1. Define \( \alpha, f : [0, 2] \to \mathbb{R} \) by

\[
\alpha(x) = f(x) = \begin{cases} 
0 & \text{if } 0 \leq x < 1 \\
1 & \text{if } 1 \leq x \leq 2
\end{cases}
\]

Prove that \( f \notin \mathcal{R}(\alpha) \) on \([0, 2]\).

2. (The Cauchy Criterion) Let \( \alpha, f : [a, b] \to \mathbb{R} \). Prove that \( f \) is integrable with respect to \( \alpha \) on \([a, b]\) (i.e. \( f \in \mathcal{R}(\alpha) \) on \([a, b]\)) if and only if for every \( \varepsilon > 0 \) there is a partition \( \mathcal{P}_\varepsilon \) of \([a, b]\) such that

\[
|S(\mathcal{P}_1, T_1, f, \alpha) - S(\mathcal{P}_2, T_2, f, \alpha)| < \varepsilon
\]

for all partitions \( \mathcal{P}_1, \mathcal{P}_2 \supset \mathcal{P}_\varepsilon \) and all choices \( T_1, T_2 \) for \( \mathcal{P}_1, \mathcal{P}_2 \), respectively.

3. Let \( a < c < b \). Let \( f, \alpha : [a, b] \to \mathbb{R} \). Prove that if \( f \in \mathcal{R}(\alpha) \) on \([a, b]\), then \( f \in \mathcal{R}(\alpha) \) on \([a, c]\).

4. (Improper integrals on \([0, 1]\)) Suppose that \( f : (0, 1] \to \mathbb{R} \) and that \( f \in \mathcal{R} \) on \([c, 1]\) for all \( 0 < c < 1 \). Define the improper integral \( \int_0^1 f(x) \, dx = \lim_{c \to 0^+} \int_c^1 f(x) \, dx \) if the limit exists (and is finite).

(a) Show that if \( f \in \mathcal{R} \) on \([0, 1]\), then the improper integral \( \lim_{c \to 0^+} \int_c^1 f(x) \, dx \) exists and equals the Riemann integral \( \int_0^1 f(x) \, dx \).

(b) Show that \( \int_0^1 \frac{1}{\sqrt{x}} \, dx \) exists as an improper integral but that \( \frac{1}{\sqrt{x}} \notin \mathcal{R} \) on \([0, 1]\).

(c) Parts (a) and (b) dealt with two different definitions of “integral” – the Riemann integral and the improper integral. The goal of both of these definitions is to give a precise meaning to “the area under a curve”. Why was the improper integral definition more successful than the Riemann integral definition in part (b)?

5. (Axiomatic definition of the integral) Let \( a < b \). Suppose that to every continuous function \( f : [a, b] \to \mathbb{R} \) and every subinterval \( [\alpha, \beta] \subset [a, b] \) there is associated a number \( I_\alpha^\beta(f) \) satisfying

(a) \( I_\alpha^\beta(sf + tg) = sI_\alpha^\beta(f) + tI_\alpha^\beta(g) \) for all \( s, t \in \mathbb{R} \)

(b) \( I_\alpha^\beta(1) = \beta - \alpha \)

(c) \( I_\alpha^\beta(f) = I_\alpha^\gamma(f) + I_\gamma^\beta(f) \) for all \( \gamma \in [\alpha, \beta] \)

(d) \( |I_\alpha^\beta(f)| \leq (\beta - \alpha) \sup_{x \in [\alpha, \beta]} |f(x)| \)

Prove that \( I_\alpha^\beta(f) = \int_\alpha^\beta f(x) \, dx \). Hint: Prove that \( \frac{d}{d\beta} \left\{ I_\alpha^\beta(f) - \int_\alpha^\beta f(x) \, dx \right\} = 0 \).

see over
6. (The Dirac Delta Function) There is a well-known “function”, the Dirac Delta function, that is very useful in the physical sciences (c.f. “point” mass, “impulse” force, spectral “line”, etc). It is “defined”, on a hand waving level, by the properties that

(i) \( \delta(x) = 0 \) except when \( x = 0 \)

(ii) \( \delta(0) \) is “so infinite” that

(iii) the area under its graph is one.

Here is a “derivation” of the the most important property of the Dirac Delta function. Let \( f \) be any continuous function. The functions \( f(x)\delta(x) \) and \( f(0)\delta(x) \) are the same since they are both zero for every \( x \neq 0 \). Consequently \( \int_{-1}^{1} f(x)\delta(x) \, dx = \int_{-1}^{1} f(0)\delta(x) \, dx = f(0) \int_{-1}^{1} \delta(x) \, dx = f(0) \).

(a) Prove that there does not exist a function \( \delta(x) \) obeying

(i) \( f(x)\delta(x) \in \mathcal{R}(x) \) on \([-1, 1]\)

(ii) \( \int_{-1}^{1} f(x)\delta(x) \, dx = f(0) \)

whenever \( f \) is continuous on \([-1, 1]\).

(b) Define the Heavyside unit function \( H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \). Prove that if the function \( f : [-1, 1] \to \mathbb{R} \) is continuous, then \( f \in \mathcal{R}(H) \) on \([-1, 1]\) and \( \int_{-1}^{1} f \, dH = f(0) \).

Remark. On a hand waving level, \( \int_{-1}^{1} f \, dH = \int_{-1}^{1} f \frac{dH}{dx} \, dx \), so “\( \delta(x) = \frac{dH}{dx} \)”.

Hence (b) has provided a rigorous procedure both for defining the delta “function” and for making sense of the often used equation \( \delta(x) = \frac{dH}{dx} \). There are other procedures for doing so, that we could use as our optional topic at the end of this course.