Products of Riemann Integrable Functions

For these notes, let $-\infty < a < b < \infty$ and $\alpha : [a,b] \to \mathbb{R}$ be nondecreasing. We shall prove

Theorem 1 *Let $f : [a,b] \to [-M,M]$ be Riemann integrable with respect to α on $[a,b]$ and let $\varphi : [-M,M] \to \mathbb{R}$ be continuous. Then $\varphi \circ f$ (which is defined by $\varphi \circ f(x) = \varphi\big(f(x)\big)$) is integrable with respect to α on $[a,b]$.*

Corollary 2 *Let $f, g : [a,b] \to \mathbb{R}$ be bounded(1) functions which are Riemann integrable with respect to α on $[a,b]$. Then fg and $|f|$ and, for any positive integer n, f^n are integrable with respect to α on $[a,b]$.*

Proof of Theorem 1: Let $\varepsilon > 0$.

The given data:

- φ is continuous on the compact set $[-M,M]$. So φ is uniformly continuous. So for any $\varepsilon' > 0$ (we shall choose one later) there is a $\delta > 0$ such that $|\varphi(x) - \varphi(y)| < \varepsilon'$ for all $x, y \in [-M,M]$ obeying $|x-y| < \delta$. Again, since φ is continuous on the compact set $[-M,M]$, it must be bounded on $[-M,M]$. So there is a constant M_φ such that $|\varphi(y)| \le M_\varphi$ for all $|y| \le M$.
- f is integrable. So, for any $\eta > 0$ (we shall choose one later), there is a partition $P_\eta = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ such that

  $$U(P_\eta, f, \alpha) - L(P_\eta, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i < \eta \tag{1}$$

  where, as usual, $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ and

  $$M_i - m_i = \sup_{x_{i-1} \le x \le x_i} f(x) - \inf_{x_{i-1} \le x \le x_i} f(x) = \sup_{x_{i-1} \le x, y \le x_i} [f(x) - f(y)]$$

The goal:

It suffices for us to prove that

$$U(P_\eta, \varphi \circ f, \alpha) - L(P_\eta, \varphi \circ f, \alpha) = \sum_{i=1}^{n} (M_i^* - m_i^*)\Delta\alpha_i < \varepsilon$$

where

$$M_i^* - m_i^* = \inf_{x_{i-1} \le x, y \le x_i} \big[\varphi\big(f(x)\big) - \varphi\big(f(y)\big)\big]$$

Set

$$A = \big\{\ 1 \le i \le n \ \big|\ M_i - m_i < \delta\ \big\} \qquad B = \big\{\ 1 \le i \le n \ \big|\ M_i - m_i \ge \delta\ \big\}$$

Control of $\sum_{i \in A} (M_i^* - m_i^*)\Delta\alpha_i$:

If $i \in A$, then, for all $x_{i-1} \le x, y \le x_i$

$$\begin{aligned} f(x) - f(y) \le M_i - m_i < \delta &\implies \varphi\big(f(x)\big) - \varphi\big(f(y)\big) < \varepsilon' \\ &\implies M_i^* - m_i^* \le \varepsilon' \end{aligned}$$

Hence

$$\sum_{i \in A} (M_i^* - m_i^*)\Delta\alpha_i \le \sum_{i \in A} \varepsilon' \Delta\alpha_i \le \varepsilon' \sum_{i=1}^{n} \Delta\alpha_i = \varepsilon'[\alpha(b) - \alpha(a)]$$

---

(1) If α is strictly increasing, then we know that integrability implies boundedness.

Control of $\sum_{i\in B}(M_i^*-m_i^*)\Delta\alpha_i$*:*

If $i\in B$, we cannot conclude that $M_i^*-m_i^*$ is small. About the best we can do is

$$x_{i-1}\le x,y\le x_i \implies \varphi\big(f(x)\big)-\varphi\big(f(y)\big)\le 2M_\varphi \implies M_i^*-m_i^*\le 2M_\varphi$$

On the other hand, we can show that $\sum_{i\in B}\Delta\alpha_i$ must be very small, because, by (1),

$$\eta>\sum_{i=1}^n(M_i-m_i)\Delta\alpha_i\ge\sum_{i\in B}(M_i-m_i)\Delta\alpha_i\ge\sum_{i\in B}\delta\Delta\alpha_i \implies \sum_{i\in B}\Delta\alpha_i<\tfrac{\eta}{\delta}$$

Hence

$$\sum_{i\in B}(M_i^*-m_i^*)\Delta\alpha_i\le\sum_{i\in B}2M_\varphi\Delta\alpha_i<2M_\varphi\tfrac{\eta}{\delta}$$

The end game:

$$\begin{aligned}U(P_\eta,\varphi\circ f,\alpha)-L(P_\eta,\varphi\circ f,\alpha)&=\sum_{i\in A}(M_i^*-m_i^*)\Delta\alpha_i+\sum_{i\in B}(M_i^*-m_i^*)\Delta\alpha_i\\&<\ \varepsilon'[\alpha(b)-\alpha(a)]\ +\ 2M_\varphi\tfrac{\eta}{\delta}\end{aligned}$$

It now suffices to choose

$$\varepsilon'=\tfrac{\varepsilon}{2[\alpha(b)-\alpha(a)]}\qquad\eta=\tfrac{\varepsilon\delta}{4M_\varphi}$$

■

Proof of Corollary 2: The integrability of $|f|$ and f^n both follow directly from Theorem 1, with $\varphi(y)=|y|$ and $\varphi(y)=y^n$, respectively. If f and g are bounded and integrable, then so is $f+g$. Hence, by Theorem 1, with $\varphi(y)=y^2$, we have that f^2, g^2 and $(f+g)^2=f^2+g^2+2fg$ are all integrable. The integrability of $fg=\frac{1}{2}\big((f+g)^2-f^2-g^2\big)$ now follows by linearity. ■