1. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a bounded function and define

\[
D = \{ x \in [a, b] \mid f \text{ is not continuous at } x \}
\]

Assume that for every \( \varepsilon > 0 \), \( D \) can be covered by a finite number of intervals whose total length is less than \( \varepsilon \). Prove that \( f \in \mathcal{R} \) on \( [a, b] \).

**Hint:** There is a partition \( I P_\varepsilon \) such that \( U(I P_\varepsilon, f) - L(I P_\varepsilon, f) \) may be written as a sum of two pieces. One piece is small by the continuity of \( f \) outside of \( D \). The other is small by the above covering property.

**Remark:** It turns out that \( f \in \mathcal{R} \) on \( [a, b] \) if and only if \( f \) is bounded and, for every \( \varepsilon > 0 \), \( D \) can be covered by a countable union of intervals whose total length is less than \( \varepsilon \).

2. (Rectifiable curves) By definition, a curve is a continuous function \( \gamma : [a, b] \rightarrow \mathbb{R}^k \). We define the length \( \Lambda_\gamma(a, b) \) of \( \gamma \) from \( a \) to \( b \) as follows. To each partition \( I P = \{x_0, \ldots, x_n\} \) of \( [a, b] \) we may associate a polygon with vertices \( \gamma(x_0), \gamma(x_1), \ldots, \gamma(x_n) \), as in the figure below. This polygon has length \( \Lambda_\gamma(I P) = \sum_{i=1}^{n} \|\gamma(x_i) - \gamma(x_{i-1})\| \) where \( \| \cdot \| \) denotes \( \mathbb{R}^k \)'s usual norm. We then define

\[
\Lambda_\gamma(a, b) = \sup \{ \Lambda_\gamma(I P) \mid I P \text{ a partition of } [a, b] \}
\]

If \( \Lambda_\gamma(a, b) < \infty \), we say that \( \gamma \) is rectifiable.

![Diagram of a polygon with vertices \( \gamma(x_0), \gamma(x_1), \ldots, \gamma(x_n) \)]

(a) Prove that \( \gamma \) is rectifiable if and only if \( \lim_{I P} \Lambda_\gamma(I P) \) exists. That is, if and only if there is a real number \( L \) such that

\[
\forall \varepsilon > 0 \exists I P_\varepsilon \text{ s.t. } I P \supset I P_\varepsilon \Rightarrow |\Lambda_\gamma(I P) - L| < \varepsilon
\]

Further prove that if \( \gamma \) is rectifiable, then \( \Lambda_\gamma(a, b) = \lim_{I P} \Lambda_\gamma(I P) \).

(b) Prove that if a curve \( \gamma : [a, b] \rightarrow \mathbb{R}^k \) is rectifiable on \( [a, b] \) and if \( a < c < b \), then \( \gamma \) is also rectifiable on \( [a, c] \) and on \( [b, c] \) and that \( \Lambda_\gamma(a, b) = \Lambda_\gamma(a, c) + \Lambda_\gamma(c, b) \).

**Remark:** If \( \gamma(x) \) has a continuous derivative on \( [a, b] \), then \( \gamma \) is rectifiable on \( [a, b] \) and \( \Lambda_\gamma(a, b) = \int_{a}^{b} \|\gamma'(x)\| \, dx \). This is proven on page 137 of Rudin.
3. (Hölder’s inequality) Let \( f, g, \alpha : [a, b] \to \mathbb{R} \) be bounded with \( \alpha \) increasing and \( f, g \in \mathcal{R}(\alpha) \) on \([a, b]\). Also let \( p, q > 0 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Assume that both \( \int_a^b |f(x)|^p \, d\alpha \) and \( \int_a^b |g(x)|^q \, d\alpha \) are nonzero. Prove that

\[
\left| \int_a^b f(x)g(x) \, d\alpha \right| \leq \left\{ \int_a^b |f(x)|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g(x)|^q \, d\alpha \right\}^{1/q}
\]

You may use the fact that if \( f, g \in \mathcal{R}(\alpha) \) on \([a, b]\), then \( fg, |f|^p, |g|^q \in \mathcal{R}(\alpha) \) on \([a, b]\). (This is proven in the supplementary notes “Products of Riemann Integrable Functions”.)

Hint: First prove that, for any \( u, v \geq 0 \), we have \( uv \leq \frac{u^p}{p} + \frac{v^q}{q} \). Then apply this with

\[
u = \frac{|f(x)|}{\left\{ \int |f(x)|^p \, d\alpha \right\}^{1/p}} \quad v = \frac{|g(x)|}{\left\{ \int |g(x)|^q \, d\alpha \right\}^{1/q}}
\]

4. (Improper integrals on \([a, \infty)\)) Suppose that \( f : [a, \infty) \to \mathbb{R} \) with \( f \in \mathcal{R} \) on \([a, b]\) for all \( b > a \).

Define the improper integral \( \int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx \) if the limit exists. If so, we say that \( \int_a^\infty f(x) \, dx \) converges. If \( \int_a^\infty |f(x)| \, dx \) converges, we say that \( \int_a^\infty f(x) \, dx \) converges absolutely.

(a) Prove that any absolutely convergent integral converges.

(b) Prove that \( \int_0^\infty \cos \frac{x}{1+x} \, dx = \int_0^\infty \sin \frac{x}{(1+x)^2} \, dx \). Show that one of these integrals converges absolutely, but the other does not.

You may use the fact that if \( f \in \mathcal{R} \) on \([a, b]\), then \( |f| \in \mathcal{R} \) on \([a, b]\).

Do not hand in problem 5.

5. (The Euler summation formula) Let \( a < b \) be real numbers and \( f : [a, b] \to \mathbb{R} \) have a continuous derivative. Prove that

\[
\sum_{\substack{n \in \mathbb{Z} \\ a < n \leq b}} f(n) = \int_a^b f(x) \, dx + \int_a^b f'(x) (\langle x \rangle) \, dx + f(a) (\langle a \rangle) - f(b) (\langle b \rangle)
\]

Here, for any \( x \in \mathbb{R} \), the symbol \( (\langle x \rangle) \) denotes the fractional part of \( x \). That is, if \( [x] \) is the greatest integer less than or equal to \( x \), then \( (\langle x \rangle) = x - [x] \).