosine Series)} The trigonometric functions \( \sin x \) and \( \cos x \) have widely used Taylor expansions about \( \alpha = 0 \). Every derivative of \( \sin x \) and \( \cos x \) is one of \( \pm \sin x \) and \( \pm \cos x \). Consequently, when we apply Theorem 1.b we always have \( |f^{(n+1)}(c)| \leq 1 \) and hence \( |R_n(\alpha, x)| \leq \frac{|x|^{n+1}}{(n+1)!} \). This converges to zero as \( n \to \infty \). Consequently, for both \( f(x) = \sin x \) and \( f(x) = \cos x \), we have \( \lim_{n \to \infty} R_n(\alpha = 0, x) = 0 \) and

\[
  f(x) = \lim_{n \to \infty} \left[ f(0) + f'(0) x + \cdots + \frac{1}{n!} f^{(n)}(0) x^n \right]
\]

First, compute all derivatives at general \( x \).

\[
\begin{align*}
  f(x) &= \sin x \quad f'(x) = \cos x \quad f''(x) = -\sin x \quad f^{(3)}(x) = -\cos x \quad f^{(4)}(x) = \sin x \quad \cdots \\
  f(x) &= \cos x \quad f'(x) = -\sin x \quad f''(x) = -\cos x \quad f^{(3)}(x) = \sin x \quad f^{(4)}(x) = \cos x \quad \cdots
\end{align*}
\]

The pattern starts over again with the fourth derivative being the same as the original function. Now set \( x = \alpha = 0 \).

\[
\begin{align*}
  f(x) &= \sin x \quad f(0) = 0 \quad f'(0) = 1 \quad f''(0) = 0 \quad f^{(3)}(0) = -1 \quad f^{(4)}(0) = 0 \quad \cdots \\
  f(x) &= \cos x \quad f(0) = 1 \quad f'(0) = 0 \quad f''(0) = -1 \quad f^{(3)}(0) = 0 \quad f^{(4)}(0) = 1 \quad \cdots
\end{align*}
\]
For \( \sin x \), all even numbered derivatives are zero. The odd numbered derivatives alternate between 1 and \(-1\). For \( \cos x \), all odd numbered derivatives are zero. The even numbered derivatives alternate between 1 and \(-1\). We conclude that

\[
\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \\
\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} \tag{1}
\]

If we wish to evaluate \( \sin x \) or \( \cos x \) for some specific value of \( x \), we may always use trig identities like \( \sin(x \pm 2\pi) = -\sin(x \pm \pi) = \cos(\frac{x}{2} - x) \) to reduce consideration to \( 0 \leq x \leq \frac{\pi}{4} < 1 \). Then the alternating series test tells us that the error introduced by truncating the series in (1) is between 0 and the first term dropped. That is, if \( 0 \leq x \leq 1 \),

\[
\sin x - \left[ x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1} \right] \text{ is between 0 and } (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
\cos x - \left[ 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{(-1)^{n-1}}{(2n-2)!}x^{2n-2} \right] \text{ is between 0 and } (-1)^n \frac{1}{(2n)!}x^{2n}
\]

This error decreases spectacularly quickly as \( n \) increases. For example

\[
\frac{1}{5!} \approx 0.0083 \quad \frac{1}{6!} \approx 0.0014 \quad \frac{1}{7!} \approx 0.0002 \quad \frac{1}{8!} \approx 0.000025 \\
\frac{1}{9!} \approx 0.000003 \quad \frac{1}{10!} \approx 0.00000028 \quad \frac{1}{11!} \approx 0.000000025 \quad \frac{1}{12!} \approx 0.000000002
\]

**Example 4 (Exponential Series)** A similar phenomenon happens with the exponential function \( f(x) = e^x \). By Theorem 1.b, for all natural numbers \( n \) and all real numbers \( x \),

\[
e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \frac{e^c}{(n+1)!}x^{n+1}
\]

for some \( c \) strictly between 0 and \( x \). Now consider any fixed real number \( x \). As \( c \) runs from 0 to \( x \), \( e^c \) runs from \( e^0 = 1 \) to \( e^x \). In particular, \( e^c \) is always between 1 and \( e^x \) and so is smaller than \( 1 + e^x \). Thus the error term

\[
|R_n(0, x)| = \left| \frac{e^c}{(n+1)!}x^{n+1} \right| \leq |x|^{n+1} \frac{|x|^{n+1}}{(n+1)!}
\]

Let’s call \( e_n(x) = \frac{|x|^{n+1}}{(n+1)!} \). I claim that as \( n \) increases towards infinity, \( e_n(x) \) decreases (quickly) towards zero. To see this, let’s compare \( e_n(x) \) and \( e_{n+1}(x) \).

\[
\frac{e_{n+1}(x)}{e_n(x)} = \frac{|x|^{n+2}}{(n+2)!} \frac{|x|^{n+1}}{(n+1)!} = \frac{|x|}{n+2}
\]
So, when $n$ is bigger than, for example $2|x|$, we have $\frac{e_{n+1}(x)}{e_n(x)} < \frac{1}{2}$. That is, increasing the index on $e_n(x)$ by one decreases the size of $e_n(x)$ by a factor of at least two. As a result $e_n(x)$ must tend to zero as $n \to \infty$. Consequently $\lim_{n \to \infty} R_n(0, x) = 0$ and

$$e^x = \lim_{n \to \infty} \left[ 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n \right] = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$