Math 321 Problem Set 4
Due Wednesday, February 8

1. Let \( a < b \). Prove that if \( f : [a, b] \to \mathbb{R} \) and \( g : [a, b] \to \mathbb{R} \) are both of bounded variation then their product \( fg \) is also of bounded variation.

2. (a) Show that a polynomial \( P(x) \) is of finite variation on every finite interval.
   (b) Develop a formula for the total variation of the polynomial \( P(x) \) on the finite interval \([a, b]\) if the zeroes of the derivative \( P'(x) \) are known. (Denote the zeroes \( a \leq r_1 < \cdots < r_n \leq b \).) The formula should only involve the numbers \( P(x) \) for various \( x \)'s.

3. Let \( f, \alpha : [a, b] \to \mathbb{R} \) with \( \alpha \) increasing and \( f \in \mathbb{R}(\alpha) \) on \([a, b] \). Set \( g(x) = \int_a^x f(t) \, d\alpha(t) \).
   Prove that \( g \) is of bounded variation with \( V_g(x) = \int_a^x |f(t)| \, d\alpha(t) \).
   Hint: Prove that, for any partition \( \mathcal{P} = \{ x_0, x_1, \ldots, x_n \} \) of \([a, b] \),
   \[
   \int_{x_{i-1}}^{x_i} |f(t)| \, d\alpha(t) - \int_{x_{i-1}}^{x_{i-1}} f(t) \, d\alpha(t) \leq (M_i - m_i)(\alpha(x_i) - \alpha(x_{i-1}))
   \]
   where, as usual,
   \[
   M_i = \sup_{t \in [x_{i-1}, x_i]} f(t) \quad \text{and} \quad m_i = \inf_{t \in [x_{i-1}, x_i]} f(t)
   \]

4. Consider the power series \( \sum_{n=0}^{\infty} x^n \).
   (a) Fix an arbitrary \( 0 < \varepsilon < \frac{1}{2} \) and \( x \in (-1, 1) \). Find explicitly a number \( N_{\varepsilon, x} \) such that
   \[
   \left| \sum_{n=0}^{m} x^n - \frac{1}{1-x} \right| \leq \varepsilon \iff m \geq N_{\varepsilon, x}
   \]
   Sketch a graph of \( N_{\varepsilon, x} \) as a function of \( x \) for fixed \( \varepsilon \).
   (b) Prove that \( \sum_{n=0}^{\infty} x^n \) converges uniformly on \([-a, a]\) for any fixed \( 0 \leq a < 1 \).
   (c) Prove that \( \sum_{n=0}^{\infty} x^n \) does not converge uniformly on \((-1, 1)\).

5. Let \( \{f_n\} \) and \( \{g_n\} \) be uniformly convergent sequences of real–valued functions on some set \( E \).
   (a) Prove that \( \{f_n + g_n\} \) converges uniformly on \( E \).
   (b) Prove that if \( \{f_n\} \) and \( \{g_n\} \) are bounded, then \( \{f_n g_n\} \) converges uniformly on \( E \).
   (c) Construct sequences \( \{f_n\} \) and \( \{g_n\} \) such that \( \{f_n\} \) and \( \{g_n\} \) converge uniformly, \( \{f_n g_n\} \) converges pointwise, but \( \{f_n g_n\} \) does not converge uniformly.