

# Standard Functions

It is a simple matter to define, and to prove many standard properties of, standard functions like  $e^x$ ,  $\sin x$ ,  $\cos x$  and  $\ln x$  by using what we have learned this year and, in particular, the following facts.

- ▷ If the radius of convergence of a power series  $\sum_{n=0}^{\infty} a_n z^n$  is  $R$  and  $r < R$ , then the series converges absolutely and uniformly on  $|z| \leq r$ .
- ▷ The radius of convergence of a differentiated power series  $\sum_{n=0}^{\infty} n a_n z^{n-1}$  is the same as that of the original series  $\sum_{n=0}^{\infty} a_n z^n$ .

## The Exponential Function

Define

$$E(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{1}$$

(So far we have just defined a function. We do not yet know that it is the number  $e$  raised to the power  $x$ .) This power series has infinite radius of convergence. We know (see Rudin, Theorem 3.50) that, given any two absolutely convergent series  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$ , the product

$$\left( \sum_{n=0}^{\infty} a_n \right) \left( \sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} c_n \quad \text{where } c_n = \sum_{m=0}^n a_m b_{n-m}$$

Consequently

$$\begin{aligned} E(x)E(y) &= \left( \sum_{n=0}^{\infty} \frac{1}{n!} x^n \right) \left( \sum_{n=0}^{\infty} \frac{1}{n!} y^n \right) = \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \frac{1}{m!} x^m \frac{1}{(n-m)!} y^{n-m} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{m=0}^n \binom{n}{m} x^m y^{n-m} \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n = E(x+y) \end{aligned} \tag{2}$$

by the binomial expansion. And, for any constant  $\alpha$ ,

$$\frac{d}{dx} E(\alpha x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha x)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d}{dx} (\alpha x)^n = \alpha \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (\alpha x)^{n-1} \stackrel{m=n-1}{=} \alpha \sum_{m=0}^{\infty} \frac{1}{m!} (\alpha x)^m = \alpha E(\alpha x) \tag{3}$$

because we are allowed to differentiate termwise.

Define the number  $e = E(1)$ . We'll now check that  $E(x) = e^x$ . Repeated application of (2) gives

$$e^n = \overbrace{E(1) \times \cdots \times E(1)}^{n \text{ factors}} = E(n)$$

for any  $n \in \mathbb{N}$ . For any natural numbers  $n$  and  $m$ , repeated application of (2) gives

$$E\left(\frac{m}{n}\right)^n = \overbrace{E\left(\frac{m}{n}\right) \times \cdots \times E\left(\frac{m}{n}\right)}^{n \text{ factors}} = E\left(n \frac{m}{n}\right) = E(m) = e^m$$

Each positive real number has a unique positive  $n^{\text{th}}$  root and  $E\left(\frac{m}{n}\right) > 0$  (for  $x > 0$ , every term in the infinite sum (1) is strictly positive) so that  $E\left(\frac{m}{n}\right) = (e^m)^{\frac{1}{n}} = e^{\frac{m}{n}}$ . At this stage, we know that  $E(x) = e^x$  for any  $x$  that is rational and strictly positive. Obviously  $E(0) = 1 = e^0$ . And, by (2),  $E(x)E(-x) = E(0) = 1$ , so that  $E(-x) = \frac{1}{E(x)}$  for any  $x \in \mathbb{R}$ . So  $E(x) = e^x$  for every  $x \in \mathbb{Q}$ . Since both  $E(x)$  and  $e^x$  are continuous functions of  $x$  and  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , this gives that  $E(x) = e^x$  for all  $x \in \mathbb{R}$ .

## The Logarithm

Since  $e > 0$ , we have  $\frac{d}{dx}e^x = e^x > 0$ . Thus the function  $x \mapsto e^x$  is strictly increasing and so has an inverse function, which we denote  $\ln x$ . As, for all  $x > 0$ ,  $e^x = 1 + x + \frac{1}{2!}x^2 + \dots > x$  we have  $\lim_{x \rightarrow \infty} e^x = \infty$  and  $\lim_{x \rightarrow -\infty} e^x = \lim_{y \rightarrow \infty} e^{-y} = \lim_{y \rightarrow \infty} \frac{1}{e^y} = 0$ . Thus the range of  $e^x$ , and hence the domain of  $\ln x$ , is  $(0, \infty)$  so that

$$e^{\ln x} = x \quad \text{for all } x > 0 \quad \text{and} \quad \ln(e^x) = x \quad \text{for all } x \in \mathbb{R} \quad (4)$$

Differentiating both sides of the left hand formula in (4) gives, by the chain rule,

$$e^{\ln x} \frac{d}{dx} \ln x = 1 \implies x \frac{d}{dx} \ln x = 1 \implies \frac{d}{dx} \ln x = \frac{1}{x} \quad (5)$$

Since  $e^0 = 1$ , the right hand formula in (4) gives  $\ln 1 = 0$ . So, by the fundamental theorem of calculus,

$$\ln x = \ln x - \ln 1 = \int_1^x \frac{d}{dt} \ln t \, dt = \int_1^x \frac{1}{t} \, dt \quad (6)$$

For any  $X, Y > 0$  we have, by (2) and (4), writing  $x = \ln X$  and  $y = \ln Y$ ,

$$\ln(XY) = \ln(e^x e^y) = \ln(e^{x+y}) = x + y = \ln X + \ln Y \quad (7)$$

We now show that  $\ln x^\alpha = \alpha \ln x$  for any  $x > 0$  and  $\alpha \in \mathbb{R}$ . Fix any  $x > 0$ . Repeated application of (7) gives  $\ln(x^n) = \overbrace{\ln x + \dots + \ln x}^{n \text{ terms}} = n \ln x$  for any  $n \in \mathbb{N}$ . So, for any natural numbers  $n$  and  $m$ ,  $m \ln(x^{\frac{n}{m}}) = \ln((x^{\frac{n}{m}})^m) = \ln(x^n) = n \ln x$ . This shows that  $\ln(x^\alpha) = \alpha \ln x$  for any  $\alpha$  that is rational and strictly positive. We already know that  $\ln 1 = 0$ . And, by (7),  $\ln x + \ln \frac{1}{x} = \ln 1 = 0$ , so that  $\ln \frac{1}{x} = -\ln x$ . So we now know that  $\ln(x^\alpha) = \alpha \ln x$  for any  $\alpha \in \mathbb{Q}$ . One often defines  $x^\alpha = \sup \{ x^\beta \mid \beta \in \mathbb{Q}, \beta \leq \alpha \}$  when  $x \geq 1$ , with an obvious variant in the case  $0 < x < 1$ . Then, both  $\alpha \ln x$  and  $\ln(x^\alpha)$  are continuous in  $\alpha$  (the continuity of  $x^\alpha$  follows from Rudin Theorem 3.20.b) so that  $\ln(x^\alpha) = \alpha \ln x$  for any  $\alpha \in \mathbb{R}$ . Alternatively, one could just define  $x^\alpha = e^{\alpha \ln x}$ , for irrational  $\alpha$ , again yielding  $\ln(x^\alpha) = \alpha \ln x$ .

## The Trig Functions Sine and Cosine

Define

$$\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad (8a)$$

$$\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \quad (8b)$$

where for any complex, nonreal  $z$ , we simply define  $e^z = E(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ . (Formulae (2) and (3) are still valid.) Note that  $(i\theta)^n = (-i\theta)^n$  for all even  $n$  and  $(i\theta)^n = -(-i\theta)^n$  for all odd  $n$ . So

$$\begin{aligned} \cos \theta &= \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{1}{n!} (i\theta)^n & \stackrel{n=2m}{=} & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \theta^{2m} \\ \sin \theta &= \frac{1}{i} \sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} \frac{1}{n!} (i\theta)^n & \stackrel{n=2m+1}{=} & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \theta^{2m+1} \end{aligned} \quad (9)$$

as in first year calculus. From either (8) or (9) we easily get

$$\frac{d}{d\theta} \sin \theta = \cos \theta \quad \frac{d}{d\theta} \cos \theta = -\sin \theta$$

Taking (8a) +  $i(8b)$  and (8a) -  $i(8b)$  gives the Euler formulae

$$e^{i\theta} = \cos \theta + i \sin \theta \quad e^{-i\theta} = \cos \theta - i \sin \theta$$

Taking the real and imaginary parts of

$$\begin{aligned} \cos(2\theta) + i \sin(2\theta) &= e^{2i\theta} = (e^{i\theta})^2 = (\cos \theta + i \sin \theta)^2 = \cos^2 \theta + 2i \sin \theta \cos \theta - \sin^2 \theta \\ \cos(3\theta) + i \sin(3\theta) &= e^{3i\theta} = (e^{i\theta})^3 = (\cos \theta + i \sin \theta)^3 = \cos^3 \theta + 3i \sin \theta \cos^2 \theta - 3 \sin^2 \theta \cos \theta - i \sin^3 \theta \end{aligned}$$

and so on, gives the trig identities

$$\begin{aligned} \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta & \sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(3\theta) &= \cos^3 \theta - 3 \sin^2 \theta \cos \theta & \sin(3\theta) &= 3 \sin^2 \theta \cos \theta - \sin^3 \theta \end{aligned}$$

and so on. Similarly

$$\cos^2 \theta + \sin^2 \theta = (\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta) = e^{i\theta} e^{-i\theta} = 1$$

and

$$\begin{aligned} \cos \theta \cos \phi &= \frac{1}{4}(e^{i\theta} + e^{-i\theta})(e^{i\phi} + e^{-i\phi}) \\ &= \frac{1}{4}(e^{i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)} + e^{-i(\theta+\phi)}) \\ &= \frac{1}{4}(e^{i(\theta+\phi)} + e^{-i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)}) \\ &= \frac{1}{2}(\cos(\theta + \phi) + \cos(\theta - \phi)) \end{aligned}$$

and, using  $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ ,

$$\begin{aligned} \sin^3 \theta &= -\frac{1}{8i}(e^{i\theta} - e^{-i\theta})^3 \\ &= -\frac{1}{8i}(e^{i3\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-i3\theta}) \\ &= \frac{3}{4} \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) - \frac{1}{4} \frac{1}{2i}(e^{i3\theta} - e^{-i3\theta}) \\ &= \frac{3}{4} \sin \theta - \frac{1}{4} \sin(3\theta) \end{aligned}$$

A disadvantage of the definitions (8) is that the periodicity of  $\sin \theta$ ,  $\cos \theta$  and  $e^{i\theta}$  do not follow trivially from the definitions. We now derive that periodicity. We start by showing that  $\cos \theta$  has exactly one zero for  $\theta \in [0, 2]$ . Of course  $\cos 0 = 1$ . Now

$$\cos 2 = 1 - \frac{1}{2!}2^2 + \frac{1}{4!}2^4 + \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n)!}2^{2n} = 1 - 2 + \frac{2}{3} + \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n)!}2^{2n} = -\frac{1}{3} + \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n)!}2^{2n} < -\frac{1}{3}$$

because  $\sum_{n=3}^{\infty} \frac{(-1)^n}{(2n)!}2^{2n}$  is an alternating series with each successive term smaller in magnitude than its predecessor (because  $\frac{1}{(2n+2)!}2^{2n+2} = \frac{1}{(2n)!}2^{2n} \frac{4}{(2n+2)(2n+1)} < \frac{1}{(2n)!}2^{2n}$  for any  $n \geq 1$ ) so that the sum of the series has the same sign as its first term, which is negative. (If you didn't understand the last argument, review the alternating series test in the Math 320 notes <http://www.math.ubc.ca/~feldman/m320/convergence.pdf>.) Since  $\cos 0 > 0$  and  $\cos 2 < 0$ ,  $\cos \theta$  must have a zero in  $(0, 2)$ , by the intermediate value theorem. If  $\cos \theta$  had two or more zeroes in  $(0, 2)$ , its derivative,  $-\sin \theta$ , would have to have a zero in  $(0, 2)$  by the mean value theorem. But this is not the case because for any  $0 < \theta < 2$ ,

$$\sin \theta = \theta - \frac{1}{3!}\theta^3 + \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1} > \theta - \frac{1}{3!}\theta^3 = \theta(1 - \frac{1}{6}\theta^2) > 0$$

Again we have used that  $\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!} \theta^{2n+1}$  is an alternating series with successively smaller terms (for  $0 < \theta < 2$ ) so that the sum has the same sign as the first term, which is positive.

So we now know that  $\cos \theta$  has exactly one zero in  $(0, 2)$ . We denote by  $\frac{\pi}{2}$  the corresponding value of  $\theta$ . Since  $1 = \cos^2 \frac{\pi}{2} + \sin^2 \frac{\pi}{2} = \sin^2 \frac{\pi}{2}$  and  $\sin \frac{\pi}{2} > 0$ , we must have  $\sin \frac{\pi}{2} = 1$ . Consequently

$$e^{\frac{\pi}{2}i} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i \implies e^{\pi i} = (e^{\frac{\pi}{2}i})^2 = -1 \text{ and } e^{2\pi i} = (e^{\frac{\pi}{2}i})^4 = 1$$

and, for any  $\theta \in \mathbb{R}$ ,

$$e^{i(\theta+2\pi)} = e^{i\theta} e^{2\pi i} = e^{i\theta}$$

That is,  $e^{i\theta}$  and its real and imaginary parts,  $\cos \theta$  and  $\sin \theta$  all have period  $2\pi$ .

Finally, we shall show that  $2\pi$  is the smallest (positive) period for  $\cos \theta$ . The trig identities

$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin \theta \quad \sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta$$

follow immediately from

$$\cos\left(\theta + \frac{\pi}{2}\right) + i \sin\left(\theta + \frac{\pi}{2}\right) = e^{i(\theta+\frac{\pi}{2})} = e^{i\frac{\pi}{2}} e^{i\theta} = i(\cos \theta + i \sin \theta)$$

These identities allow us to easily determine the sign and monotonicity behaviour of  $\sin$  and  $\cos$  on  $[0, 2\pi]$ , from the corresponding behaviour on  $[0, \frac{\pi}{2}]$ , which we already know:

- As  $\theta$  increases from 0 to  $\frac{\pi}{2}$ ,  $\cos \theta$  decreases strictly monotonically (since  $\frac{d}{d\theta} \cos \theta = -\sin \theta < 0$  for all  $\theta \in (0, \frac{\pi}{2})$ ) from 1 to 0 and  $\sin \theta$  increases strictly monotonically from 0 to 1.
- Then, as  $\theta$  increases from  $\frac{\pi}{2}$  to  $\pi$ ,  $\cos \theta = -\sin(\theta - \frac{\pi}{2})$  decreases strictly monotonically from 0 to  $-1$  and  $\sin \theta = \cos(\theta - \frac{\pi}{2})$  decreases strictly monotonically from 1 to 0.
- Then, as  $\theta$  increases from  $\pi$  to  $\frac{3}{2}\pi$ ,  $\cos \theta = -\cos(\theta - \pi)$  increases strictly monotonically from  $-1$  to 0 and  $\sin \theta = -\sin(\theta - \pi)$  decreases strictly monotonically from 0 to  $-1$ .
- Then, as  $\theta$  increases from  $\frac{3}{2}\pi$  to  $2\pi$ ,  $\cos \theta = \sin(\theta - \frac{3}{2}\pi)$  increases strictly monotonically from 0 to 1 and  $\sin \theta = -\cos(\theta - \frac{3}{2}\pi)$  increases strictly monotonically from  $-1$  to 0.

Thus  $\cos \theta$  can never take the value 1 for  $\theta$  in  $(0, \frac{\pi}{2}]$  or in  $[\frac{\pi}{2}, \pi]$  or in  $[\pi, \frac{3\pi}{2}]$  or in  $[\frac{3\pi}{2}, 2\pi)$ . So  $\theta = 2\pi$  is the smallest strictly positive value of  $\theta$  for which  $\cos \theta = 1$  and  $2\pi$  is not only a period of  $\cos \theta$ , it is the primitive (i.e. smallest) period of  $\cos \theta$ .