A Definition of the Riemann–Stieltjes Integral

Let $a < b$ and let $f, \alpha : [a,b] \to \mathbb{R}$. In these notes I will state one of several closely related, but not 100% equivalent, standard definitions of the Riemann–Stieltjes integral $\int_a^b f(x)\,d\alpha(x)$. Let's start by reviewing the first year Calculus definition of the Riemann integral $\int_a^b f(x)\,dx$. The definition goes in several steps.

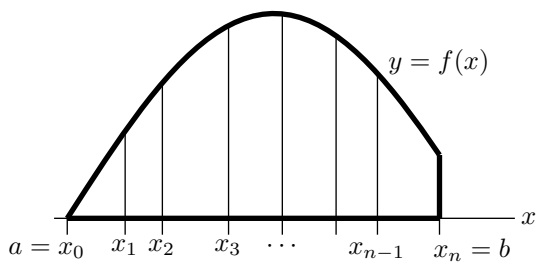
Step 1: Slice up the interval $[a,b]$ into a finite number of subintervals.
A *partition* of $[a,b]$ is a finite set $\mathbb{P} = \{x_0, x_1, \cdots, x_n\}$ of points with

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

The *mesh*

$$\|P\| = \max_{1 \le i \le n} |x_i - x_{i-1}|$$

of this partition is the width of the widest subinterval

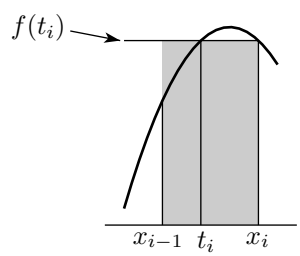


Step 2: For each $1 \le i \le n$, choose a point to represent the subinterval $[x_{i-1}, x_i]$.
A *choice* for the partition $\mathbb{P}$ is a finite set $\mathbb{T} = \{t_1, \cdots, t_n\}$ of points with

$$x_{i-1} \le t_i \le x_i \qquad \text{for each } 1 \le i \le n$$

Step 3: Calculate an approximate value for the integral, by approximating the ith strip by a rectangle of width $x_i - x_{-1}$ and height $f(t_i)$.
The *Riemann sum* with partition $\mathbb{P}$ and choice $\mathbb{T}$ is

$$S(\mathbb{P}, \mathbb{T}, f) = \sum_{i=1}^{n} f(t_i)\,(x_i - x_{i-1})$$



Step 4: Make the partition finer and finer.
A partition $\mathbb{P}'$ is said to be *finer* than the partition $\mathbb{P}$ if $\mathbb{P} \subset \mathbb{P}'$.

Step 5: Hope the Riemann sums converge.
A function $f : [a,b] \to \mathbb{R}$ is said to be *Riemann integrable* on $[a,b]$, denoted "$f \in \mathcal{R}$ on $[a,b]$", if there exists an $I \in \mathbb{R}$ such that

$$\forall\, \varepsilon > 0 \;\; \exists \text{ a partition } \mathbb{P}_\varepsilon \text{ of } [a,b] \;\text{ s.t. }\; \big|S(\mathbb{P}, \mathbb{T}, f) - I\big| < \varepsilon$$
$$\text{for all partitions } \mathbb{P} \text{ of } [a,b] \text{ finer than } \mathbb{P}_\varepsilon$$
$$\text{and all choices } \mathbb{T} \text{ for } \mathbb{P}$$

If so, I is denoted $\int_a^b f(x)\,dx$ and we write

$$\lim_{\mathbb{P}} S(\mathbb{P}, \mathbb{T}, f) = \int_a^b f(x)\,dx$$

To generalize this to the Riemann–Stieltjes integral $\int_a^b f(x)\,d\alpha(x)$ we simply replace the strip width $x_i - x_{i-1}$ with $\alpha(x_i) - \alpha(x_{i-1})$. That is, Steps 1, 2 and 4 are left as is and Steps 3 and 5 are replaced with

Step 3_α: Calculate an approximate value for the integral.
The *Riemann–Stieltjes sum* with partition $\mathbb{P}$ and choice $\mathbb{T}$ is

$$S(\mathbb{P},\mathbb{T},f,\alpha)=\sum_{i=1}^{n} f(t_i)\,\big[\alpha(x_i)-\alpha(x_{i-1})\big]$$

Step 5_α: Hope the Riemann–Stieltjes sums converge.
A function $f:[a,b]\to\mathbb{R}$ is said to be *Riemann integrable with respect to* α on $[a,b]$, denoted "$f\in\mathcal{R}(\alpha)$ on $[a,b]$", if there exists an $I\in\mathbb{R}$ such that

$$\begin{aligned}\forall\,\varepsilon>0\ \ \exists\,\text{a partition } \mathbb{P}_\varepsilon \text{ of } [a,b]\ \text{ s.t. }\ & \big|S(\mathbb{P},\mathbb{T},f,\alpha)-I\big|<\varepsilon\\ & \text{for all partitions } \mathbb{P} \text{ of } [a,b] \text{ finer than } \mathbb{P}_\varepsilon\\ & \text{and all choices } \mathbb{T} \text{ for } \mathbb{P}\end{aligned}$$

If so, I is denoted $\int_a^b f(x)\,d\alpha(x)$ or $\int_a^b f\,d\alpha$ and we write

$$\lim_{\mathbb{P}} S(\mathbb{P},\mathbb{T},f,\alpha)=\int_a^b f(x)\,d\alpha(x)$$

Remark.

1. If $\alpha(x)=x$, then the Riemann–Stieltjes integral $\int_a^b f(x)\,d\alpha(x)$ reduces to the Riemann integral $\int_a^b f(x)\,dx$.
2. We shall eventually prove that
   (a) If α is monotonic and f is continuous, then $f\in\mathcal{R}(\alpha)$.
   (b) If α is continuous and f is monotonic, then $f\in\mathcal{R}(\alpha)$.
   (c) If α is strictly monotonic and f is unbounded, then $f\notin\mathcal{R}(\alpha)$.
3. Observe that

$$\begin{aligned}S(\mathbb{P},\mathbb{T},f,\alpha)&=\sum_{i=1}^{n} f(t_i)\,\frac{\alpha(x_i)-\alpha(x_{i-1})}{x_i-x_{i-1}}\,\big[x_i-x_{i-1}\big]\\ &\approx\sum_{i=1}^{n} f(t_i)\,\alpha'(t_i)\,\big[x_i-x_{i-1}\big]\end{aligned}$$

We shall eventually prove that if α has a continuous derivative, then

$$\int_a^b f(x)\,d\alpha(x)=\int_a^b f(x)\alpha'(x)\,dx$$

Example.

1. If $\alpha(b)\neq\alpha(a)$ and

$$f(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

then $f\notin\mathcal{R}(\alpha)$ on $[a,b]$ because, for any choice $\mathbb{T}$ consisting only of rational points,

$$S(\mathbb{P},\mathbb{T},f,\alpha)=\sum_{i=1}^{n} f(t_i)\,\big[\alpha(x_i)-\alpha(x_{i-1})\big]=\sum_{i=1}^{n} 1\times\big[\alpha(x_i)-\alpha(x_{i-1})\big]=\alpha(b)-\alpha(a)$$

while, for any choice $\mathbb{T}$ consisting only of irrational points,

$$S(\mathbb{P},\mathbb{T},f,\alpha)=\sum_{i=1}^{n}f(t_i)\left[\alpha(x_i)-\alpha(x_{i-1})\right]=\sum_{i=1}^{n}0\times\left[\alpha(x_i)-\alpha(x_{i-1})\right]=0$$

So if $\varepsilon<\frac{\alpha(b)-\alpha(a)}{2}$, there is no number I such that both of these partial sums are within ε of I, no matter what $\mathbb{P}$ is.

2. If $\alpha(x)=x$ and

$$f(x)=\begin{cases}\frac{1}{n} & \text{if } x \text{ is rational with } x=\frac{m}{n} \text{ in lowest terms}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

then $f\in\mathcal{R}$ on $[a,b]$ and $\int_a^b f(x)\,dx=0$.

Proof: Let $\varepsilon>0$. There are only finitely many, say N, points in $[a,b]$ with $f(x)>\frac{\varepsilon}{2(b-a)}$. Let $\mathbb{P}$ be any partition with mesh $\|\mathbb{P}\|<\frac{\varepsilon}{4N}$.

- For any subinterval that does contain a point x where $f(x)>\frac{\varepsilon}{2(b-a)}$

$$|f(t_i)|\,[x_i-x_{i-1}]\le 1\cdot\|\mathbb{P}\|<\frac{\varepsilon}{4N}$$

  There are at most $2N$ such subintervals, because each of the N points with $f(x)>\frac{\varepsilon}{2(b-a)}$ can appear in at most 2 subintervals.
- For any subinterval that does not contain a point x where $f(x)>\frac{\varepsilon}{2(b-a)}$

$$|f(t_i)|\,[x_i-x_{i-1}]\le\frac{\varepsilon}{2(b-a)}[x_i-x_{i-1}]$$

so that

$$\left|S(\mathbb{P},\mathbb{T},f)\right|<2N\cdot\frac{\varepsilon}{4N}+\sum_i\frac{\varepsilon}{2(b-a)}[x_i-x_{i-1}]\le\varepsilon$$

■

3. The constant function $1\in\mathcal{R}(\alpha)$ on $[a,b]$ and

$$\int_a^b d\alpha=\alpha(b)-\alpha(a)$$

This is true for any function $\alpha:[a,b]\to\mathbb{R}$, because

$$S(\mathbb{P},\mathbb{T},1,\alpha)=\sum_{i=1}^{n}1\left[\alpha(x_i)-\alpha(x_{i-1}\right]=\alpha(b)-\alpha(a)$$

for any partition $\mathbb{P}$ and any choice $\mathbb{T}$.