# Products of Riemann Integrable Functions

For these notes, let $M > 0$, $-\infty < a < b < \infty$ and $\alpha : [a,b] \to \mathbb{R}$ be nondecreasing. We shall prove

**Theorem 1** *Let $f : [a,b] \to [-M,M]$ be Riemann integrable with respect to $\alpha$ on $[a,b]$ and let $\varphi : [-M,M] \to \mathbb{R}$ be continuous. Then $\varphi \circ f$ (which is defined by $\varphi \circ f(x) = \varphi\big(f(x)\big)$ ) is integrable with respect to $\alpha$ on $[a,b]$.*

**Corollary 2** *Let $f, g : [a,b] \to \mathbb{R}$ be bounded[(1)] functions which are Riemann integrable with respect to $\alpha$ on $[a,b]$. Then $fg$ and $|f|$ and, for any positive integer $n$, $f^n$ are integrable with respect to $\alpha$ on $[a,b]$.*

**Proof of Theorem 1:** Let $\varepsilon > 0$.

*The given data:*

- $\varphi$ is continuous on the compact set $[-M,M]$. So $\varphi$ is uniformly continuous. So for any $\varepsilon' > 0$ (we shall choose one later) there is a $\delta > 0$ such that $|\varphi(x) - \varphi(y)| < \varepsilon'$ for all $x, y \in [-M,M]$ obeying $|x-y| < \delta$. Again, since $\varphi$ is continuous on the compact set $[-M,M]$, it must be bounded on $[-M,M]$. So there is a constant $M_\varphi$ such that $|\varphi(y)| \le M_\varphi$ for all $|y| \le M$.
- $f$ is integrable. So, for any $\eta > 0$ (we shall choose one later), there is a partition $P_\eta = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ such that

$$U(P_\eta, f, \alpha) - L(P_\eta, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i < \eta \tag{1}$$

  where, as usual, $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ and

$$M_i - m_i = \sup_{x_{i-1} \le x \le x_i} f(x) \; - \inf_{x_{i-1} \le x \le x_i} f(x) = \sup_{x_{i-1} \le x, y \le x_i} [f(x) - f(y)]$$

*The goal:*

It suffices for us to prove that

$$U(P_\eta, \varphi \circ f, \alpha) - L(P_\eta, \varphi \circ f, \alpha) = \sum_{i=1}^{n} (M_i^* - m_i^*)\Delta\alpha_i < \varepsilon$$

where

$$M_i^* - m_i^* = \sup_{x_{i-1} \le x, y \le x_i} \big[\varphi\big(f(x)\big) - \varphi\big(f(y)\big)\big]$$

(1) If $\alpha$ is strictly increasing, then we know that integrability implies boundedness.

 

Set

$$A = \left\{ \ 1 \le i \le n \ \middle| \ M_i - m_i < \delta \ \right\} \qquad B = \left\{ \ 1 \le i \le n \ \middle| \ M_i - m_i \ge \delta \ \right\}$$

*Control of* $\sum_{i\in A}(M_i^* - m_i^*)\Delta\alpha_i$*:*

If $i \in A$, then, for all $x_{i-1} \le x, y \le x_i$

$$\begin{aligned} f(x) - f(y) \le M_i - m_i < \delta \implies & \varphi\big(f(x)\big) - \varphi\big(f(y)\big) < \varepsilon' \\ \implies & M_i^* - m_i^* \le \varepsilon' \end{aligned}$$

Hence

$$\sum_{i\in A}(M_i^* - m_i^*)\Delta\alpha_i \le \sum_{i\in A}\varepsilon'\Delta\alpha_i \le \varepsilon'\sum_{i=1}^{n}\Delta\alpha_i = \varepsilon'[\alpha(b) - \alpha(a)]$$

*Control of* $\sum_{i\in B}(M_i^* - m_i^*)\Delta\alpha_i$*:*

If $i \in B$, we cannot conclude that $M_i^* - m_i^*$ is small. About the best we can do is

$$x_{i-1} \le x, y \le x_i \implies \varphi\big(f(x)\big) - \varphi\big(f(y)\big) \le 2M_\varphi \implies M_i^* - m_i^* \le 2M_\varphi$$

On the other hand, we can show that $\sum_{i\in B}\Delta\alpha_i$ must be very small, because, by (1),

$$\eta > \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i \ge \sum_{i\in B}(M_i - m_i)\Delta\alpha_i \ge \sum_{i\in B}\delta\Delta\alpha_i \implies \sum_{i\in B}\Delta\alpha_i < \tfrac{\eta}{\delta}$$

Hence

$$\sum_{i\in B}(M_i^* - m_i^*)\Delta\alpha_i \le \sum_{i\in B}2M_\varphi\Delta\alpha_i < 2M_\varphi\tfrac{\eta}{\delta}$$

*The end game:*

$$\begin{aligned} U(P_\eta, \varphi\circ f, \alpha) - L(P_\eta, \varphi\circ f, \alpha) &= \sum_{i\in A}(M_i^* - m_i^*)\Delta\alpha_i + \sum_{i\in B}(M_i^* - m_i^*)\Delta\alpha_i \\ &< \quad \varepsilon'[\alpha(b) - \alpha(a)] \quad + \quad 2M_\varphi\tfrac{\eta}{\delta} \end{aligned}$$

It now suffices to choose

$$\varepsilon' = \tfrac{\varepsilon}{2[\alpha(b)-\alpha(a)]} \qquad \eta = \tfrac{\varepsilon\delta}{4M_\varphi}$$

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**Proof of Corollary 2:** The integrability of $|f|$ and $f^n$ both follow directly from Theorem 1, with $\varphi(y) = |y|$ and $\varphi(y) = y^n$, respectively. If $f$ and $g$ are bounded and integrable, then so is $f+g$. Hence, by Theorem 1, with $\varphi(y) = y^2$, we have that $f^2$, $g^2$ and $(f+g)^2 = f^2+g^2+2fg$ are all integrable. The integrability of $fg = \frac{1}{2}\big((f+g)^2 - f^2 - g^2\big)$ now follows by linearity. ■