Taylor's Theorem

Theorem 1 (Taylor's Theorem) *Let $a < b$, $n \in \mathbb{N} \cup \{0\}$, and $f : [a, b] \to \mathbb{R}$. Assume that $f^{(n)}$ exists and is continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b). Let $\alpha \in [a, b]$ and define the Taylor polynomial of degree n with expansion point α to be*

$$P_n(x) = \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(\alpha)\,(x-\alpha)^k$$

Then, for all $x \in [a, b]$,

$$f(x) = P_n(x) + R_n(\alpha, x)$$

where the error term $R_n(\alpha, x)$ is given by

(a) (integral form) $R_n(\alpha, x) = \int_\alpha^x \frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n\,dt$, if $f^{(n+1)}$ is integrable.

(b) (Lagrange form) $R_n(\alpha, x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-\alpha)^{n+1}$ for some c strictly between α and x.

(c) (Cauchy form) $R_n(\alpha, x) = \frac{1}{n!} f^{(n+1)}(c)\,(x-c)^n(x-\alpha)$ for some c strictly between α and x.

(d) If $r \in \mathbb{N}$, then $R_n(\alpha, x) = \frac{1}{rn!} f^{(n+1)}(c)\,(x-c)^{n-r+1}(x-\alpha)^r$ for some c strictly between α and x.

(e) $R_n(\alpha, x) = \frac{1}{n!} Q^{(n)}(\alpha)(x-\alpha)^{n+1}$ where

$$Q(t) = \begin{cases} \frac{f(t)-f(x)}{t-x} & \text{if } t \neq x \\ f'(x) & \text{if } t = x \end{cases}$$

Proof: Fix any evaluation and expansion points $x, \alpha \in [a, b]$. Define the function $S(t)$ by

$$f(x) = f(t) + (x-t)f'(t) + \tfrac{1}{2}(x-t)^2 f''(t) + \cdots + \tfrac{1}{n!}(x-t)^n f^{(n)}(t) + S(t) \qquad (1)$$

Observe that substituting $t = \alpha$ into (1) gives $f(x) = P_n(x) + S(\alpha)$. So we wish to find

$$R_n(\alpha, x) = S(\alpha)$$

The function $S(t)$ is determined by its derivative and its value at one point. Finding a value of $S(t)$ for one value of t is easy. Substitute $t = x$ into (1) to yield $S(x) = 0$. To find $S'(t)$, apply $\frac{d}{dt}$ to both sides of (1). Recalling that x is just a constant parameter,

$$\begin{aligned} 0 &= f'(t) + \big[-f'(t) + (x-t)f''(t)\big] + \big[-(x-t)f''(t) + \tfrac{1}{2}(x-t)^2 f^{(3)}(t)\big] \\ &\qquad + \cdots + \big[-\tfrac{1}{(n-1)!}(x-t)^{n-1} f^{(n)}(t) + \tfrac{1}{n!}(x-t)^n f^{(n+1)}(t)\big] + S'(t) \\ &= \tfrac{1}{n!}(x-t)^n f^{(n+1)}(t) + S'(t) \end{aligned}$$

so that

$$S'(t) = -\tfrac{1}{n!} f^{(n+1)}(t)\,(x-t)^n$$

(a) By the fundamental theorem of calculus

$$S(\alpha) = -\big[S(x) - S(\alpha)\big] = -\int_\alpha^x S'(t)\ dt = \int_\alpha^x \tfrac{1}{n!} f^{(n+1)}(t)\,(x-t)^n\ dt$$

(c) By the mean value theorem, there is a c strictly between α and x such that

$$\begin{aligned} S(\alpha) = S(\alpha) - S(x) = S'(c)\ (\alpha - x) &= -\tfrac{1}{n!} f^{(n+1)}(c)\,(x-c)^n(\alpha - x) \\ &= \tfrac{1}{n!} f^{(n+1)}(c)\,(x-c)^n(x-\alpha) \end{aligned}$$

(b) By the generalized mean value theorem (see the notes entitled "The Mean Value Theorem") with $F(t) = S(t)$ and $G(t) = (x-t)^{n+1}$, there is a c strictly between α and x such that

$$\begin{aligned} S(\alpha) = S(\alpha) - S(x) &= \tfrac{S'(c)}{G'(c)}\big(G(\alpha) - G(x)\big) \\ &= -\tfrac{1}{n!} f^{(n+1)}(c)\,(x-c)^n \tfrac{1}{-(n+1)(x-c)^n}\ (x-\alpha)^{n+1} \\ &= \tfrac{1}{(n+1)!} f^{(n+1)}(c)(x-\alpha)^{n+1} \end{aligned}$$

Don't forget, when computing $G'(c)$, that G is a function of t with x just a fixed parameter.

(d) By the generalized mean value theorem with $G(t) = (x-t)^r$, there is a c strictly between α and x such that

$$\begin{aligned} S(\alpha) = S(\alpha) - S(x) &= \tfrac{S'(c)}{G'(c)}\big(G(\alpha) - G(x)\big) \\ &= -\tfrac{1}{n!} f^{(n+1)}(c)\,(x-c)^n \tfrac{1}{-r(x-c)^{r-1}}\ (x-\alpha)^r \\ &= \tfrac{1}{rn!} f^{(n+1)}(c)(x-c)^{n-r+1}(x-\alpha)^r \end{aligned}$$

(e) We'll only consider the case that $x \neq \alpha$. For $x = \alpha$, the error $R_n(\alpha, x)$ is obviously zero and we'll just take it as a convention that $\frac{1}{n!} Q^{(n)}(\alpha)(x-\alpha)^{n+1} = 0$ even if $Q^{(n)}(\alpha)$ is not defined. Since f is n times differentiable, so is $Q(t)$, at least for all $t \neq x$. In particular $Q(t)$ is n times differentiable at $t = \alpha$. From the definition of Q we have that

$$\begin{aligned} f(t) &= f(x) + (t-x)\,Q(t) & \Longrightarrow \quad f(\alpha) &= f(x) - (x-\alpha)\,Q(\alpha) \\ f'(t) &= Q(t) + (t-x)\,Q'(t) & \Longrightarrow \quad f'(\alpha) &= Q(\alpha) - (x-\alpha)\,Q'(\alpha) \\ f^{(2)}(t) &= 2Q'(t) + (t-x)\,Q^{(2)}(t) & \Longrightarrow \quad f^{(2)}(\alpha) &= 2Q'(\alpha) - (x-\alpha)\,Q^{(2)}(\alpha) \\ &\vdots & &\vdots \\ f^{(k)}(t) &= kQ^{(k-1)}(t) + (t-x)\,Q^{(k)}(t) & \Longrightarrow \quad f^{(k)}(\alpha) &= kQ^{(k-1)}(\alpha) - (x-\alpha)\,Q^{(k)}(\alpha) \end{aligned}$$

for $k \le n$. So

$$\begin{aligned} f(x) - P(x) &= f(x) - f(\alpha) - \sum_{k=1}^{n} \tfrac{1}{k!} f^{(k)}(\alpha)\,(x-\alpha)^k \\ &= (x-\alpha)Q(\alpha) - \sum_{k=1}^{n} \left\{ \tfrac{1}{(k-1)!} Q^{(k-1)}(\alpha)\,(x-\alpha)^k - \tfrac{1}{k!} Q^{(k)}(\alpha)\,(x-\alpha)^{k+1} \right\} \end{aligned}$$

The sum telescopes leaving

$$f(x) - P(x) = \tfrac{1}{n!} Q^{(n)}(\alpha)\,(x-\alpha)^{n+1}$$

as desired.

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Remark 2 It is rarely necessary, or even possible, to evaluate $R_n(\alpha, x)$ exactly. It is usually sufficient to find a number M such that $\left|f^{(n+1)}(c)\right| \le M$ for all c between the expansion point α and the x of interest. Both parts (a) and (b) of Taylor's Theorem then imply that

$$|R_n(\alpha, x)| \le \tfrac{1}{(n+1)!} M |x-\alpha|^{n+1}$$

Example 3 (Sine and Cosine Series) The trigonometric functions $\sin x$ and $\cos x$ have widely used Taylor expansions about $\alpha = 0$. Every derivative of $\sin x$ and $\cos x$ is one of $\pm \sin x$ and $\pm \cos x$. Consequently, when we apply Theorem 1.b we always have $\left|f^{(n+1)}(c)\right| \le 1$ and hence $|R_n(\alpha, x)| \le \frac{|x|^{n+1}}{(n+1)!}$. This converges to zero as $n \to \infty$. Consequently, for both $f(x) = \sin x$ and $f(x) = \cos x$, we have $\lim\limits_{n\to\infty} R_n(\alpha = 0, x) = 0$ and

$$f(x) = \lim_{n\to\infty} \left[f(0) + f'(0)\,x + \cdots + \tfrac{1}{n!} f^{(n)}(0)\,x^n \right]$$

First, compute all derivatives at general x.

$$\begin{array}{llllll} f(x) = \sin x & f'(x) = \cos x & f''(x) = -\sin x & f^{(3)}(x) = -\cos x & f^{(4)}(x) = \sin x & \cdots \\ f(x) = \cos x & f'(x) = -\sin x & f''(x) = -\cos x & f^{(3)}(x) = \sin x & f^{(4)}(x) = \cos x & \cdots \end{array}$$

The pattern starts over again with the fourth derivative being the same as the original function. Now set $x = \alpha = 0$.

$$\begin{array}{llllll} f(x) = \sin x & f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f^{(3)}(0) = -1 & f^{(4)}(0) = 0 & \cdots \\ f(x) = \cos x & f(0) = 1 & f'(0) = 0 & f''(0) = -1 & f^{(3)}(0) = 0 & f^{(4)}(0) = 1 & \cdots \end{array}$$

For $\sin x$, all even numbered derivatives are zero. The odd numbered derivatives alternate between 1 and -1. For $\cos x$, all odd numbered derivatives are zero. The even numbered derivatives alternate between 1 and -1. We conclude that

$$\begin{aligned} \sin x &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots = \sum_{n=0}^{\infty}(-1)^n \tfrac{1}{(2n+1)!}x^{2n+1} \\ \cos x &= 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots = \sum_{n=0}^{\infty}(-1)^n \tfrac{1}{(2n)!}x^{2n} \end{aligned} \tag{1}$$

If we wish to evaluate $\sin x$ or $\cos x$ for some specific value of x, we may always use trig identities like $\sin x = \sin(x \pm 2\pi) = -\sin(x \pm \pi) = \cos\left(\frac{\pi}{2} - x\right)$ to reduce consideration to $0 \le x \le \frac{\pi}{4} < 1$. Then the alternating series test tells us that the error introduced by truncating the series in (1) is between 0 and the first term dropped. That is, if $0 \le x \le 1$,

$$\begin{aligned} &\sin x - \left[x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots + \tfrac{(-1)^{n-1}}{(2n-1)!}x^{2n-1}\right] \quad \text{is between 0 and } (-1)^n \tfrac{x^{2n+1}}{(2n+1)!} \\ &\cos x - \left[1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots + \tfrac{(-1)^{n-1}}{(2n-2)!}x^{2n-2}\right] \quad \text{is between 0 and } (-1)^n \tfrac{1}{(2n)!}x^{2n} \end{aligned}$$

This error decreases spectacularly quickly as n increases. For example

$$\begin{array}{llll} \frac{1}{5!} \approx 0.0083 & \frac{1}{6!} \approx 0.0014 & \frac{1}{7!} \approx 0.0002 & \frac{1}{8!} \approx 0.000025 \\ \frac{1}{9!} \approx 0.000003 & \frac{1}{10!} \approx 0.00000028 & \frac{1}{11!} \approx 0.000000025 & \frac{1}{12!} \approx 0.000000002 \end{array}$$

Example 4 (Exponential Series) A similar phenomenon happens with the exponential function $f(x) = e^x$. By Theorem 1.b, for all natural numbers n and all real numbers x,

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \cdots + \tfrac{1}{n!}x^n + \tfrac{e^c}{(n+1)!}x^{n+1}$$

for some c strictly between 0 and x. Now consider any fixed real number x. As c runs from 0 to x, e^c runs from $e^0 = 1$ to e^x. In particular, e^c is always between 1 and e^x and so is smaller than $1 + e^x$. Thus the error term

$$|R_n(0,x)| = \left|\frac{e^c}{(n+1)!}x^{n+1}\right| \le [e^x + 1]\frac{|x|^{n+1}}{(n+1)!}$$

Let's call $e_n(x) = \frac{|x|^{n+1}}{(n+1)!}$. I claim that as n increases towards infinity, $e_n(x)$ decreases (quickly) towards zero. To see this, let's compare $e_n(x)$ and $e_{n+1}(x)$.

$$\frac{e_{n+1}(x)}{e_n(x)} = \frac{\frac{|x|^{n+2}}{(n+2)!}}{\frac{|x|^{n+1}}{(n+1)!}} = \frac{|x|}{n+2}$$

So, when n is bigger than, for example $2|x|$, we have $\frac{e_{n+1}(x)}{e_n(x)} < \frac{1}{2}$. That is, increasing the index on $e_n(x)$ by one decreases the size of $e_n(x)$ by a factor of at least two. As a result $e_n(x)$ must tend to zero as $n \to \infty$. Consequently $\lim\limits_{n\to\infty} R_n(0,x) = 0$ and

$$e^x = \lim_{n\to\infty} \left[1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \cdots + \tfrac{1}{n!}x^n\right] = \sum_{n=0}^{\infty} \tfrac{1}{n!}x^n$$