Riemann–Stieltjes Integrals with $\alpha$ a Step Function

**Definition.** A step function on the interval $[a, b]$ is a function $\alpha : [a, b] \to \mathbb{R}$ that

1. has finitely many points of discontinuity
   
   \[ a \leq s_1 < s_2 < \cdots < s_n \leq b \]

2. and is constant on the subintervals $[a, s_1), (s_1, s_2), (s_2, s_3), \ldots, (s_{n-1}, s_n)$ and $(s_n, b]$.

**Theorem.** Let $\alpha : [a, b] \to \mathbb{R}$ be a step function with discontinuities at $s_1 < \cdots < s_n$, where $a \leq s_1$ and $s_n \leq b$. Let $f : [a, b] \to \mathbb{R}$ be continuous at each $s_j$, $1 \leq j \leq n$. Then $f \in R(\alpha)$ on $[a, b]$ and

\[ \int_a^b f \, d\alpha = \sum_{j=1}^n f(s_j) \left[ \alpha(s_j^+) - \alpha(s_j^-) \right] \]

where

\[ \alpha(s_+) = \lim_{t \to s^+} \alpha(t) \quad \alpha(s_-) = \lim_{t \to s^-} \alpha(t) \]

and, by convention,

- if $s_1 = a$ then $\alpha(s_1^-) = \alpha(a)$
- if $s_n = b$ then $\alpha(s_n^+) = \alpha(b)$

**Proof:** Let $\varepsilon > 0$. Choose the partition $P_\varepsilon = \{ a = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_m = b \}$ so that

1. $\{s_1, \ldots, s_n\} \subseteq P_\varepsilon$
2. the norm or mesh of $P_\varepsilon = \|P_\varepsilon\| = \max_{1 \leq i \leq m} |\bar{x}_i - \bar{x}_{i-1}| < \delta$ with
   \[ \delta = \min \{|s_2 - s_1|, \cdots, |s_n - s_{n-1}|, \delta_0\} \]

   and $\delta_0$ is given by

   Insert (*), given below, here.

Now let $P = \{a = x_0, x_1, \ldots, x_p = b\} \supset P_\varepsilon$ be any partition finer than $P_\varepsilon$ and $T = \{t_1, \cdots, t_p\}$ be any choice for $P$ and consider each term in

\[ S(P, T, f, \alpha) = \sum_{i=1}^p f(t_i) \left[ \alpha(x_i) - \alpha(x_{i-1}) \right] \]
For each $1 \leq i \leq p$, either

1. neither $x_i$ nor $x_{i-1}$ is in $\{s_1, \ldots, s_n\}$, in which case both $x_i$ and $x_{i-1}$ lie in a subinterval of $[a, b]$ (either $[a, s_1)$, or $(s_{j-1}, s_j)$ for some $2 \leq j \leq n$, or $(s_n, b]$) on which $\alpha$ is required to be constant. In this case $\alpha(x_i) - \alpha(x_{i-1}) = 0$.

or (2) there is a $1 \leq j \leq n$ with $x_i = s_j$. In this case $\alpha(x_i) - \alpha(x_{i-1}) = \alpha(s_j) - \alpha(s_{j-1})$.

or (3) there is a $1 \leq j \leq n$ with $x_{i-1} = s_j$. In this case $\alpha(x_i) - \alpha(x_{i-1}) = \alpha(s_j) - \alpha(s_j)$.

These three possibilities are illustrated below, with the points of $P$ indicated by hash marks.

So

$$S(P, T, f, \alpha) = \left\{\text{case (2) terms}\right\} + \left\{\text{case (3) terms}\right\}$$

$$= \sum_{j=1}^{n} \left\{f(t_{i_j}) [\alpha(s_j) - \alpha(s_{j-1})] + f(t_{i_j})' [\alpha(s_j) - \alpha(s_{j-1})] + f(t_{i_j}) [\alpha(s_j) + \alpha(s_j)]\right\}$$

(1)

Here $t_{i_j}$ lies in the subinterval of $P$ whose right hand end point is $s_j$ and $t_{i_j}'$ lies in the subinterval of $P$ whose left hand end point is $s_j$. Because $\|P\| < \delta$, we have $s_j - \delta < t_{i_j} \leq s_j$ and $s_j \leq t_{i_j}' < s_j + \delta$. We may write the value of the integral given in the statement of the theorem in a form quite like that of $S(P, T, f, \alpha)$:

$$\sum_{j=1}^{n} f(s_j) [\alpha(s_j) - \alpha(s_{j-1})] = \sum_{j=1}^{n} \left\{f(s_j) [\alpha(s_j) - \alpha(s_{j-1})] + f(s_j) [\alpha(s_j) + \alpha(s_j)]\right\}$$

(2)

(The two $f(s_j)\alpha(s_j)$ terms cancel.) Subtracting (2) from (1) and using the triangle inequality gives

$$\left|S(P, T, f, \alpha) - \sum_{j=1}^{n} f(s_j) [\alpha(s_j) + \alpha(s_j)]\right|$$

$$\leq \sum_{j=1}^{n} \left\{|f(t_{i_j}) - f(s_j)| |\alpha(s_j) - \alpha(s_{j-1})| + |f(t_{i_j}) - f(s_j)| |\alpha(s_j) + \alpha(s_j)|\right\}$$

Now

(*) for each $1 \leq j \leq n$, $f$ is continuous at $s_j$ so that there is a $\delta_j > 0$ such that

$$|f(t) - f(s_j)| < \frac{\varepsilon}{\sum_{k=1}^{n} |\alpha(s_k) - \alpha(s_k)| + |\alpha(s_k) + \alpha(s_k)|$$

for all $t$ with $|t - s_j| < \delta_j$. Choose $\delta_0 = \min\{\delta_1, \ldots, \delta_j\}$.

Consequently

$$\left|S(P, T, f, \alpha) - \sum_{j=1}^{n} f(s_j) [\alpha(s_j) + \alpha(s_j)]\right| < \varepsilon$$

as desired.
Remark.

1. We can weaken the hypotheses of the above theorem to requiring that
   - either $f$ or $\alpha$ is continuous from the left at $s_j$ and
   - either $f$ or $\alpha$ is continuous from the right at $s_j$.
for each $1 \leq j \leq n$. On the other hand, if both $f$ and $\alpha$ are discontinuous from the same side of some $s_j$, then $f \notin \mathcal{R}(\alpha)$. See Problem Set 2, #1.

2. The above theorem provides one rigorous formulation of the Dirac Delta Function. See Problem Set 2, #6.