1. Let $R$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and let $0 \leq r < R$. Prove that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$.

**Solution.** Let $\varepsilon > 0$. Pick any $r'$ such that $r < r' < R$. Since $R$ is the radius of convergence, $R^{-1} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$ and there exists an $N \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} \leq \frac{1}{r'}$, or equivalently $|a_n| \leq r'^{-n}$, for all $n \geq N$. Hence there is a constant $C$ such that $|a_n| \leq Cr'^{-n}$ for all $n \geq 0$. So if $|x| \leq r$,

$$\left| \sum_{n=0}^{m} a_n x^n - \sum_{n=0}^{\infty} a_n x^n \right| \leq \sum_{n=m+1}^{\infty} a_n x^n \leq \sum_{n=m+1}^{\infty} C r'^{-n} r^n = C \frac{(r')^{m+1}}{1 - r'} \leq \varepsilon$$

if $m \geq \frac{\ln[C(1 - \varepsilon)]}{\ln r'} - 1$ (see Problem Set 4, #4) which is some fixed number, depending on $\varepsilon$, but not on $x$.

2. Give examples of each of the following. Do not use the examples of chapter 7 of Rudin. Make your examples as simple as possible (for example, step functions). You will be penalized for unnecessary complexity. Sketch graphs of most of the functions involved.

(a) $f_n \to f$ pointwise, but not uniformly or in the mean
(b) $f_n \to f$ in the mean, but not pointwise or uniformly
(c) $f_n \to f$ uniformly, but not in the mean
(d) $f_n \to f$ pointwise, all of the $f_n$’s differentiable, but $f$ not continuous
(e) $f_n \to f$ pointwise, all of the $f_n$’s bounded in magnitude by 1 and integrable on $[-1, 1]$, but $f$ not integrable on $[-1, 1]$
(f) $f_n \to f$ pointwise, all of the $f_n$’s and $f$ integrable, but $\int_{a}^{b} f_n(x) \, dx \not\to \int_{a}^{b} f(x) \, dx$
(g) $f_n \to f$ uniformly, all of the $f_n$’s and $f$ integrable, but $\int_{-\infty}^{\infty} f_n(x) \, dx \not\to \int_{-\infty}^{\infty} f(x) \, dx$
(h) $f_n \to f$ uniformly, all of the $f_n$’s differentiable, $f_n' \to g$ pointwise, but $f$ not differentiable
(i) $f_n \to f$ uniformly, all of the $f_n$’s and $f$ differentiable, $f_n' \to g$ pointwise, but $f' \not\to g$

**Solution.** (a) We will use functions on the interval $[0, 1]$. Set $f(x) = 0$ and

$$f_n(x) = \begin{cases} n & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$

$$\|f_n - f\|_{\infty} = n \to 0 \text{ as } n \to \infty$$

$$\|f_n - f\|_{2} = \left[ \int_{\frac{1}{n}}^{\frac{2}{n}} n^2 \, dx \right]^{1/2} = \sqrt{n} \to 0 \text{ as } n \to \infty$$

We have pointwise convergence since $f_n(0) = 0$ for all $n$ and $f_n(x) = 0$ for all $n > \frac{2}{x}$.

(b) We will use functions on the interval $[-1, 1]$. Set $f(x) = 0$ and

$$f_n(x) = \begin{cases} n^{1/4} & \text{if } -\frac{1}{2n} \leq x \leq \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

$$\|f_n - f\|_{\infty} = n^{1/4} \to 0 \text{ as } n \to \infty$$

$$\|f_n - f\|_{2} = \left[ \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n^{1/2} \, dx \right]^{1/2} = \frac{1}{n^{1/4}} \to 0 \text{ as } n \to \infty$$

We do not have pointwise convergence since $f_n(0) = n^{1/4}$ diverges as $n \to \infty$.

(c) We will use functions $f_n, f : \mathbb{R} \to \mathbb{R}$. Set $f(x) = 0$ and

$$f_n(x) = \begin{cases} n^{1/n} & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

$$\|f_n - f\|_{\infty} = n^{1/n} \to 0 \text{ as } n \to \infty$$

$$\|f_n - f\|_{2} = \left[ \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{1}{n} \, dx \right]^{1/2} = 1 \to 0 \text{ as } n \to \infty$$
(d) We will use functions \( f_n, f : \mathbb{R} \to \mathbb{R} \). Set

\[
  f(x) = \begin{cases} 
    1 & \text{if } x \leq 0 \\
    0 & \text{if } x > 0
  \end{cases} =

\[
  f_n(x) = \begin{cases} 
    1 & \text{if } x \leq 0 \\
    1 - nx & \text{if } 0 \leq x \leq \frac{1}{n} \\
    0 & \text{if } x \geq \frac{1}{n}
  \end{cases} =

Then \( f \) is not continuous, but the \( f_n \)'s are continuous and converge pointwise to \( f \).

(e) We will use functions \( f, f_n : [-1, 1] \to [-1, 1] \). Then

\[
  f_n(x) = \begin{cases} 
    1 & \text{if } x = \frac{p}{q} \text{ for some } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ with } q \leq n \\
    0 & \text{otherwise}
  \end{cases} \overset{\text{ptwise}}{\longrightarrow} f(x) = \begin{cases} 
    1 & \text{if } x \in \mathbb{Q} \\
    0 & \text{if } x \notin \mathbb{Q}
  \end{cases}

Note that \( f_n(x) = 0 \) except for finitely many \( x \)'s and so is integrable. We showed in class that \( f \) is not integrable. As another example, we can take the \( f, f_n \) of part (d) with

\[
  \alpha(x) = \begin{cases} 
    1 & \text{if } x \geq 0 \\
    0 & \text{if } x < 0
  \end{cases}
\]

Then every \( f_n \in \mathcal{R}(\alpha) \) on \([-1, 1]\) but \( f \notin \mathcal{R}(\alpha) \) on \([-1, 1]\).

(f) Take the \( f, f_n \) of part (a). Then

\[
  \int_0^1 f_n(x) \, dx = 1 \to 0 = \int_0^1 f(x) \, dx
\]

(g) We use functions \( f_n, f : \mathbb{R} \to \mathbb{R} \). Set \( f(x) = 0 \) and

\[
  f_n(x) = \begin{cases} 
    \frac{1}{n} & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n} \\
    0 & \text{otherwise}
  \end{cases}
\]

\[
  \|f - f_n\|_\infty = \frac{1}{n} \to 0 \text{ as } n \to \infty
\]

\[
  \int_{-\infty}^{\infty} f_n(x) \, dx = 1 \to 0 = \int_{-\infty}^{\infty} f(x) \, dx
\]

(h) We use functions \( f_n, f, g : \mathbb{R} \to \mathbb{R} \). Set

\[
  f_n(x) = \sqrt{x^2 + \frac{1}{n}}
\]

\[
  f(x) = |x|
\]

\[
  f'_n(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}
\]

\[
  g(x) = \begin{cases} 
    1 & \text{if } x > 0 \\
    0 & \text{if } x = 0 \\
    -1 & \text{if } x < 0
  \end{cases}
\]

Note that \( f_n \) converges uniformly to \( f \) (since, for all \( x > 0 \), \( \frac{d}{dx} [f_n(x) - f(x)] = f'_n(x) - 1 < 0 \), we have that \( \|f_n - f\|_\infty \leq f_n(0) - f(0) = \frac{1}{n} \) and that \( f'_n \) converges pointwise to \( g \) but that \( f \) is not differentiable at \( x = 0 \).
(i) We use functions \( f_n, f, g : \mathbb{R} \to \mathbb{R} \).

\[
\begin{align*}
    f_n(x) &= \begin{cases} 
      0 & \text{if } x \leq -\frac{1}{n} \\
      \frac{n}{2} x^2 + x + \frac{1}{2n} & \text{if } -\frac{1}{n} \leq x \leq 0 \\
      -\frac{n}{2} x^2 + x + \frac{1}{2n} & \text{if } 0 \leq x \leq \frac{1}{n} \\
      \frac{1}{n} & \text{if } x \geq \frac{1}{n}
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    f(x) &= 0 \\
    f'(x) &= \begin{cases} 
      0 & \text{if } x \leq -\frac{1}{n} \\
      nx + 1 & \text{if } -\frac{1}{n} \leq x \leq 0 \\
      -nx + 1 & \text{if } 0 \leq x \leq \frac{1}{n} \\
      0 & \text{if } x \geq \frac{1}{n}
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    g(x) &= \begin{cases} 
      1 & \text{if } x = 0 \\
      0 & \text{if } x \neq 0
    \end{cases}
\end{align*}
\]

Note that \( f_n \) converges uniformly to 0 (since \( \|f_n\|_{\infty} = \frac{1}{n} \)) and that \( f'_n \) converges pointwise to \( g \) which is not equal to \( f' = 0 \).

3. Consider \( f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x} \).

(a) Find all values of \( x \) in \( \mathbb{R} \) for which the series is defined and converges – denote this set \( D \).

(b) Is \( f \) continuous on \( D \)? Prove or disprove.

(c) Is \( f \) bounded on \( D \)? Prove or disprove.

(d) Does the above series converge uniformly on \( D \)? Prove or disprove.

**Solution.** (a) If \( x = 0 \), the series reduces to \( \sum_{n=1}^{\infty} 1 \), which clearly diverges. If \( x = -\frac{1}{m} \) for some \( m \in \mathbb{N} \), then the \( n = m \) term in the series is not defined. Set \( D' = \{ 0 \} \cup \{ -\frac{1}{m^2} \mid m \in \mathbb{N} \} \) and \( D = \mathbb{R} \setminus D' \). I will now prove that if \( x \in D \), then the series converges at \( x \). In fact, I will prove that, for every \( X > 0 \), the series converges uniformly on \( \{ x \in \mathbb{R} \mid |x| \geq X \} \cap D \). So let \( X > 0 \). Then there is an \( N \in \mathbb{N} \) such that for all \( n \geq N \) and \( |x| \geq X \), \( n^2|x| \geq N^2X \geq 2 \), so that

\[
|1 + n^2x| \geq n^2|x| - 1 = \frac{1}{2} n^2|x| + \frac{1}{2} n^2|x| - 1 \geq \frac{1}{2} n^2X + \frac{1}{2} \times 2 - 1 = \frac{1}{2} n^2X
\]

and hence

\[
\left| \frac{1}{1+n^2x} \right| \leq \frac{2}{X n^2}
\]

Since \( \sum_{n=N}^{\infty} \frac{1}{X n^2} \) converges, the original series converges uniformly on \( \{ x \in \mathbb{R} \mid |x| \geq X \} \cap D \), by the Weierstrass M–test. This confirms that \( D = \mathbb{R} \setminus D' \).

(b) Yes, \( f \) is continuous on \( D \). Let \( X > 0 \). On \( \{ x \in \mathbb{R} \mid |x| \geq X \} \cap D \), \( f(x) \) is a uniform limit of continuous partial sums and hence is continuous. Since this is the case for all \( X > 0 \) and \( 0 \notin D \), \( f \) is continuous on \( D \).

(c) No, \( f \) is not bounded. Let \( N \in \mathbb{N} \) and consider \( x = \frac{1}{\sqrt{N}} \). Then, for \( n \leq N \), \( 1 + n^2x \leq 2 \). Consequently,

\[
f\left( \frac{1}{\sqrt{N}} \right) = \sum_{n=0}^{\infty} \frac{1}{1+n^2/2} \geq \sum_{n=0}^{N} \frac{1}{1+n^2/2} \geq \sum_{n=0}^{N} \frac{1}{2} = \frac{1}{2} (N + 1)
\]

As this is the case for all \( N \in \mathbb{N} \), \( f \) cannot be bounded.
4. Let $\alpha : [a, b] \to \mathbb{R}$ be nondecreasing and let $f : [a, b] \times [c, d] \to \mathbb{R}$. Prove:
(a) If $f$ is continuous, then $g(y) = \int_a^b f(x, y) \, d\alpha(x)$ is continuous.
(b) If $\frac{\partial f}{\partial y}$ is continuous, then $g(y) = \int_a^b f(x, y) \, d\alpha(x)$ is differentiable with $g'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y) \, d\alpha(x)$.

Solution. Both parts are trivial if $\alpha(b) = \alpha(a)$, since then $\alpha$ is constant and $g$ is identically zero. So assume that $\alpha(b) > \alpha(a)$.

(a) Since $f$ is a continuous function defined on a compact set, it is uniformly continuous. That is, for each $\epsilon > 0$ there is a $\delta > 0$ such that $|f(\tilde{u}) - f(\tilde{v})| < \frac{\epsilon}{\alpha(b) - \alpha(a)}$ for all $\tilde{u}, \tilde{v} \in [a, b] \times [c, d]$ obeying $|\tilde{u} - \tilde{v}| < \delta$. Consequently, if $|y - y'| < \delta$ (the same $\delta$), then

$$|g(y) - g(y')| = \left| \int_a^b \left[ f(x, y) - f(x, y') \right] \, d\alpha(x) \right| \leq [\alpha(b) - \alpha(a)] \sup_{x \in [a, b]} |f(x, y) - f(x, y')|$$

$$\leq [\alpha(b) - \alpha(a)] \frac{\epsilon}{\alpha(b) - \alpha(a)} = \epsilon$$

(b) Since $\frac{\partial f}{\partial y}$ is a continuous function defined on a compact set, it is uniformly continuous. That is, for each $\epsilon > 0$ there is a $\delta > 0$ such that $\left| \frac{\partial f}{\partial y}(\tilde{u}) - \frac{\partial f}{\partial y}(\tilde{v}) \right| < \frac{\epsilon}{\alpha(b) - \alpha(a)}$ for all $\tilde{u}, \tilde{v} \in [a, b] \times [c, d]$ obeying $|\tilde{u} - \tilde{v}| < \delta$. Consequently, for each fixed $x \in [a, b]$, if $|y - y'| < \delta$ (the same $\delta$), then, by the Mean Value Theorem (the usual MVT in one dimension),

$$\left| \frac{f(x, y') - f(x, y)}{y' - y} \right| - \frac{\partial f}{\partial y}(x, y) \right| = \left| \frac{\partial f}{\partial y}(x, y'') - \frac{\partial f}{\partial y}(x, y) \right| \leq \frac{\epsilon}{\alpha(b) - \alpha(a)}$$

and

$$\left| \frac{g(y') - g(y)}{y' - y} \right| - \int_a^b \frac{\partial f}{\partial y}(x, y) \, d\alpha(x) \right| = \left| \int_a^b \left\{ \frac{f(x, y') - f(x, y)}{y' - y} - \frac{\partial f}{\partial y}(x, y) \right\} \, d\alpha(x) \right| \leq \frac{\epsilon}{\alpha(b) - \alpha(a)} [\alpha(b) - \alpha(a)] = \epsilon$$

This verifies the definition that $\lim_{y' \to y} \frac{g(y') - g(y)}{y' - y}$ exists and equals $\int_a^b \frac{\partial f}{\partial y}(x, y) \, d\alpha(x)$.

5. Prove that if $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous, then

$$\int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx = \int_c^d \left( \int_a^b f(x, y) \, dy \right) \, dx$$

Hint: Calculate $\frac{\partial}{\partial y} \int_c^d \left( \int_a^b f(x, y) \, dy \right) \, dx$ and $\frac{\partial}{\partial t} \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dy$.

Solution. By the last question, $\int_c^d f(x, y) \, dy$ is continuous as a function of $x$ so that $\int_c^d f(x, y) \, dy$ exists for all $t \in [a, b]$ and, by the fundamental theorem of calculus, is differentiable with respect to $t$ with derivative $\int_c^d f(t, y) \, dy$.

Also by the fundamental theorem of calculus, $\frac{\partial}{\partial y} \int_a^b f(x, y) \, dy$ exists and equals $f(t, y)$, which is continuous. Note that $\int_a^b f(x, y) \, dx$ is a function of $t$ and $y$ – call it $h(t, y)$. Hence, by part (b) of the last question, $h(t, y) \, dy$ exists and is differentiable with respect to $t$ with derivative $\int_c^d \frac{\partial h}{\partial t}(t, y) \, dy = \int_c^d f(t, y) \, dy$.

Hence $\int_c^d \left( \int_a^b f(x, y) \, dy \right) \, dy$ and $\int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx$ have the same derivative with respect to $t$ and are both zero when $t = a$. So they are equal for all $t$, including $t = b$. 

4
Do not hand in problem 6.

6. Find \( \{ a_{m,n} \mid m, n \in \mathbb{N} \} \) obeying
   
   \( \text{(a) } \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} \text{ exists, but } \lim_{m \to \infty} a_{m,n} \text{ does not exist for any } m \in \mathbb{N}. \)
   
   \( \text{(b) } \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} \text{ exists, } \lim_{n \to \infty} a_{m,n} \text{ exists for any } n \in \mathbb{N}, \text{ but } \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} \text{ does not exist.} \)
   
   \( \text{(c) } \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} \text{ and } \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} \text{ exist and are equal, but } \lim_{n \to \infty} a_{n,n} \text{ does not exist.} \)
   
   \( \text{(d) } \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n}, \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} \text{ and } \lim_{n \to \infty} a_{n,n} \text{ all exist, but are different.} \)

Solution. (a) Let

\[
a_{m,n} = \begin{cases} 
1 & \text{if } m = n \\
0 & \text{if } m > n \\
0 & \text{if } m < n, n - m \text{ odd} \\
1 & \text{if } m < n, n - m \text{ even}
\end{cases}
\]

Pictorially

\[
\begin{array}{c|ccccccc}
\text{n} & 1 & 0 & 1 & 0 & \cdots & \rightarrow \\
\downarrow & 0 & 1 & 0 & 1 & \cdots & \rightarrow \\
& 0 & 0 & 1 & 0 & \cdots & \rightarrow \\
& & 0 & 0 & 1 & \\
& & \vdots & \vdots & \vdots & \\
& & \downarrow & \downarrow & \downarrow & \\
\rightarrow & 0 & 0 & 0 & \rightarrow & 0 \\
\end{array}
\]

For any fixed \( n, a_{m,n} = 0 \) for all \( m > n \) (the slash through the \( n \) just means that I am thinking of it as being held fixed) and so converges to 0 as \( m \to \infty \). Hence \( \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = \lim_{n \to \infty} 0 = 0 \). On the other hand, for each fixed \( m \), the sequence \( a_{\Phi,n} \) has two subsequential limits (namely 0 and 1) and hence diverges.

(b) Let

\[
a_{m,n} = \begin{cases} 
m & \text{if } m < n \\
0 & \text{if } m \geq n
\end{cases}
\]

Pictorially

\[
\begin{array}{c|c|c|c|c|c|c}
\text{n} & 0 & 1 & 1 & 1 & \cdots & \rightarrow \\
\downarrow & 0 & 0 & 2 & 2 & \cdots & \rightarrow \\
& 0 & 0 & 0 & 3 & \cdots & \rightarrow \\
& & 0 & 0 & 0 & \\
& & \vdots & \vdots & \vdots & \vdots & \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \\
\rightarrow & 0 & 0 & 0 & \rightarrow & 0 & \#
\end{array}
\]

For any fixed \( n, a_{m,n} = 0 \) for all \( m > n \) and so converges to 0 as \( m \to \infty \). Hence \( \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = \lim_{n \to \infty} 0 = 0 \). For each fixed \( m, a_{\Phi,n} = m \) for all \( n > m \) and so converges to \( m \). But the sequence \( \{ m \} \) diverges to \( +\infty \).

(c) Let

\[
a_{m,n} = \begin{cases} 
m & \text{if } m = n \\
0 & \text{if } m \neq n
\end{cases}
\]
For any fixed $n$, $a_{m,n} = 0$ for all $m > n$ and so converges to 0 as $m \to \infty$. Hence \( \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = \lim_{n \to \infty} 0 = 0 \).
Similarly, for each fixed $m$, $a_{n,m} = 0$ for all $n > m$ and so converges to 0. Hence \( \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} = \lim_{m \to \infty} 0 = 0 \). But the sequence $d_m = a_{m,m}$ diverges to $+\infty$.

(d) Let

\[
    a_{m,n} = \begin{cases} 
        1 & \text{if } m = n \\
        0 & \text{if } m > n \\
        2 & \text{if } n > m 
    \end{cases}
\]

For any fixed $n$, $a_{m,n} = 0$ for all $m > n$ and so converges to 0 as $m \to \infty$. Hence \( \lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = \lim_{n \to \infty} 0 = 0 \).
Similarly, for each fixed $m$, $a_{n,m} = 2$ for all $n > m$ and so converges to 2. Hence \( \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n} = \lim_{m \to \infty} 2 = 2 \). But the sequence $d_m = a_{m,m} = 1$ converges to 1.