Self–Adjoint Matrices — Problem Solutions

Problem M.1 Let $V \subset \mathbb{C}^n$. Prove that $V^\perp$ is a linear subspace of $\mathbb{C}^n$.

Solution. If $\mathbf{v}, \mathbf{w} \in V^\perp$ and $\lambda \in \mathbb{C}$, then

$$\begin{aligned} \langle \mathbf{v}+\mathbf{w}, \mathbf{u}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{w}, \mathbf{u}\rangle = 0 \text{ for all } \mathbf{u} \in V \implies \mathbf{v}+\mathbf{w} \in V^\perp \\ \langle \lambda\mathbf{v}, \mathbf{u}\rangle = \lambda \langle \mathbf{v}, \mathbf{u}\rangle = 0 \text{ for all } \mathbf{u} \in V \implies \lambda\mathbf{v} \in V^\perp \end{aligned}$$

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Problem M.2 Let V be a linear subspace of $\mathbb{C}^n$. Prove that each $\mathbf{w} \in \mathbb{C}^n$ has a unique decomposition of the form $\mathbf{w} = \mathbf{v} + \mathbf{v}_\perp$ with $\mathbf{v} \in V$ and $\mathbf{v}_\perp \in V^\perp$.

Solution. Uniqueness: If $\mathbf{v}+\mathbf{v}_\perp = \mathbf{v}'+\mathbf{v}'_\perp$ with $\mathbf{v}, \mathbf{v}' \in V$ and $\mathbf{v}_\perp, \mathbf{v}'_\perp \in V^\perp$, then $\mathbf{v}-\mathbf{v}' = \mathbf{v}'_\perp - \mathbf{v}_\perp \in V \cap V^\perp$. In particular

$$\langle \mathbf{v}-\mathbf{v}', \mathbf{v}-\mathbf{v}'\rangle = \langle \mathbf{v}-\mathbf{v}', \mathbf{v}'_\perp - \mathbf{v}_\perp\rangle = 0 \implies \mathbf{v}-\mathbf{v}' = \mathbf{0} \implies \mathbf{v} = \mathbf{v}' \text{ and } \mathbf{v}'^\perp = \mathbf{v}^\perp$$

Existence: Fix any $\mathbf{w} \in \mathbb{C}^n$ and consider the function $f : V \to [0,\infty)$ defined by

$$f(\mathbf{v}') = \|\mathbf{v}' - \mathbf{w}\|^2$$

This is a continuous function of $\mathbf{v}'$ which tends to ∞ as $\|\mathbf{v}'\| \to \infty$. So f achieves its minimum value, say at some $\mathbf{v}' = \mathbf{v} \in V$. Set $\mathbf{v}_\perp = \mathbf{w} - \mathbf{v}$. Clearly $\mathbf{w} = \mathbf{v} + \mathbf{v}_\perp$ and $\mathbf{v} \in V$ so it remains only to prove that $\mathbf{v}^\perp \in V^\perp$.

If $\mathbf{v}^\perp \notin V^\perp$, then there is some $\mathbf{v}'' \in V$ with $\langle \mathbf{v}^\perp, \mathbf{v}''\rangle \neq 0$. We may assume, without loss of generality, that Re $\langle \mathbf{v}^\perp, \mathbf{v}''\rangle \neq 0$. (Otherwise, replace $\mathbf{v}''$ by $i\mathbf{v}''$.) Since $\mathbf{v}$ is a global minimum for f, we necessarily have

$$\begin{aligned} 0 &= \tfrac{d}{d\alpha} f\big(\mathbf{v}+\alpha\mathbf{v}''\big)\big|_{\alpha=0} = \tfrac{d}{d\alpha} \langle \mathbf{v}+\alpha\mathbf{v}''-\mathbf{w}, \mathbf{v}+\alpha\mathbf{v}''-\mathbf{w}\rangle\,\big|_{\alpha=0} \\ &= \langle \mathbf{v}'', \mathbf{v}-\mathbf{w}\rangle + \langle \mathbf{v}-\mathbf{w}, \mathbf{v}''\rangle = 2\text{Re}\,\langle \mathbf{v}^\perp, \mathbf{v}''\rangle \\ &\neq 0 \end{aligned}$$

which contradicts the assumption that $\mathbf{v}^\perp \notin V^\perp$. ■

Problem M.3 Let A and B be any $n \times n$ matrices. Prove that $B = A^*$ if and only if $\langle B\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, A\mathbf{w}\rangle$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$.

Solution. We have

$$\begin{aligned} \langle B\mathbf{v}, \mathbf{w}\rangle &= \sum_{i,j=1}^n B_{i,j} v_j \overline{w_i} = \sum_{i,j=1}^n \overline{w_i}\; B_{i,j}\; v_j \\ \langle \mathbf{v}, A\mathbf{w}\rangle &= \sum_{i,j=1}^n v_j \overline{A_{j,i} w_i} = \sum_{i,j=1}^n \overline{w_i}\; \overline{A_{j,i}}\; v_j \end{aligned}$$

Thus $\langle B\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, A\mathbf{w}\rangle$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ if and only if $B_{i,j} = \overline{A_{j,i}}$ for all $1 \leq i, j \leq n$. ■

Problem M.4 Let A be any $n \times n$ matrix. Let V be any linear subspace of $\mathbb{C}^n$ and $V^\perp$ its orthogonal complement. Prove that if $AV \subset V$ (i.e. $\mathbf{w} \in V \Rightarrow A\mathbf{w} \in V$), then $A^*V^\perp \subset V^\perp$.

Solution. Let $\mathbf{w} \in V^\perp$. We are to show that $A^*\mathbf{w} \in V^\perp$. To do so it suffices to prove that, for all $\mathbf{v} \in V$, $A^*\mathbf{w} \perp \mathbf{v}$. But

$$\langle \mathbf{v}, A^*\mathbf{w} \rangle = \langle A\mathbf{v}, \mathbf{w} \rangle = 0$$

since $A\mathbf{v} \in V$ and $\mathbf{w} \in V^\perp$. ■

Problem M.5 Let A be a normal matrix. Let λ be an eigenvalue of A and V the eigenspace of A of eigenvalue λ. Prove that V is the eigenspace of A^* of eigenvalue $\bar{\lambda}$.

Solution. Let $\mathbf{v} \in V$ and set $\mathbf{v}' = A^*\mathbf{v}$. As A is normal

$$A\mathbf{v}' = AA^*\mathbf{v} = A^*A\mathbf{v} = A^*(\lambda\mathbf{v}) = \lambda A^*\mathbf{v} = \lambda\mathbf{v}' \Rightarrow \mathbf{v}' \in V$$

Hence both A and A^* map V into V. By Problem M.2, any vector $\mathbf{w} \in \mathbb{C}^n$ has a unique decomposition $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ with $\mathbf{w}_1 \in V$ and $\mathbf{w}_2 \in V^\perp$. Hence

$$\begin{aligned}
\langle A^*\mathbf{v}, \mathbf{w} \rangle &= \langle A^*\mathbf{v}, \mathbf{w}_1 \rangle + \langle A^*\mathbf{v}, \mathbf{w}_2 \rangle = \langle A^*\mathbf{v}, \mathbf{w}_1 \rangle && \text{since } A^*\mathbf{v} \in V,\ \mathbf{w}_2 \in V^\perp \\
&= \langle \mathbf{v}, A\mathbf{w}_1 \rangle = \langle \mathbf{v}, \lambda\mathbf{w}_1 \rangle = \bar{\lambda} \langle \mathbf{v}, \mathbf{w}_1 \rangle && \text{since } \mathbf{w}_1 \in V \\
&= \langle \bar{\lambda}\mathbf{v}, \mathbf{w}_1 \rangle = \langle \bar{\lambda}\mathbf{v}, \mathbf{w}_1 \rangle + \langle \bar{\lambda}\mathbf{v}, \mathbf{w}_2 \rangle && \text{since } \bar{\lambda}\mathbf{v} \in V,\ \mathbf{w}_2 \in V^\perp \\
&= \langle \bar{\lambda}\mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

This is true for all $\mathbf{w} \in \mathbb{C}^n$ so $A^*\mathbf{v} = \bar{\lambda}\mathbf{v}$. This shows that the eigenspace of A^* for eigenvalue $\bar{\lambda}$ contains all of V.

Applying the above conclusion with A replaced by to $B = A^*$ and λ replaced by $\mu = \bar{\lambda}$ shows that the eigenspace of $B^* = A$ for eigenvalue $\bar{\mu} = \bar{\bar{\lambda}} = \lambda$, namely V, contains the eigenspace of $B = A^*$ for eigenvalue $\mu = \bar{\lambda}$. ■

Problem M.6 Let A be a normal matrix. Let $\mathbf{v}$ and $\mathbf{w}$ be eigenvectors of A with different eigenvalues. Prove that $\mathbf{v} \perp \mathbf{w}$.

Solution. Let $A\mathbf{v} = \lambda\mathbf{v}$ and $A\mathbf{w} = \mu\mathbf{w}$. By Problem M.5, $A^*\mathbf{v} = \bar{\lambda}\mathbf{v}$ and $A^*\mathbf{w} = \bar{\mu}\mathbf{w}$. Hence

$$\begin{aligned}
\langle A\mathbf{v}, \mathbf{w} \rangle &= \langle \lambda\mathbf{v}, \mathbf{w} \rangle = \lambda \langle \mathbf{v}, \mathbf{w} \rangle \\
= \langle \mathbf{v}, A^*\mathbf{w} \rangle &= \langle \mathbf{v}, \bar{\mu}\mathbf{w} \rangle = \mu \langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

So

$$\lambda \langle \mathbf{v}, \mathbf{w} \rangle = \mu \langle \mathbf{v}, \mathbf{w} \rangle \Rightarrow (\lambda - \mu) \langle \mathbf{v}, \mathbf{w} \rangle = 0 \Rightarrow \langle \mathbf{v}, \mathbf{w} \rangle = 0$$

since $\lambda - \mu \neq 0$. ■

Problem M.7 Let A be a self-adjoint matrix. Prove that

a) A is normal

b) Every eigenvalue of A is real.

Solution. (a) is obvious since AA^* and A^*A both equal A^2.
b) If $A\mathbf{v} = \lambda\mathbf{v}$, then, by Problem M.5, $A^*\mathbf{v} = \bar{\lambda}\mathbf{v}$. As $A = A^*$, this forces $\lambda\mathbf{v} = \bar{\lambda}\mathbf{v}$ and hence $\lambda = \bar{\lambda}$, since $\mathbf{v} \neq \mathbf{0}$. ■

Problem M.8 Let U be a unitary matrix. Prove that

a) U is normal

b) Every eigenvalue λ of U obeys $|\lambda| = 1$, i.e. is of modulus one.

Solution. a) Recall that, for any $m \times n$ matrix A, if A has a right inverse, R (that is $AR = \mathbb{1}_{m\times m}$) and a left inverse, L (that is $LA = \mathbb{1}_{n\times n}$), then $R = L$ and $n = m$. The proof that $R = L$ is

$$R = \mathbb{1}_{n\times n} R = LAR = L\mathbb{1}_{m\times m} = L$$

Also recall that if a square matrix has a left inverse then it also has a right inverse and vice versa. That's a consequence of the following facts.

- For any $m \times n$ matrix, A, the dimension of the range (i.e. the set of all vectors of the form $A\mathbf{v}$) and the dimension of the kernel (i.e. the set of all vectors $\mathbf{v}$ for which $A\mathbf{v} = \mathbf{0}$) add up to n. To see this, let $\mathbf{e}_1, \cdots \mathbf{e}_p$ be any basis for the kernel of A and extend $\mathbf{e}_1, \cdots \mathbf{e}_p$ to a basis $\mathbf{e}_1, \cdots \mathbf{e}_p$, $\mathbf{f}_1, \cdots, \mathbf{f}_q$ of $\mathbb{C}^n$. Then $p + q = n$, the dimension of the kernel is p and, since the range is the set of all vectors of the form $\alpha_1 A\mathbf{f}_1 + \cdots \alpha_q A\mathbf{f}_q$ and the vectors $A\mathbf{f}_1, \cdots, A\mathbf{f}_q$ are independent, the dimension of the range is q.
- An $m \times n$ matrix A has a left inverse if and only if the dimension of its kernel is 0. That is, if and only if A is one–to–one. If A has a left inverse L, the dimension of the kernel must be 0 because $\mathbf{v} \neq 0 \implies LA\mathbf{v} = \mathbf{v} \neq \mathbf{0} \implies A\mathbf{v} \neq \mathbf{0}$. Conversely, if the dimension of the kernel of A is 0 and $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is any basis for $\mathbb{C}^n$, then $\mathbf{f}_1 = A\mathbf{e}_1, \cdots, \mathbf{f}_n = A\mathbf{e}_n$ are independent and we may extend $\mathbf{f}_1, \cdots, \mathbf{f}_n$ to a basis, $\mathbf{f}_1, \cdots, \mathbf{f}_m$, for $\mathbb{C}^m$ and choose

$$L\Big(\sum_{j=1}^{m} \alpha_j \mathbf{f}_j\Big) = \sum_{j=1}^{n} \alpha_j \mathbf{e}_j$$

- An $m \times n$ matrix A has a right inverse if and only if the dimension of its range is m. That is, if and only if A is onto. If A has a right inverse, then any $\mathbf{v} \in \mathbb{C}^m$ is in the range of A because $\mathbf{v} = A(R\mathbf{v})$. Conversely, if A is onto and $\mathbf{f}_1, \cdots, \mathbf{f}_m$ is any basis for $\mathbb{C}^m$, then for each $1 \le j \le m$, there is some $\mathbf{e}_j \in \mathbb{C}^n$ with $A\mathbf{e}_j = \mathbf{f}_j$ and we may choose

$$R\Big(\sum_{j=1}^{m} \alpha_j \mathbf{f}_j\Big) = \sum_{j=1}^{m} \alpha_j \mathbf{e}_j$$

If U is unitary, then U is square and has U^* as its right inverse. Hence U also has a left inverse and that left inverse is again U^* so that

$$UU^* = \mathbb{1} = U^*U$$

b) If $U\mathbf{v} = \lambda\mathbf{v}$, then, by Problem M.5, $U^*\mathbf{v} = \bar{\lambda}\mathbf{v}$. Hence

$$\mathbf{v} = U^*U\mathbf{v} = U^*\big(\lambda\mathbf{v}\big) = \lambda U^*\mathbf{v} = \lambda\bar{\lambda}\mathbf{v}$$

As $\mathbf{v} \neq \mathbf{0}$, $\bar{\lambda}\lambda = 1$. ■