Reduction to the Riemann Integral

Theorem. *Let $a < b$. Let $f : [a, b] \to \mathbb{R}$ be bounded and $\alpha : [a, b] \to \mathbb{R}$ have a continuous derivative on $[a, b]$. Then*

$$f\alpha' \in \mathcal{R} \text{ on } [a,b] \iff f \in \mathcal{R}(\alpha) \text{ on } [a,b]$$

and, if either integral exists,

$$\int_a^b f(x)\alpha'(x)\ dx = \int_a^b f(x)\ d\alpha(x)$$

Corollary. *In the special case that the function f is the constant function $f(x) = 1$, this theorem reduces to*

$$\int_a^b \alpha'(x)\ dx = \int_a^b d\alpha(x) = \alpha(b) - \alpha(a)$$

which is half of the fundamental theorem of calculus.

Proof: For any partition $\mathbb{P} = \{a = x_0, \cdots, x_n = b\}$ of $[a, b]$ and any choice $\mathbb{T} = \{t_1, \cdots, t_n\}$ for $\mathbb{P}$

$$S(\mathbb{P}, \mathbb{T}, f\alpha', x) = \sum_{i=1}^n f(t_i)\alpha'(t_i)\big[x_i - x_{i-1}\big]$$

and, by the mean value theorem, there is, for each $1 \le i \le n$, some $v_i \in \big[x_{i-1}, x_i\big]$ such that

$$\begin{aligned} S(\mathbb{P}, \mathbb{T}, f, \alpha) &= \sum_{i=1}^n f(t_i)\big[\alpha(x_i) - \alpha(x_{i-1})\big] \\ &= \sum_{i=1}^n f(t_i)\,\alpha'(v_i)\big[x_i - x_{i-1}\big] \end{aligned}$$

So

$$\Big|S(\mathbb{P}, \mathbb{T}, f\alpha', x) - S(\mathbb{P}, \mathbb{T}, f, \alpha)\Big| \le \sum_{i=1}^n \big|f(t_i)\big|\ \big|\alpha'(t_i) - \alpha'(v_i)\big|\ \big[x_i - x_{i-1}\big]$$

Now let $\varepsilon > 0$.

- Since f is assumed to be bounded, there is an $M > 0$ such that $|f(t)| \le M$ for all $a \le t \le b$.
- Since α' is assumed to exist and be continuous on $[a, b]$, it is uniformly continuous on $[a, b]$. Hence, there is a $\delta > 0$ such that $\big|\alpha'(t) - \alpha'(v)\big| \le \frac{\varepsilon}{2M[b-a]}$ for all $t, v \in [a, b]$ with $|t - v| < \delta$.

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In particular, if $\|\mathbb{P}\| < \delta$, then $|t_i - v_i| < \delta$ and hence $\left|\alpha'(t_i) - \alpha'(v_i)\right| \le \frac{\varepsilon}{2M[b-a]}$ for all $1 \le i \le n$. So, if $\|\mathbb{P}\| < \delta$, we have

$$\begin{aligned}\Big|S(\mathbb{P},\mathbb{T},f\alpha',x) - S(\mathbb{P},\mathbb{T},f,\alpha)\Big| &\le \sum_{i=1}^{n} M\ \tfrac{\varepsilon}{2M[b-a]}\ \big[x_i - x_{i-1}\big] \\ &= \tfrac{\varepsilon}{2[b-a]} \sum_{i=1}^{n} \big[x_i - x_{i-1}\big] = \tfrac{\varepsilon}{2}\end{aligned}$$

In the event that $f \in \mathcal{R}(\alpha)$ on $[a,b]$, there is a partition $\mathbb{P}'_\varepsilon$ such that $\mathbb{P} \supset \mathbb{P}'_\varepsilon \implies \left|S(\mathbb{P},\mathbb{T},f,\alpha) - \int_a^b f d\alpha\right| < \frac{\varepsilon}{2}$ for all choices $\mathbb{T}$ for $\mathbb{P}$. Choose a partition $\mathbb{P}_\varepsilon$ for $[a,b]$ by adding sufficiently many points to $\mathbb{P}'_\varepsilon$ that $\|\mathbb{P}_\varepsilon\| < \delta$. Then, by the triangle inequality, for any partition $\mathbb{P} \supset \mathbb{P}_\varepsilon$ and any choice $\mathbb{T}$ for $\mathbb{P}$,

$$\begin{aligned}&\left|S(\mathbb{P},\mathbb{T},f\alpha',x) - \int_a^b f d\alpha\right| \\ &\qquad \le \Big|S(\mathbb{P},\mathbb{T},f\alpha',x) - S(\mathbb{P},\mathbb{T},f,\alpha)\Big| + \left|S(\mathbb{P},\mathbb{T},f,\alpha) - \int_a^b f d\alpha\right| \\ &\qquad < \varepsilon\end{aligned}$$

as desired. The argument in the case that we assume $f\alpha' \in \mathcal{R}$ on $[a,b]$ is identical.

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