Properties of the Riemann–Stieltjes Integral

Theorem (Linearity Properties)
Let \( a < c < d < b \) and \( A, B \in \mathbb{R} \) and \( f, g, \alpha, \beta : [a, b] \to \mathbb{R} \).

(a) If \( f, g \in \mathcal{R}(\alpha) \) on \([a, b]\), then \( Af + Bg \in \mathcal{R}(\alpha) \) on \([a, b]\) and
\[
\int_a^b [Af + Bg] \, d\alpha = A \int_a^b f \, d\alpha + B \int_a^b g \, d\alpha
\]

(b) If \( f \in \mathcal{R}(\alpha) \cap \mathcal{R}(\beta) \) on \([a, b]\), then \( f \in \mathcal{R}(A\alpha + B\beta) \) on \([a, b]\) and
\[
\int_a^b f \, d(A\alpha + B\beta) = A \int_a^b f \, d\alpha + B \int_a^b f \, d\beta
\]

(c) If \( f \in \mathcal{R}(\alpha) \) on \([a, c]\) and on \([c, b]\), then \( f \in \mathcal{R}(\alpha) \) on \([a, b]\) and
\[
\int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha
\]

(d) If \( f \in \mathcal{R}(\alpha) \) on \([a, b]\) then \( f \in \mathcal{R}(\alpha) \) on \([c, d] \subset [a, b]\).

Proof: (a) Let \( \varepsilon > 0 \). Then, for any partition \( P \) of \([a, b]\) and choice \( T \) for \( P \),
\[
\left| S(P, T, Af + Bg, \alpha) - A \int_a^b f \, d\alpha - B \int_a^b g \, d\alpha \right| \\
= \left| AS(P, T, f, \alpha) + BS(P, T, g, \alpha) - A \int_a^b f \, d\alpha - B \int_a^b g \, d\alpha \right| \\
\leq |A| \left| S(P, T, f, \alpha) - \int_a^b f \, d\alpha \right| + |B| \left| S(P, T, g, \alpha) - \int_a^b g \, d\alpha \right|
\]
Assume that \( A \) and \( B \) are nonzero. (The cases that \( A \) and/or \( B \) are zero are similar, but easier.) Since \( f \in \mathcal{R}(\alpha) \) on \([a, b]\) there is a partition \( P_{f, \varepsilon} \) such that
\[
\left| S(P, T, f, \alpha) - \int_a^b f \, d\alpha \right| < \frac{\varepsilon}{2|A|} \quad \text{whenever } P \supset P_{f, \varepsilon}
\]
and since \( g \in \mathcal{R}(\alpha) \) on \([a, b]\) there is a partition \( P_{g, \varepsilon} \) such that
\[
\left| S(P, T, g, \alpha) - \int_a^b g \, d\alpha \right| < \frac{\varepsilon}{2|B|} \quad \text{whenever } P \supset P_{g, \varepsilon}
\]
It now suffices to set \( P_\varepsilon = P_{f, \varepsilon} \cup P_{g, \varepsilon} \) and observe that
\[
\left| S(P, T, Af + Bg, \alpha) - A \int_a^b f \, d\alpha - B \int_a^b g \, d\alpha \right| < \varepsilon \quad \text{whenever } P \supset P_\varepsilon
\]
(b) See Problem Set 1, #3.
(c) See Problem Set 1, #2.
(d) See Problem Set 2, #3.
Theorem (Integration by Parts)
Let \(a < b\) and \(f, \alpha : [a, b] \to \mathbb{R}\). If \(f \in \mathcal{R}(\alpha)\) on \([a, b]\), then \(\alpha \in \mathcal{R}(f)\) on \([a, b]\) and
\[
\int_a^b f(x) \, d\alpha(x) + \int_a^b \alpha(x) \, df(x) = f(b)\alpha(b) - f(a)\alpha(a)
\]
\[
= \int_a^b d(f\alpha)
\]

Proof: We want to show that
\[
\int_a^b \alpha \, df = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f \, d\alpha
\]
For any partition \(\mathcal{P} = \{a = x_0, x_1, x_2, x_3, \ldots, x_n = b\}\) of \([a, b]\) and choice \(\mathcal{T} = \{t_1, t_2, t_3, \ldots, t_n\}\) for \(\mathcal{P}\),
\[
S(\mathcal{P}, \mathcal{T}, \alpha, f) - \left\{f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f \, d\alpha\right\}
\]
\[
= \sum_{i=1}^n \alpha(t_i) [f(x_i) - f(x_{i-1})] - \sum_{i=1}^n \alpha(x_i) f(x_i) + \sum_{i=1}^n \alpha(x_{i-1}) f(x_{i-1}) + \int_a^b f \, d\alpha
\]
(The 1 \(\leq i \leq n - 1\) terms of the second sum cancel the
2 \(\leq i \leq n\) terms of the third sum.)
\[
= -\sum_{i=1}^n f(x_i) \left[\alpha(x_i) - \alpha(t_i)\right] + \sum_{i=1}^n f(x_{i-1}) \left[\alpha(t_i) - \alpha(x_{i-1})\right] + \int_a^b f \, d\alpha
\]
\[
= -S(\mathcal{P} \cup \mathcal{T}, \mathcal{T}', f, \alpha) + \int_a^b f \, d\alpha
\]
where
\(\mathcal{T}' = \{x_0, x_1, x_1, x_2, x_2, x_3, \ldots, x_n-1, x_n-1, x_n\}\)
is a choice for the partition
\[
\mathcal{P} \cup \mathcal{T} = \{a, t_0, t_1, x_1, t_1, x_1, t_2, x_2, t_2, x_2, \ldots, t_n, x_n\}
\]
Now let \(\varepsilon > 0\). Since \(f \in \mathcal{R}(\alpha)\) for \([a, b]\) there is a partition \(\mathcal{P}_\varepsilon\) such that
\[
\left|S(\hat{\mathcal{P}}, \hat{\mathcal{T}}, f, \alpha) - \int_a^b f \, d\alpha\right| < \varepsilon
\]
for all partitions \(\hat{\mathcal{P}}\) finer than \(\mathcal{P}_\varepsilon\). If the partition \(\mathcal{P}\) above is finer than \(\mathcal{P}_\varepsilon\) then the partition
\(\mathcal{P} \cup \mathcal{T}\) is also finer than \(\mathcal{P}_\varepsilon\) and we have
\[
\left|S(\mathcal{P}, \mathcal{T}, \alpha, f) - \left\{f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f \, d\alpha\right\}\right| = \left|S(\mathcal{P} \cup \mathcal{T}, \mathcal{T}', f, \alpha) - \int_a^b f \, d\alpha\right| < \varepsilon
\]
\(\blacksquare\)
Theorem (The Change of Variables \( x = g(y) \))

Let
\( \circ \ a < b \text{ and } c < d, \)
\( \circ \ g : [c, d] \to [a, b] \text{ be continuous, strictly monotonic and obey } g(c) = a \text{ and } g(d) = b \text{ and } \)
\( \circ \ f, \alpha : [a, b] \to \mathbb{R}. \)

Set
\[
    h(y) = f(g(y)) \quad \beta(y) = \alpha(g(y))
\]

If \( f \in \mathcal{R}(\alpha) \text{ on } [a, b], \text{ then } h \in \mathcal{R}(\beta) \text{ on } [c, d] \text{ and }
\[
    \int_a^b f(x) \, d\alpha(x) = \int_c^d h(y) \, d\beta(y)
\]

Proof: For any partition \( \mathcal{P} = \{ c = y_0, \cdots, y_n = d \} \) of \( [c, d] \) and any choice \( \mathcal{T} = \{ s_1, \cdots, s_n \} \) for \( \mathcal{P} \)
\[
    \left| S(\mathcal{P}, \mathcal{T}, h, \beta) - \int_a^b f \, d\alpha \right| = \left| \sum_{i=1}^n h(s_i) [\beta(y_i) - \beta(y_{i-1})] - \int_a^b f \, d\alpha \right|
\]
\[
    = \left| \sum_{i=1}^n f(g(s_i)) [\alpha(g(y_i)) - \alpha(g(y_{i-1}))] - \int_a^b f \, d\alpha \right|
\]
\[
    = \left| S(g(\mathcal{P}), g(\mathcal{T}), f, \alpha) - \int_a^b f \, d\alpha \right|
\]
where
\[
    g(\mathcal{P}) = \{ g(y) \mid y \in \mathcal{P} \}
\]
\[
    = \{ g(y_0) = g(c) = a, \quad g(y_1), \cdots, \quad g(y_n) = g(d) = b \}
\]
is a partition of \( [a, b] \) because \( g \) is assumed to be strictly monotonic, so that
\[
    y_{i-1} < y_i \implies x_{i-1} < x_i
\]
and is assumed to obey \( x_0 = g(y_0) = a \) and \( x_n = g(y_n) = b \) and
\[
    g(\mathcal{T}) = \{ g(s_1), \cdots, g(s_n) \}
\]
is a choice for \( g(\mathcal{P}) \) because \( g \) is assumed to be strictly monotonic so that
\[
    y_{i-1} \leq s_i \leq y_i \implies x_{i-1} = g(y_{i-1}) \leq g(s_i) = t_i \leq g(y_i) = x_i
\]

Now let \( \varepsilon > 0 \). We have assumed that \( f \in \mathcal{R}(\alpha) \text{ on } [a, b], \) so there is a partition \( \mathcal{P}_\varepsilon' \) of \( [a, b] \) such that \( \mathcal{P}_\varepsilon' \supset \mathcal{P}_\varepsilon \implies |S(\mathcal{P}_\varepsilon', \mathcal{T}', f, \alpha) - \int_a^b f \, d\alpha| < \varepsilon \) for all choices \( \mathcal{T}' \) for \( \mathcal{P}_\varepsilon' \). The assumptions that we have made on \( g \) guarantee that the inverse function \( g^{-1} : [a, b] \to [c, d] \) exists and that \( g^{-1}(\mathcal{P}_\varepsilon') \) is a partition of \( [c, d] \). We choose \( \mathcal{P}_\varepsilon' = g^{-1}(\mathcal{P}_\varepsilon) \). Then
\[
    \mathcal{P} \supset \mathcal{P}_\varepsilon \implies \mathcal{P}' = g(\mathcal{P}) \supset g(\mathcal{P}_\varepsilon) = \mathcal{P}_\varepsilon'
\]
\[
    \implies |S(\mathcal{P}', \mathcal{T}', h, \beta) - \int_a^b f \, d\alpha| = |S(g(\mathcal{P}), g(\mathcal{T}), f, \alpha) - \int_a^b f \, d\alpha| < \varepsilon
\]
as desired.

\( \Box \)
Theorem (Basic Bounds)
Let \(a < b\) and \(f, g, \alpha : [a, b] \to \mathbb{R}\). Assume that \(f, g \in R(\alpha)\) on \([a, b]\) and \(\alpha\) is monotonic.

(a) If \(f(x) \leq g(x)\) for all \(x \in [a, b]\), then

\[
\int_a^b f \, d\alpha \leq \int_a^b g \, d\alpha
\]

(a) If \(|f(x)| \leq g(x)\) for all \(x \in [a, b]\), then

\[
\left| \int_a^b f \, d\alpha \right| \leq \int_a^b g \, d\alpha
\]

Proof: Let \(\varepsilon > 0\). Since \(f \in R(\alpha)\) on \([a, b]\) there is a partition \(P_{f, \varepsilon}\) such that

\[
\left| S(P, T, f, \alpha) - \int_a^b f \, d\alpha \right| < \varepsilon
\]

whenever \(P \supset P_{f, \varepsilon}\) and since \(g \in R(\alpha)\) on \([a, b]\) there is a partition \(P_{g, \varepsilon}\) such that

\[
\left| S(P, T, g, \alpha) - \int_a^b g \, d\alpha \right| < \varepsilon
\]

whenever \(P \supset P_{g, \varepsilon}\). Set \(P_\varepsilon = P_{f, \varepsilon} \cup P_{g, \varepsilon}\).

(a) We have

\[
\int_a^b f \, d\alpha \leq S(P_\varepsilon, T, f, \alpha) + \varepsilon
\]

\[
= \sum_{i=1}^{n} f(t_i)\left[\alpha(x_i) - \alpha(x_{i-1})\right] + \varepsilon
\]

\[
\leq \sum_{i=1}^{n} g(t_i)\left[\alpha(x_i) - \alpha(x_{i-1})\right] + \varepsilon
\]

(since \(f(t_i) \leq g(t_i)\) and \(\alpha(x_i) - \alpha(x_{i-1}) \geq 0\))

\[
= S(P_\varepsilon, T, g, \alpha) + \varepsilon
\]

\[
\leq \int_a^b g \, d\alpha + 2\varepsilon
\]

As \(\int_a^b f \, d\alpha \leq \int_a^b g \, d\alpha + 2\varepsilon\) is true for all \(\varepsilon > 0\), we also have \(\int_a^b f \, d\alpha \leq \int_a^b g \, d\alpha\).
(b) We have

\[ \left| \int_a^b f \, d\alpha \right| \leq |S(\Pi_{\varepsilon}, T, f, \alpha)| + \varepsilon \]

\[ \leq \sum_{i=1}^{n} |f(t_i)| \left[ \alpha(x_i) - \alpha(x_{i-1}) \right] + \varepsilon \]

\[ \leq \sum_{i=1}^{n} g(t_i) \left[ \alpha(x_i) - \alpha(x_{i-1}) \right] + \varepsilon \]

(since \(|f(t_i)| \leq g(t_i)\) and \(\alpha(x_i) - \alpha(x_{i-1}) \geq 0\))

\[ = S(\Pi_{\varepsilon}, T, g, \alpha) + \varepsilon \]

\[ \leq \int_a^b g \, d\alpha + 2\varepsilon \]

As \( \left| \int_a^b f \, d\alpha \right| \leq \int_a^b g \, d\alpha + 2\varepsilon \) is true for all \( \varepsilon > 0 \), we also have \( \left| \int_a^b f \, d\alpha \right| \leq \int_a^b g \, d\alpha \).