1. Let $R$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and let $0 \leq r < R$. Prove that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$.

**Solution.** Let $\varepsilon > 0$. Pick any $r'$ such that $r < r' < R$. Since $R$ is the radius of convergence, $R = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$ and there exists an $N \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} \leq \frac{r}{r'}$, or equivalently $|a_n| \leq r'^{-n}$, for all $n \geq N$. Hence there is a constant $C$ such that $|a_n| \leq C r'^{-n}$ for all $n \geq 0$. So if $|x| \leq r$,

$$\sum_{n=0}^{\infty} a_n x^n \leq \sum_{n=m+1}^{\infty} C r'^{-n} x^n = C r^{-m+1} \frac{1}{1-r'^{-1}} \leq \varepsilon$$

(see Problem Set 4, #4) which is some fixed number, depending on $\varepsilon$, but not on $x$.

2. Give examples of each of the following. Do not use the examples of chapter 7 of Rudin. Make your examples as simple as possible (for example, step functions). You will be penalized for unnecessary complexity. Sketch graphs of most of the functions involved.

(a) $f_n \to f$ pointwise, but not uniformly or in the mean
(b) $f_n \to f$ in the mean, but not pointwise or uniformly
(c) $f_n \to f$ uniformly, but not in the mean
(d) $f_n \to f$ pointwise, all of the $f_n$’s continuous, but $f$ not continuous
(e) $f_n \to f$ pointwise, all of the $f_n$’s bounded in magnitude by 1 and integrable on $[-1, 1]$, but $f$ not integrable on $[-1, 1]$
(f) $f_n \to f$ pointwise, all of the $f_n$’s and $f$ integrable, but $\int_a^b f_n(x) \, dx \to \int_a^b f(x) \, dx$
(g) $f_n \to f$ uniformly, all of the $f_n$’s and $f$ integrable, but $\int_{-\infty}^{\infty} f_n(x) \, dx \to \int_{-\infty}^{\infty} f(x) \, dx$
(h) $f_n \to f$ uniformly, all of the $f_n$’s differentiable, $f_n' \to g$ pointwise, but $f$ not differentiable
(i) $f_n \to f$ uniformly, all of the $f_n$’s and $f$ differentiable, $f_n' \to g$ pointwise, but $f' \neq g$

**Solution.** (a) We will use functions on the interval $[0, 1]$. Set $f(x) = 0$ and

$$f_n(x) = \begin{cases} n & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$

$$\|f_n - f\|_{\infty} = n \to 0 \text{ as } n \to \infty$$

$$\|f_n - f\|_2 = \left[ \int_{\frac{1}{n}}^{\frac{2}{n}} n^2 \, dx \right]^{1/2} = \sqrt{n} \to 0 \text{ as } n \to \infty$$

We have pointwise convergence since $f_n(0) = 0$ for all $n$ and $f_n(x) = 0$ for all $n > \frac{2}{x}$.

(b) We will use functions on the interval $[-1, 1]$. Set $f(x) = 0$ and

$$f_n(x) = \begin{cases} n^{1/4} & \text{if } -\frac{1}{2n} \leq x \leq \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

$$\|f_n - f\|_{\infty} = n^{1/4} \to 0 \text{ as } n \to \infty$$

$$\|f_n - f\|_2 = \left[ \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n^{1/2} \, dx \right]^{1/2} = \frac{1}{n^{1/4}} \to 0 \text{ as } n \to \infty$$

We do not have pointwise convergence since $f_n(0) = n^{1/4}$ diverges as $n \to \infty$.

(c) We will use functions $f_n, f : \mathbb{R} \to \mathbb{R}$. Set $f(x) = 0$ and

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{n}} & \text{if } -\frac{1}{\sqrt{n}} \leq x \leq \frac{1}{\sqrt{n}} \\ 0 & \text{otherwise} \end{cases}$$

$$\|f_n - f\|_{\infty} = \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty$$

$$\|f_n - f\|_2 = \left[ \int_{-\frac{1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} \frac{1}{\sqrt{n}} \, dx \right]^{1/2} = 1 \to 0 \text{ as } n \to \infty$$
(d) We will use functions \( f_n, f : \mathbb{R} \to \mathbb{R} \). Set
\[
f(x) = \begin{cases} 
1 & \text{if } x \leq 0 \\
0 & \text{if } x > 0
\end{cases}
\]
\[
f_n(x) = \begin{cases} 
1 & \text{if } x \leq 0 \\
1 - nx & \text{if } 0 \leq x \leq \frac{1}{n} \\
0 & \text{if } x \geq \frac{1}{n}
\end{cases}
\]
Then \( f \) is not continuous, but the \( f_n \)'s are continuous and converge pointwise to \( f \).

(e) We will use functions \( f, f_n : [-1, 1] \to [-1, 1] \). Then
\[
f_n(x) = \begin{cases} 
1 & \text{if } x = \frac{p}{q} \text{ for some } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ with } q \leq n \\
0 & \text{otherwise}
\end{cases}
\]
Note that \( f_n(x) = 0 \) except for finitely many \( x \)'s and so is integrable. We showed in class that \( f \) is not integrable. As another example, we can take the \( f, f_n \) of part (d) with
\[
\alpha(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
0 & \text{if } x < 0
\end{cases}
\]
Then every \( f_n \in \mathcal{R}(\alpha) \) on \([-1, 1]\) but \( f \notin \mathcal{R}(\alpha) \) on \([-1, 1]\).

(f) Take the \( f, f_n \) of part (a). Then
\[
\int_0^1 f_n(x) \, dx = 1 \to 0 = \int_0^1 f(x) \, dx
\]

(g) We use functions \( f_n, f : \mathbb{R} \to \mathbb{R} \). Set \( f(x) = 0 \) and
\[
f_n(x) = \begin{cases} 
\frac{1}{n} & \text{if } -\frac{n}{2} \leq x \leq \frac{n}{2} \\
0 & \text{otherwise}
\end{cases}
\]
\[
\|f - f_n\|_{\infty} = \frac{1}{n} \to 0 \text{ as } n \to \infty
\]
\[
\int_{-\infty}^{\infty} f_n(x) \, dx = 1 \to 0 = \int_{-\infty}^{\infty} f(x) \, dx
\]

(h) We use functions \( f_n, f, g : \mathbb{R} \to \mathbb{R} \). Set
\[
f_n(x) = \sqrt{x^2 + \frac{1}{n}}
\]
\[
f(x) = |x|
\]
\[
f'_n(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}
\]
\[
g(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0
\end{cases}
\]
Note that \( f_n \) converges uniformly to \( f \) (since, for all \( x > 0 \), \( \frac{d}{dx} [f_n(x) - f(x)] = f'_n(x) - 1 < 0 \), we have that \( \|f_n - f\|_{\infty} \leq f_n(0) - f(0) = \frac{1}{n} \)) and that \( f'_n \) converges pointwise to \( g \) but that \( f \) is not differentiable at \( x = 0 \).
(i) We use functions $f_n, f, g : \mathbb{R} \to \mathbb{R}$.

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq -\frac{1}{n} \\ \frac{x}{n}^2 + x + \frac{1}{n} & \text{if } -\frac{1}{n} \leq x \leq 0 \\ -\frac{x}{n}^2 + x + \frac{1}{n} & \text{if } 0 \leq x \leq \frac{1}{n} \\ \frac{1}{n} & \text{if } x \geq \frac{1}{n} \end{cases}$$

$$f(x) = 0$$

$$f'_n(x) = \begin{cases} 0 & \text{if } x \leq -\frac{1}{n} \\ n x + 1 & \text{if } -\frac{1}{n} \leq x \leq 0 \\ -n x + 1 & \text{if } 0 \leq x \leq \frac{1}{n} \\ 0 & \text{if } x \geq \frac{1}{n} \end{cases}$$

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Note that $f_n$ converges uniformly to 0 (since $\|f_n\|_\infty = \frac{1}{n}$) and that $f'_n$ converges pointwise to $g$ which is not equal to $f' = 0$. Two other possible choices are

- $f_n(x) = \frac{x^{n+1}}{n+1}$, $f'_n(x) = x^n$, $f(x) = 0$, $g(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } 0 \leq x < 1 \end{cases}$, with $0 \leq x \leq 1$

- $f_n(x) = \frac{1}{n} e^{-nx}$, $f'_n(x) = e^{-nx}$, $f(x) = 0$, $g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \leq 1 \end{cases}$, with $0 \leq x \leq 1$

3. Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+x^n}$.

(a) Find all values of $x$ in $\mathbb{R}$ for which the series is defined and converges – denote this set $D$.

(b) Is $f$ continuous on $D$? Prove or disprove.

(c) Is $f$ bounded on $D$? Prove or disprove.

(d) Does the above series converge uniformly on $D$? Prove or disprove.

**Solution.** (a) If $x = 0$, the series reduces to $\sum_{n=1}^{\infty} 1$, which clearly diverges. If $x = -\frac{1}{m}$ for some $m \in \mathbb{N}$, then the $n = m$ term in the series is not defined. Set $D' = \{0\} \cup \{ -\frac{1}{m} \mid m \in \mathbb{N} \}$ and $D = \mathbb{R} \setminus D'$. I will now prove that if $x \in D$, then the series converges at $x$. In fact, I will prove that, for every $X > 0$, the series converges uniformly on $\{ x \in \mathbb{R} \mid |x| \geq X \} \cap D$. So let $X > 0$. Then there is an $N \in \mathbb{N}$ such that, for all $n \geq N$ and $|x| \geq X$, $n^2|x| \geq N^2X \geq 2$, so that

$$|1 + n^2x| \geq n^2|x| - 1 = \frac{n^2}{2}|x| + \frac{n^2}{2}|x| - 1 \geq \frac{1}{2}n^2X + \frac{1}{2} \times 2 - 1 = \frac{1}{2}n^2X$$

and hence

$$|\frac{1}{1+x^n}| \leq \frac{2}{X} n^2$$

Since $\sum_{n=N}^{\infty} \frac{1}{n^2}$ converges, the original series converges uniformly on $\{ x \in \mathbb{R} \mid |x| \geq X \} \cap D$, by the Weierstrass $M$–test. This confirms that $D = \mathbb{R} \setminus D'$.

(b) Yes, $f$ is continuous on $D$. Let $X > 0$. On $\{ x \in \mathbb{R} \mid |x| \geq X \} \cap D$, $f(x)$ is a uniform limit of continuous partial sums and hence is continuous. Since this is the case for all $X > 0$ and $0 \notin D$, $f$ is continuous on $D$.

(c) No, $f$ is not bounded. Let $N \in \mathbb{N}$ and consider $x = \frac{1}{N}$. Then, for $n \leq N$, $1 + n^2 x \leq 2$. Consequently,

$$f\left(\frac{1}{N}\right) = \sum_{n=0}^{\infty} \frac{1}{1+n^2/N} \geq \sum_{n=0}^{N} \frac{1}{1+n^2/N} \geq \sum_{n=0}^{N} \frac{1}{2} = \frac{1}{2}(N+1)$$

3
As this is the case for all \( N \in \mathbb{N} \), \( f \) cannot be bounded.

(d) No, \( f \) does not converge uniformly on \( D \). If the series converged uniformly, \( f \) would be bounded on \((0, \infty)\), because each partial sum is bounded on \((0, \infty)\). This would contradict the result of part (c).

4. Let \( f : [a, b] \rightarrow \mathbb{R} \) be nondecreasing and let \( f : [a, b] \times [c, d] \rightarrow \mathbb{R} \). Prove:

(a) If \( f \) is continuous, then \( g(y) = \int_a^b f(x, y) \, dx \) is continuous.

(b) If \( \frac{\partial f}{\partial y} \) is continuous, then \( g(y) = \int_a^b f(x, y) \, dx \) is differentiable with \( g'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y) \, dx \).

**Solution.** Both parts are trivial if \( \alpha(b) = \alpha(a) \), since then \( \alpha \) is constant and \( g \) is identically zero. So assume that \( \alpha(b) > \alpha(a) \).

(a) Since \( f \) is a continuous function defined on a compact set, it is uniformly continuous. That is, for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \(|f(u) - f(v)| < \frac{\varepsilon}{\alpha(b) - \alpha(a)}\) for all \( u, v \in [a, b] \times [c, d] \) obeying \(|u - v| < \delta\). Consequently, if \(|y - y'| < \delta\) (the same \( \delta \)), then

\[
|g(y) - g(y')| = \left| \int_a^b [f(x, y) - f(x, y')] \, dx \right| \leq \alpha(b) - \alpha(a) \sup_{x \in [a,b]} |f(x, y) - f(x, y')| \leq \frac{\varepsilon}{\alpha(b) - \alpha(a)} = \varepsilon
\]

(b) Since \( \frac{\partial f}{\partial y} \) is a continuous function defined on a compact set, it is uniformly continuous. That is, for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \(|\frac{\partial f}{\partial y}(u) - \frac{\partial f}{\partial y}(v)| < \frac{\varepsilon}{\alpha(b) - \alpha(a)}\) for all \( u, v \in [a, b] \times [c, d] \) obeying \(|u - v| < \delta\). Consequently, for each fixed \( x \in [a, b] \), if \(|y - y'| < \delta\) (the same \( \delta \)), then, by the Mean Value Theorem (the usual MVT in one dimension),

\[
\left| \frac{f(x, y') - f(x, y)}{y' - y} - \frac{\partial f}{\partial y}(x, y) \right| = \left| \frac{\partial f}{\partial y}(x, y'') - \frac{\partial f}{\partial y}(x, y) \right| \quad \text{for some } y'' \text{ between } y' \text{ and } y
\]

and

\[
\left| \frac{g(y') - g(y)}{y' - y} - \int_a^b \frac{\partial f}{\partial y}(x, y) \, dx \right| = \left| \int_a^b \left\{ \frac{f(x, y') - f(x, y)}{y' - y} - \frac{\partial f}{\partial y}(x, y) \right\} \, dx \right| \leq \frac{\varepsilon}{\alpha(b) - \alpha(a)} [\alpha(b) - \alpha(a)] = \varepsilon
\]

This verifies the definition that \( \lim_{y' \to y} \frac{g(y') - g(y)}{y' - y} \) exists and equals \( \int_a^b \frac{\partial f}{\partial y}(x, y) \, dx \).

5. Prove that if \( f : [a, b] \times [c, d] \rightarrow \mathbb{R} \) is continuous, then

\[
\int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy = \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx
\]

Hint: Calculate \( \frac{\partial}{\partial t} \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy \) and \( \frac{\partial}{\partial t} \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx \).

**Solution.** By the last question, \( \int_c^d f(x, y) \, dy \) is continuous as a function of \( x \) so that \( \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx \) exists for all \( t \in [a, b] \) and, by the fundamental theorem of calculus, is differentiable with respect to \( t \) with derivative \( \int_c^d f(t, y) \, dy \).

Also by the fundamental theorem of calculus, \( \frac{\partial}{\partial t} \int_a^b f(x, y) \, dx \) exists and equals \( f(t, y) \), which is continuous. Note that \( \int_a^b f(x, y) \, dx \) is a function of \( t \) and \( y - \) call it \( h(t, y) \). Hence, by part (b) of the last question, \( \int_c^d h(t, y) \, dy \) exists and is differentiable with respect to \( t \) with derivative \( \int_c^d \frac{\partial h}{\partial t}(t, y) \, dy = \int_c^d f(t, y) \, dy \).

Hence \( \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy \) and \( \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx \) have the same derivative with respect to \( t \) and are both zero when \( t = a \). So they are equal for all \( t \), including \( t = b \).
Do not hand in problem 6.

6. Find \( \{ a_{m,n} \mid m, n \in \mathbb{N} \} \) obeying

(a) \( \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} \) exists, but \( \lim_{m \to \infty} a_{m,n} \) does not exist for any \( m \in \mathbb{N} \).

(b) \( \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} \) exists, \( \lim_{n \to \infty} a_{m,n} \) exists for any \( n \in \mathbb{N} \), but \( \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} \) does not exist.

(c) \( \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} \) and \( \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} \) exist and are equal, but \( \lim_{n \to \infty} a_{n,n} \) does not exist.

(d) \( \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} \), \( \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} \) and \( \lim_{n \to \infty} a_{n,n} \) all exist, but are different.

Solution. (a) Let

\[
a_{m,n} = \begin{cases} 
1 & \text{if } m = n \\
0 & \text{if } m > n \\
0 & \text{if } m < n, \ n - m \text{ odd} \\
1 & \text{if } m < n, \ n - m \text{ even}
\end{cases}
\]

Pictorially

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For any fixed \( n \), \( a_{m,\neq} = 0 \) for all \( m > n \) (the slash through the \( n \) just means that I am thinking of it as being held fixed) and so converges to 0 as \( m \to \infty \). Hence \( \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = \lim_{n \to \infty} 0 = 0 \). On the other hand, for each fixed \( m \), the sequence \( a_{\neq,n} \) has two subsequential limits (namely 0 and 1) and hence diverges.

(b) Let

\[
a_{m,n} = \begin{cases} 
m & \text{if } m < n \\
0 & \text{if } m \geq n
\end{cases}
\]

Pictorially

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For any fixed \( n \), \( a_{m,\neq} = 0 \) for all \( m > n \) and so converges to 0 as \( m \to \infty \). Hence \( \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = \lim_{n \to \infty} 0 = 0 \). For each fixed \( m \), \( a_{\neq,n} = m \) for all \( n > m \) and so converges to \( m \). But the sequence \( \{m\} \) diverges to \( +\infty \).

(c) Let

\[
a_{m,n} = \begin{cases} 
m & \text{if } m = n \\
0 & \text{if } m \neq n
\end{cases}
\]
Pictorially

\[
\begin{array}{c|ccccccc}
  n \rightarrow & 1 & 0 & 0 & 0 & \ldots & \rightarrow & 0 \\
\downarrow & 0 & 2 & 0 & 0 & \ldots & \rightarrow & 0 \\
\downarrow & 0 & 0 & 3 & 0 & \ldots & \rightarrow & 0 \\
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0 & 0 & 0 & 0 & \rightarrow & 0 \\
\end{array}
\]

For any fixed \( n \), \( a_{m,n} = 0 \) for all \( m > n \) and so converges to 0 as \( m \to \infty \). Hence \( \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = \lim_{m \to \infty} 0 = 0 \). Similarly, for each fixed \( m \), \( a_{n,m} = 0 \) for all \( n > m \) and so converges to 0. Hence \( \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} = \lim_{n \to \infty} 0 = 0 \). But the sequence \( d_m = a_{m,m} \) diverges to \( +\infty \).

(d) Let

\[
a_{m,n} = \begin{cases} 
1 & \text{if } m = n \\
0 & \text{if } m > n \\
2 & \text{if } n > m
\end{cases}
\]

Pictorially

\[
\begin{array}{c|ccccccc}
  m \rightarrow & 1 & 2 & 2 & 2 & \ldots & \rightarrow & 2 \\
\downarrow & 0 & 1 & 2 & 2 & \ldots & \rightarrow & 2 \\
\downarrow & 0 & 0 & 1 & 2 & \ldots & \rightarrow & 2 \\
\vdots \vdots \vdots & \vdots \vdots \vdots & \vdots \vdots \vdots & \vdots \vdots \vdots & \vdots \vdots \vdots & \vdots \vdots \vdots & \vdots \vdots \vdots \\
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0 & 0 & 0 & 0 & \rightarrow & 0 \\
\end{array}
\]

For any fixed \( n \), \( a_{m,n} = 0 \) for all \( m > n \) and so converges to 0 as \( m \to \infty \). Hence \( \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = \lim_{n \to \infty} 0 = 0 \). Similarly, for each fixed \( m \), \( a_{n,m} = 2 \) for all \( n > m \) and so converges to 2. Hence \( \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} = \lim_{m \to \infty} 2 = 2 \). But the sequence \( d_m = a_{m,m} = 1 \) converges to 1.