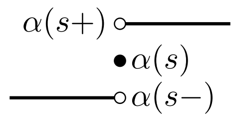
Riemann–Stieltjes Integrals with α a Step Function

Theorem. *Let $\alpha : [a,b] \to \mathbb{R}$ be a step function with discontinuities at $s_1 < \cdots < s_n$, where $a \le s_1$ and $s_n \le b$. Let $f : [a,b] \to \mathbb{R}$ be continuous at each s_j, $1 \le j \le n$. Then $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and*

$$\int_a^b f\,d\alpha = \sum_{j=1}^{n} f(s_j)\,\big[\alpha(s_j+) - \alpha(s_j-)\big]$$

where

$$\alpha(s_+) = \lim_{\substack{t\to s\\ t>s}} \alpha(t) \qquad \alpha(s_-) = \lim_{\substack{t\to s\\ t<s}} \alpha(t)$$



and, by convention,

$$\textit{if } s_1 = a \quad \alpha(s_1-) = \alpha(a)$$
$$\textit{if } s_n = b \quad \alpha(s_n+) = \alpha(b)$$

Proof: Let $\varepsilon > 0$. Choose the partition $P_\varepsilon = \big\{a = \tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_m = b\big\}$ so that

(1) $\{s_1, \cdots, s_n\} \subset P_\varepsilon$

(2) the norm or mesh of $P_\varepsilon = \|P_\varepsilon\| = \max_{1\le i\le m} \big|\tilde{x}_i - \tilde{x}_{i-1}\big| < \delta$ with $\delta = \min\big\{|s_2 - s_1|, \cdots, |s_n - s_{n-1}|, \delta_0\big\}$ and δ_0 is given by

Insert $(*)$ **here**.

Now let $P = \{a = x_0, x_1, \cdots, x_p = b\} \supset P_\varepsilon$ be any partitition finer than P_ε and $T = \{t_1, \cdots, t_p\}$ be any choice for P and consider each term in

$$S(P,T,f,\alpha) = \sum_{i=1}^{p} f(t_i)\big[\alpha(x_i) - \alpha(x_{i-1})\big]$$

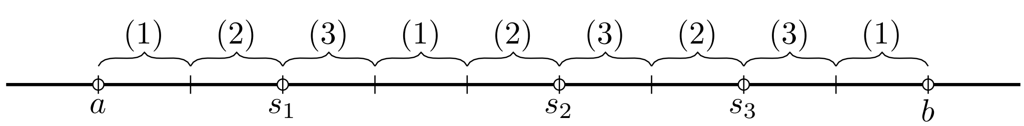
For each $1 \le i \le p$, either

(1) neither x_i nor x_{i-1} is in $\{s_1, \cdots, s_n\}$, in which case both x_i and x_{i-1} lie in a subinterval of $[a,b]$ (either $[a, s_1)$, or (s_{j-1}, s_j) for some $2 \le j \le n$, or $(s_n, b]$) on which α is required to be constant. In this case $\alpha(x_i) - \alpha(x_{i-1}) = 0$.

or (2) there is a $1 \le j \le n$ with $x_i = s_j$. In this case $\alpha(x_i) - \alpha(x_{i-1}) = \alpha(s_j) - \alpha(s_j-)$.

or (3) there is a $1 \le j \le n$ with $x_{i-1} = s_j$. In this case $\alpha(x_i) - \alpha(x_{i-1}) = \alpha(s_j+) - \alpha(s_j)$.

These three possibilities are illustrated below, with the points of P indicated by hash marks.



So

$$S(P,T,f,\alpha) = \qquad \big\{\text{case (2) terms}\big\} \quad + \quad \big\{\text{case (3) terms}\big\}$$

$$= \sum_{j=1}^{n} \Big\{ f(t_{i_j})\big[\alpha(s_j) - \alpha(s_j-)\big] + f(t_{i'_j})\big[\alpha(s_j+) - \alpha(s_j)\big] \Big\} \tag{1}$$

Here t_{i_j} lies in the subinterval of P whose right hand end point is s_j and $t_{i'_j}$ lies in the subinterval of P whose left hand end point is s_j. Because $\|P\| < \delta$, we have $s_j - \delta < t_{i_j} \le s_j$ and $s_j \le t_{i'_j} < s_j + \delta$. We may write the value of the integral given in the statement of the theorem in a form quite like that of $S(P,T,f,\alpha)$:

$$\sum_{j=1}^{n} f(s_j)\big[\alpha(s_j+) - \alpha(s_j-)\big] = \sum_{j=1}^{n} \Big\{ f(s_j)\big[\alpha(s_j) - \alpha(s_j-)\big] + f(s_j)\big[\alpha(s_j+) - \alpha(s_j)\big] \Big\} \tag{2}$$

(The two $f(s_j)\alpha(s_j)$ terms cancel.) Subtracting (2) from (1) and using the triangle inequality gives

$$\Big| S(P,T,f,\alpha) - \sum_{j=1}^{n} f(s_j)\big[\alpha(s_j+) - \alpha(s_j-)\big] \Big|$$

$$\le \sum_{j=1}^{n} \Big\{ \big|f(t_{i_j}) - f(s_j)\big| \, \big|\alpha(s_j) - \alpha(s_j-)\big| + \big|f(t_{i'_j}) - f(s_j)\big| \, \big|\alpha(s_j+) - \alpha(s_j)\big| \Big\}$$

Now

($*$) for each $1 \le j \le n$, f is continuous at s_j so that there is a $\delta_j > 0$ such that

$$\big|f(t) - f(s_j)\big| < \frac{\varepsilon}{\sum_{k=1}^{n} \big\{ |\alpha(s_k) - \alpha(s_k-)| + |\alpha(s_k+) - \alpha(s_k)| \big\}}$$

for all t with $|t - s_j| < \delta_j$. Choose $\delta_0 = \min\{\delta_1, \cdots \delta_j\}$.

Consequently

$$\Big| S(P,T,f,\alpha) - \sum_{j=1}^{n} f(s_j)\big[\alpha(s_j+) - \alpha(s_j-)\big] \Big| < \varepsilon$$

as desired.

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