Products of Riemann Integrable Functions

For these notes, let $M > 0$, $-\infty < a < b < \infty$ and $\alpha : [a, b] \to \mathbb{R}$ be nondecreasing. We shall prove

**Theorem 1** Let $f : [a, b] \to [-M, M]$ be Riemann integrable with respect to $\alpha$ on $[a, b]$ and let $\varphi : [-M, M] \to \mathbb{R}$ be continuous. Then $\varphi \circ f$ (which is defined by $\varphi \circ f(x) = \varphi(f(x))$) is integrable with respect to $\alpha$ on $[a, b]$.

**Corollary 2** Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions which are Riemann integrable with respect to $\alpha$ on $[a, b]$. Then $fg$ and $|f|$ and, for any positive integer $n$, $f^n$ are integrable with respect to $\alpha$ on $[a, b]$.

**Proof of Theorem 1:** Let $\varepsilon > 0$.

The given data:

- $\varphi$ is continuous on the compact set $[-M, M]$. So $\varphi$ is uniformly continuous. So for any $\varepsilon' > 0$ (we shall choose one later) there is a $\delta > 0$ such that $|\varphi(x) - \varphi(y)| < \varepsilon'$ for all $x, y \in [-M, M]$ obeying $|x - y| < \delta$. Again, since $\varphi$ is continuous on the compact set $[-M, M]$, it must be bounded on $[-M, M]$. So there is a constant $M_\varphi$ such that $|\varphi(y)| \leq M_\varphi$ for all $|y| \leq M$.

- $f$ is integrable. So, for any $\eta > 0$ (we shall choose one later), there is a partition $P_\eta = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that

$$U(P_\eta, f, \alpha) - L(P_\eta, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha_i < \eta$$

(1)

where, as usual, $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ and

$$M_i - m_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) - \inf_{x_{i-1} \leq x \leq x_i} f(x) = \sup_{x_{i-1} \leq x, y \leq x_i} [f(x) - f(y)]$$

The goal:

It suffices for us to prove that

$$U(P_\eta, \varphi \circ f, \alpha) - L(P_\eta, \varphi \circ f, \alpha) = \sum_{i=1}^{n} (M_i^* - m_i^*) \Delta \alpha_i < \varepsilon$$

where

$$M_i^* - m_i^* = \sup_{x_{i-1} \leq x, y \leq x_i} [\varphi(f(x)) - \varphi(f(y))]$$

(1) If $\alpha$ is strictly increasing, then we know that integrability implies boundedness.
Set
\[ A = \{ 1 \leq i \leq n \mid M_i - m_i < \delta \} \quad B = \{ 1 \leq i \leq n \mid M_i - m_i \geq \delta \} \]

**Control of \( \sum_{i \in A} (M_i^* - m_i^*) \Delta \alpha_i \):**

If \( i \in A \), then, for all \( x_{i-1} \leq x, y \leq x_i \)
\[ f(x) - f(y) \leq M_i - m_i < \delta \implies \varphi(f(x)) - \varphi(f(y)) < \varepsilon' \]
\[ \implies M_i^* - m_i^* \leq \varepsilon' \]

Hence
\[ \sum_{i \in A} (M_i^* - m_i^*) \Delta \alpha_i \leq \sum_{i \in A} \varepsilon' \Delta \alpha_i \leq \varepsilon' \sum_{i=1}^n \Delta \alpha_i = \varepsilon'[\alpha(b) - \alpha(a)] \]

**Control of \( \sum_{i \in B} (M_i^* - m_i^*) \Delta \alpha_i \):**

If \( i \in B \), we cannot conclude that \( M_i^* - m_i^* \) is small. About the best we can do is
\[ x_{i-1} \leq x, y \leq x_i \implies \varphi(f(x)) - \varphi(f(y)) \leq 2M_{\varphi} \implies M_i^* - m_i^* \leq 2M_{\varphi} \]

On the other hand, we can show that \( \sum_{i \in B} \Delta \alpha_i \) must be very small, because, by (1),
\[ \eta > \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i \geq \sum_{i \in B} (M_i - m_i) \Delta \alpha_i \geq \sum_{i \in B} \delta \Delta \alpha_i \implies \sum_{i \in B} \Delta \alpha_i < \frac{n}{\delta} \]

Hence
\[ \sum_{i \in B} (M_i^* - m_i^*) \Delta \alpha_i \leq \sum_{i \in B} 2M_{\varphi} \Delta \alpha_i < 2M_{\varphi} \frac{n}{\delta} \]

**The end game:**
\[
U(P_\eta, \varphi \circ f, \alpha) - L(P_\eta, \varphi \circ f, \alpha) = \sum_{i \in A} (M_i^* - m_i^*) \Delta \alpha_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta \alpha_i \\
< \varepsilon'[\alpha(b) - \alpha(a)] + 2M_{\varphi} \frac{n}{\delta} 
\]

It now suffices to choose
\[ \varepsilon' = \frac{\varepsilon}{2[\alpha(b) - \alpha(a)]} \quad \eta = \frac{\varepsilon \delta}{4M_{\varphi}} \]

**Proof of Corollary 2:** The integrability of \(|f|\) and \(f^n\) both follow directly from Theorem 1, with \( \varphi(y) = |y| \) and \( \varphi(y) = y^n \), respectively. If \( f \) and \( g \) are bounded and integrable, then so is \( f + g \). Hence, by Theorem 1, with \( \varphi(y) = y^2 \), we have that \( f^2, g^2 \) and \( (f + g)^2 = f^2 + g^2 + 2fg \) are all integrable. The integrability of \( fg = \frac{1}{2}((f + g)^2 - f^2 - g^2) \) now follows by linearity.  

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