MATH 321 Problem Set 6 Solutions

1. (Space filling curve) Define $\Phi(t) = (x(t), y(t)) : [0, 1] \rightarrow [0, 1] \times [0, 1]$ where $x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1} t)$, $y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n} t)$ and $f(t)$ is any continuous function obeying

(i) $0 \leq f(t) \leq 1$
(ii) $f(t+2) = f(t)$ for all $t \in \mathbb{R}$ (i.e. $f$ is periodic of period 2)
(iii) $f(t) = 0$ for all $0 \leq t \leq \frac{1}{3}$
(iv) $f(t) = 1$ for all $\frac{2}{3} \leq t \leq 1$

Prove that $\Phi$ is continuous and maps $[0, 1]$ onto $[0, 1] \times [0, 1]$.

Hint: Every $(x_0, y_0) \in [0, 1] \times [0, 1]$ has the form $x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}$, $y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$ with every $a_i \in \{0, 1\}$. These are just the binary expansions (i.e. base 2 analogues of the decimal expansions) of $x_0$ and $y_0$. Show that if $t_0 = \sum_{j=1}^{\infty} 3^{-j-1}(2a_j)$, then $f(3^k t_0) = a_k$ so that $\Phi(t_0) = (x_0, y_0)$.

Solution. Step 1 – $\Phi$ is continuous: Since $0 \leq f(t) \leq 1$ for all $t$, we have

$$|2^{-n} f(3^{2n-1} t)|, |2^{-n} f(3^{2n} t)| \leq 2^{-n}$$

which we denote $M_n$

Since $\sum_{n=1}^{\infty} M_n < \infty$, the Weierstrass $M$-test assures us that the series defining $x(t)$ and $y(t)$ converge uniformly. As uniform limits of continuous functions are continuous, $x(t)$ and $y(t)$ are continuous. Hence so is $\Phi(t)$.

Step 2 – $\Phi$ is onto: Let $(x_0, y_0) \in [0, 1] \times [0, 1]$. Write $(x_0, y_0)$ in the form $x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}$, $y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$ with every $a_i \in \{0, 1\}$. Define $t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)$. Then, for each $k \in \mathbb{N}$

$$3^k t_0 = 2 \sum_{i=1}^{k-1} 3^{k-i-1} a_i + \frac{2}{3} a_k + \sum_{i=k+1}^{\infty} \frac{2}{3^{i-1}} a_i$$

(with the sum $\sum_{i=1}^{k-1} 3^{k-i-1} a_i$ being zero when $k = 1$). Since $f$ has period 2

$$f(3^k t_0) = f\left(\frac{2}{3} a_k + \sum_{i=k+1}^{\infty} \frac{2}{3^{i-1}} a_i\right)$$

Since $0 \leq \sum_{i=k+1}^{\infty} \frac{2}{3^{i-1}} a_i \leq 2 \sum_{i=2}^{\infty} \frac{1}{3^i} = \frac{1}{3}$, the argument $\frac{2}{3} a_k + \sum_{i=k+1}^{\infty} \frac{2}{3^{i-1}} a_i$ takes a value in $[0, \frac{1}{3}]$ when $a_k = 0$ and takes a value in $[\frac{2}{3}, 1]$ when $a_k = 1$. As $f$ takes the value 0 on all of $[0, \frac{1}{3}]$ and takes the value 1 on all of $[\frac{2}{3}, 1]$,

$$f(3^k t_0) = a_k$$

Hence

$$x(t_0) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1} t_0) = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1} = x_0$$

$$y(t_0) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n} t_0) = \sum_{n=1}^{\infty} 2^{-n} a_{2n} = y_0$$

Hence $\Phi$ is onto.

Remark. Just for general interest, here is a proof that every $z \in [0, 1]$ has a binary expansion $z = \sum_{n=1}^{\infty} 2^{-n} b_n$ where each $b_n$ is either 0 or 1. Since $1 = \sum_{n=1}^{\infty} 2^{-n}$, we may assume that $z \in [0, 1)$. Define $b_n$ inductively by $b_1 = \lfloor 2z \rfloor$, where $\lfloor x \rfloor$ is the integer part of $x$, and, for $n \geq 2$,

$$b_n = \left\lfloor 2^n \left(z - \sum_{m=1}^{n-1} 2^{-m} b_m\right) \right\rfloor$$
I claim that $0 \leq z - \sum_{m=1}^{n-1} 2^{-m}b_m < 2^{-n+1}$. Before proving this, observe that it implies both that $b_n$ is either 0 or 1 and also that $\sum_{m=1}^{\infty} 2^{-m}b_m$ converges to $z$. We prove the claim by induction. Since $z \in [0,1)$, it is true for $n = 1$. Now assume that it is true for $n$. Then $0 \leq 2^n \{ z - \sum_{m=1}^{n-1} 2^{-m}b_m \} < 2$. Since $b_n$ is the integer part of $2^n \{ z - \sum_{m=1}^{n-1} 2^{-m}b_m \}$,

$$0 \leq 2^n \{ z - \sum_{m=1}^{n-1} 2^{-m}b_m \} - b_n < 1 \implies 0 \leq z - \sum_{m=1}^{n-1} 2^{-m}b_m - 2^{-n}b_n < 2^{-n}$$

which is the claim for $n + 1$.

2. Define $I : C([0,1]) \to C([0,1])$ by $I(f)(x) = \int_0^x f(t) \, dt$.

(a) Think of $C([0,1])$ as a metric space with metric $d(f,g) = \| f - g \|_\infty$. Prove that $I$ is uniformly continuous as a function from the metric space $C([0,1])$ to itself. That is, prove that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\| I(f) - I(g) \|_\infty < \varepsilon$ for all $f, g \in C([0,1])$ obeying $\| f - g \|_\infty < \delta$.

(b) Is $I$ one–to–one? Justify your answer.

(c) Is $I$ onto? Justify your answer.

**Solution.** (a) For any $f, g \in C([0,1])$,

$$d(I(f), I(g)) = \| I(f) - I(g) \|_\infty = \sup_{0 \leq x \leq 1} \left| \int_0^x [f(t) - g(t)] \, dt \right| \leq \sup_{0 \leq x \leq 1} \int_0^x |f(t) - g(t)| \, dt$$

$$\leq \sup_{0 \leq x \leq 1} \int_0^x |f(t) - g(t)| \, dt = \sup_{0 \leq x \leq 1} x \| f - g \|_\infty = \| f - g \|_\infty = d(f, g)$$

Thus, if $\varepsilon > 0$, then $d(I(f), I(g)) < \varepsilon$, for all $f, g \in C([0,1])$ with $d(f, g) < \delta \equiv \varepsilon$. So $I$ is uniformly continuous.

(b) Yes, $I$ is one–to–one. If $I(f) = I(g)$, then $I(f-g) = 0$. So it suffices to prove that if $I(h)$ is the zero function, then $h$ is the zero function. But if $\int_0^x h(x) \, dx$ is identically zero, then so is its derivative. By the fundamental theorem of calculus (and the continuity of $h$), that derivative is $h(x)$.

(c) No $I$ is not onto. Every function in the range of $I$ is of the form $g = I(f)$ for some $f \in C([0,1])$. By the fundamental theorem of calculus, the derivative of $g = I(f)$ is $f$, which is continuous. So every function in the range of $I$ is differentiable with continuous derivative. But not every $g \in C([0,1])$ is differentiable with continuous derivative. So $I$ cannot be onto. In particular $g(x) = |x - \frac{1}{2}|$ is in $C([0,1])$, but cannot be in the range of $I$.

3. Give an example of a subset, $F$, of $C([0,1])$ which is pointwise bounded but not bounded.

**Solution.** Let, for each $n \in \mathbb{N}$,

$$f_n(x) = \begin{cases} \frac{1}{x} & \text{if } \frac{1}{n} \leq x \leq 1 \\ n^2x & \text{if } 0 \leq x \leq \frac{1}{n} \end{cases}$$

Then $F = \{ f_n \mid n \in \mathbb{N} \} \subset C([0,1])$.

- First, fix any $0 < x \leq 1$. Choose any $m \in \mathbb{N}$ with $x \geq \frac{1}{m}$. If $n \geq m$, then, as $x \geq \frac{1}{m} \geq \frac{1}{n}$, we have $f_n(x) = \frac{1}{x} \leq m$. If $n \leq m$, then $f_n(x) \leq n \leq m$. (This is true for all $x$). Thus, for all $n \in \mathbb{N}$, we have $0 \leq f_n(x) \leq m$.

- For $x = 0$ we have $f_n(x) = 0$ for all $n \in \mathbb{N}$.

We have now shown that, for each fixed $0 \leq x \leq 1$, the set $\{ f(x) \mid f \in F \}$ is bounded. On the other hand $f_n\left(\frac{1}{n}\right) = n$ so that $\{ f_n(x) \mid 0 \leq x \leq 1, f \in F \}$ is not bounded.
4. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. Set, for each \( n \in \mathbb{N} \), \( f_n(t) = f(nt) \). Suppose that \( \{f_n\}_{n \in \mathbb{N}} \) is equicontinuous. What conclusion can you draw about \( f \)?

**Solution.** The function \( f \) is constant. To prove this, it suffices to prove that, for every \( s \in \mathbb{R} \) and every \( \varepsilon > 0 \), we have \( |f(s) - f(0)| < \varepsilon \). So fix any \( s \in \mathbb{R} \) and \( \varepsilon > 0 \). By hypothesis, there is a \( \delta > 0 \) such that \( |f(nt) - f(0)| = |f_n(t) - f_n(0)| < \varepsilon \) for all \( |t| < \delta \) and all \( n \in \mathbb{N} \). In particular, if \( t = \frac{s}{n} \) and \( n \) is large enough that \( |s| < n\delta \), we have \( |f(s) - f(0)| = |f(nt) - f(0)| < \varepsilon \).

5. Let \( \{f_n\}_{n \in \mathbb{N}} \) be an equicontinuous sequence of functions on a compact metric space \( K \). Prove that if \( \{f_n\}_{n \in \mathbb{N}} \) converges pointwise on \( K \), then \( \{f_n\}_{n \in \mathbb{N}} \) converges uniformly on \( K \).

**Solution.** Let \( \varepsilon > 0 \). Since \( \{f_n\}_{n \in \mathbb{N}} \) is equicontinuous there is a \( \delta > 0 \) such that \( |f_n(x) - f_n(y)| < \frac{\varepsilon}{p} \) for all \( n \in \mathbb{N} \) and all \( x, y \in K \) obeying \( d(x, y) < \delta \). Since \( K \) is compact, it is totally bounded, so that it is covered by a finite number of open balls of radius \( \delta \). Suppose that those balls are centred at \( x_1, \ldots, x_p \). For each \( 1 \leq j \leq p \), \( \{f_n(x_j)\}_{n \in \mathbb{N}} \) converges, so that there is an \( N_j \in \mathbb{N} \) such that \( |f_n(x_j) - f_n(x_j)| < \frac{\varepsilon}{p} \) for all \( n, m \geq N_j \). Hence if \( n, m \geq N = \max_{1 \leq j \leq p} N_j \) and \( x \in K \), then, choosing \( x_j \) such that \( |x - x_j| < \delta \),

\[
|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_j)| + |f_n(x_j) - f_m(x_j)| + |f_m(x_j) - f_m(x)|
\]

\[
< \frac{\varepsilon}{p} + \frac{\varepsilon}{p} + \frac{\varepsilon}{p} = \frac{3p}{p} \varepsilon
\]

Hence \( \|f_n - f_m\|_{\infty} \leq \frac{3p}{p} \varepsilon < \varepsilon \) for all \( n, m \geq N \). That is, the sequence \( \{f_n\}_{n \in \mathbb{N}} \) is Cauchy in the space \( \mathcal{C}(K) \) of continuous, bounded functions on \( K \) equipped with the supremum metric. (All continuous functions on the compact set \( K \) are automatically bounded.) Since \( \mathcal{C}(K) \) is complete, the sequence converges uniformly.

6. A family \( \mathcal{F} \) of functions \( f : K \to \mathbb{C} \), defined on the metric space \( K \), is said to be equicontinuous at \( x_0 \in K \) if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( d_K(x, x_0) < \delta \) implies \( |f(x) - f(x_0)| < \varepsilon \) for all \( f \in \mathcal{F} \). Prove that if \( K \) is compact and \( \mathcal{F} \) is equicontinuous at each \( x_0 \in K \), then \( \mathcal{F} \) is equicontinuous on \( K \) (that is, for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( |f(x) - f(y)| < \varepsilon \) for all \( f \in \mathcal{F} \) and all \( x, y \in K \) with \( d_K(x, y) < \delta \)).

**Solution.** Let \( \varepsilon > 0 \). By hypothesis, for each \( p \in K \) there is a \( \delta(p) \) such that \( |f(x) - f(p)| < \frac{\varepsilon}{2} \) for all \( x \in K \) with \( d_K(x, p) < \delta(p) \) and all \( f \in \mathcal{F} \). Since \( K \) is compact, the open cover \( \{B_{\delta(p)/2}(p)\}_{p \in K} \) has a finite open subcover, \( B_{\delta(p_1)/2}(p_1) \cup \cdots \cup B_{\delta(p_n)/2}(p_n) \). Set

\[
\delta = \frac{1}{2} \min \left\{ \delta(p_1), \ldots, \delta(p_n) \right\}
\]

I claim that this \( \delta \) does the job. Let \( f \in \mathcal{F} \) and \( x, y \in K \) with \( d_K(x, y) < \delta \). Choose \( 1 \leq j \leq n \) such that \( x \in B_{\delta(p_j)/2}(p_j) \). So \( d_K(x, p_j) < \frac{1}{2} \delta(p_j) \). Since \( d_K(x, y) < \delta \leq \frac{1}{2} \delta(p_j) \), we also have

\[
d_K(y, p_j) \leq d_K(y, x) + d_K(x, p_j) < \delta + \frac{1}{2} \delta(p_j) \leq \frac{3}{2} \delta(p_j) + \frac{1}{2} \delta(p_j) = \delta(p_j)
\]

Thus both \( x \) and \( y \) are in \( B_{\delta(p_j)}(p_j) \) and

\[
|f(x) - f(y)| \leq |f(x) - f(p_j)| + |f(p_j) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

as desired.
7. Let \( \phi : [0, 1] \times \mathbb{R} \to \mathbb{R} \) be bounded and continuous. For each \( n \in \mathbb{N} \), let \( F_n : [0, 1] \to \mathbb{R} \) satisfy

\[
F_n(0) = \frac{1}{n} \quad F_n'(t) = \phi(t, F_n(t)) \quad \text{for all} \quad t \in [0, 1]
\]

Prove that there is a subsequence \( \{ F_{n_k} \} \) that converges uniformly to a function \( F : [0, 1] \to \mathbb{R} \) obeying

\[
F(0) = 0 \quad F'(t) = \phi(t, F(t)) \quad \text{for all} \quad t \in [0, 1]
\]

**Solution.** Since

\[
\sup \left\{ |F_n(0)| \mid n \in \mathbb{N} \right\} = \frac{1}{n} = 1
\]

and

\[
\sup \left\{ |F_n'(t)| \mid n \in \mathbb{N}, \ t \in [0, 1] \right\} \leq \sup \left\{ |\phi(t, y)| \mid t \in [0, 1], \ y \in \mathbb{R} \right\} < \infty
\]

the sequence \( \{ F_n \} \) has a uniformly convergent subsequence \( \{ F_{n_k} \} \), by the corollary to the Arzelà–Ascoli theorem. Call the limit \( F \). Since the subsequence converges pointwise and \( \phi \) is continuous, we have

\[
F(0) = \lim_{k \to \infty} F_{n_k}(0) = \lim_{k \to \infty} \frac{1}{n_k} = 0 \quad \text{and} \quad \lim_{k \to \infty} \phi(t, F_{n_k}(t)) = \phi(t, F(t)).
\]

To complete the proof, it suffices to prove that \( F_{n_k}'(t) \) converges uniformly, because this will imply that \( F(t) \) is differentiable with

\[
F'(t) = \lim_{k \to \infty} F_{n_k}'(t) = \lim_{k \to \infty} \phi(t, F_{n_k}(t)) = \phi(t, F(t))
\]

**Proof that \( F_{n_k}'(t) \) converges uniformly:** Let \( \varepsilon > 0 \). Since \([0, 1]\) is compact and \( F_{n_k}(t) \) is continuous, it is bounded. Since \( \{ F_{n_k}(t) \} \) converges uniformly, there is an \( M > 0 \) such that \( |F_{n_k}(t)| \leq M \) for all \( k \in \mathbb{N} \) and all \( t \in [0, 1] \). Since \( \phi(t, y) \) is continuous and \([0, 1] \times [-M, M] \) is compact, \( \phi \) is uniformly continuous on \([0, 1] \times [-M, M] \). In particular, there is a \( \delta > 0 \) such that

\[
t \in [0, 1], \ y, y' \in [-M, M], \ |y - y'| < \delta \implies |\phi(t, y) - \phi(t, y')| < \varepsilon
\]

Since \( \{ F_{n_k}(t) \} \) converges uniformly, there is an \( N \in \mathbb{N} \) such that \( |F_{n_k}(t) - F_{n_\ell}(t)| < \delta \) for all \( k, \ell \geq N \). Hence

\[
k, \ell \geq N, \ t \in [0, 1] \implies |F_{n_k}(t) - F_{n_\ell}(t)| < \delta
\]

\[
\implies |\phi(t, F_{n_k}(t)) - \phi(t, F_{n_\ell}(t))| < \varepsilon
\]

\[
\implies |F_{n_k}'(t) - F_{n_\ell}'(t)| < \varepsilon
\]

So \( \{ F_{n_k}'(t) \} \) satisfies the Cauchy criterion uniformly and hence converges uniformly.