1. Let \( a < b \). Prove that if \( f : [a, b] \to \mathbb{R} \) and \( g : [a, b] \to \mathbb{R} \) are both of bounded variation then their product \( fg \) is also of bounded variation.

2. (a) Show that a polynomial \( P(x) \) is of bounded variation on every finite interval.
(b) Develop a formula for the total variation of the polynomial \( P(x) \) on the finite interval \( [a, b] \) if the zeroes of the derivative \( P'(x) \) are known. (Denote the zeroes \( a \leq r_1 < \cdots < r_n \leq b \).) The formula should only involve the numbers \( P(x) \) for various \( x \)'s.

3. Let \( f, \alpha : [a, b] \to \mathbb{R} \) with \( \alpha \) increasing and \( f \in \mathbb{R}(\alpha) \) on \( [a, b] \). Set \( g(x) = \int_a^x f(t) \, d\alpha(t) \).

Prove that \( g \) is of bounded variation with \( V_g(x) = \int_a^x |f(t)| \, d\alpha(t) \).

Hint: Prove that, for any partition \( \mathcal{P} = \{x_0, x_1, \ldots, x_n\} \) of \( [a, b] \),

\[
\int_{x_{i-1}}^{x_i} |f(t)| \, d\alpha(t) - \left| \int_{x_{i-1}}^{x_i} f(t) \, d\alpha(t) \right| \leq (M_i - m_i)(\alpha(x_i) - \alpha(x_{i-1})),
\]

where, as usual,

\[
M_i = \sup_{t \in [x_{i-1}, x_i]} f(t) \quad \text{and} \quad m_i = \inf_{t \in [x_{i-1}, x_i]} f(t)
\]

4. Consider the power series \( \sum_{n=0}^{\infty} x^n \).

(a) Fix an arbitrary \( 0 < \varepsilon < \frac{1}{2} \) and \( x \in (-1, 1) \). Find explicitly a number \( N_{\varepsilon, x} \) such that

\[
\left| \sum_{n=0}^{m} x^n - \frac{1}{1-x} \right| \leq \varepsilon \iff m \geq N_{\varepsilon, x}
\]

Sketch a graph of \( N_{\varepsilon, x} \) as a function of \( x \) for fixed \( \varepsilon \).

(b) Prove that \( \sum_{n=0}^{\infty} x^n \) converges uniformly on \( [-a, a] \) for any fixed \( 0 \leq a < 1 \).

(c) Prove that \( \sum_{n=0}^{\infty} x^n \) does not converge uniformly on \( (-1, 1) \).

5. Let \( \{f_n\} \) and \( \{g_n\} \) be uniformly convergent sequences of real–valued functions on some set \( E \).

(a) Prove that \( \{f_n + g_n\} \) converges uniformly on \( E \).

(b) Prove that if \( \{f_n\} \) and \( \{g_n\} \) are bounded, then \( \{f_ng_n\} \) converges uniformly on \( E \).

(c) Construct sequences \( \{f_n\} \) and \( \{g_n\} \) such that \( \{f_n\} \) and \( \{g_n\} \) converge uniformly,\n\( \{f_ng_n\} \) converges pointwise, but \( \{f_ng_n\} \) does not converge uniformly.