Reduction to the Riemann Integral

**Theorem.** Let $a < b$. Let $f : [a, b] \to \mathbb{R}$ be bounded and $\alpha : [a, b] \to \mathbb{R}$ have a continuous derivative on $[a, b]$. Then

$$f\alpha' \in \mathcal{R} \text{ on } [a, b] \iff f \in \mathcal{R}(\alpha) \text{ on } [a, b]$$

and, if either integral exists,

$$\int_a^b f(x)\alpha'(x) \, dx = \int_a^b f(x) \, d\alpha(x)$$

**Corollary.** In the special case that the function $f$ is the constant function $f(x) = 1$, this theorem reduces to

$$\int_a^b \alpha'(x) \, dx = \int_a^b d\alpha(x) = \alpha(b) - \alpha(a)$$

which is half of the fundamental theorem of calculus.

**Proof:** For any partition $\mathcal{P} = \{a = x_0, \cdots, x_n = b\}$ of $[a, b]$ and any choice $\mathcal{T} = \{t_1, \cdots, t_n\}$ for $\mathcal{P}$

$$S(\mathcal{P}, \mathcal{T}, f\alpha', x) = \sum_{i=1}^{n} f(t_i)\alpha'(t_i) [x_i - x_{i-1}]$$

and, by the mean value theorem, there is, for each $1 \leq i \leq n$, some $v_i \in [x_{i-1}, x_i]$ such that

$$S(\mathcal{P}, \mathcal{T}, f, \alpha) = \sum_{i=1}^{n} f(t_i) [\alpha(x_i) - \alpha(x_{i-1})]$$

$$= \sum_{i=1}^{n} f(t_i) \alpha'(v_i) [x_i - x_{i-1}]$$

So

$$|S(\mathcal{P}, \mathcal{T}, f\alpha', x) - S(\mathcal{P}, \mathcal{T}, f, \alpha)| \leq \sum_{i=1}^{n} |f(t_i)| |\alpha' (t_i) - \alpha'(v_i)| [x_i - x_{i-1}]$$

Now let $\varepsilon > 0$.

- Since $f$ is assumed to be bounded, there is an $M > 0$ such that $|f(t)| \leq M$ for all $a \leq t \leq b$.
- Since $\alpha'$ is assumed to exist and be continuous on $[a, b]$, it is uniformly continuous on $[a, b]$. Hence, there is a $\delta > 0$ such that $|\alpha'(t) - \alpha'(v)| \leq \frac{\varepsilon}{2M(b-a)}$ for all $t, v \in [a, b]$ with $|t - v| < \delta$. 

© Joel Feldman. 2017. All rights reserved. January 20, 2017 Reduction to the Riemann Integral
In particular, if \( \|P\| < \delta \), then \( |t_i - v_i| < \delta \) and hence \( |\alpha'(t_i) - \alpha'(v_i)| \leq \frac{\varepsilon}{2M[b-a]} \) for all \( 1 \leq i \leq n \). So, if \( \|P\| < \delta \), we have
\[
\left| S(P, T, f, \alpha) - \int_a^b f d\alpha \right| \leq \sum_{i=1}^n \frac{\varepsilon}{2M[b-a]} [x_i - x_{i-1}]
\leq \frac{\varepsilon}{2} \sum_{i=1}^n [x_i - x_{i-1}] = \frac{\varepsilon}{2}
\]

In the event that \( f \in \mathcal{R}(\alpha) \) on \( [a, b] \), there is a partition \( P' \) such that \( P \supset P' \Rightarrow \left| S(P, T, f, \alpha) - \int_a^b f d\alpha \right| < \frac{\varepsilon}{2} \) for all choices \( T \) for \( P \). Choose a partition \( P \) for \( [a, b] \) by adding sufficiently many points to \( P' \) that \( \|P\| < \delta \). Then, by the triangle inequality, for any partition \( P \supset P' \) and any choice \( T \) for \( P \),
\[
\left| S(P, T, f, \alpha) - \int_a^b f d\alpha \right| \leq \left| S(P, T, f, \alpha) - S(P, T, f, \alpha) \right| + \left| S(P, T, f, \alpha) - \int_a^b f d\alpha \right| < \varepsilon
\]
as desired. The argument in the case that we assume \( f \in \mathcal{R} \) on \( [a, b] \) is identical.