A Definition of the Riemann–Stieltjes Integral

Let $a < b$ and let $f, \alpha : [a, b] \to \mathbb{R}$. In these notes I will state one of several closely related, but not 100% equivalent, standard definitions of the Riemann–Stieltjes integral $\int_a^b f(x) \, d\alpha(x)$. Let’s start by reviewing the first year Calculus definition of the Riemann integral $\int_a^b f(x) \, dx$. The definition goes in several steps.

**Step 1:** Slice up the interval $[a, b]$ into a finite number of subintervals.
A partition of $[a, b]$ is a finite set $P = \{x_0, x_1, \cdots, x_n\}$ of points with

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

**Step 2:** For each $1 \leq i \leq n$, choose a point to represent the subinterval $[x_{i-1}, x_i]$.
A choice for the partition $P$ is a finite set $T = \{t_1, \cdots, t_n\}$ of points with

$$x_{i-1} \leq t_i \leq x_i \quad \text{for each} \quad 1 \leq i \leq n$$

**Step 3:** Calculate an approximate value for the integral, by approximating the $i$th strip by a rectangle of width $x_i - x_{i-1}$ and height $f(t_i)$.
The Riemann sum with partition $P$ and choice $T$ is

$$S(P, T, f) = \sum_{i=1}^{n} f(t_i) (x_i - x_{i-1})$$

**Step 4:** Make the partition finer and finer.
A partition $P'$ is said to be finer than the partition $P$ if $P' \supset P$.

**Step 5:** Hope the Riemann sums converge.
A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable on $[a, b]$, denoted “$f \in \mathcal{R}$ on $[a, b]$”, if there exists an $I \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \ \exists \text{a partition } P_{\varepsilon} \text{ of } [a, b] \quad \text{s.t.} \quad |S(P, T, f) - I| < \varepsilon$$

for all partitions $P$ of $[a, b]$ finer than $P_{\varepsilon}$

and all choices $T$ for $P$. 

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If so, $I$ is denoted $\int_{a}^{b} f(x) \, dx$ and we write
\[
\lim_{P \to \infty} S(P, T, f) = \int_{a}^{b} f(x) \, dx
\]
To generalize this to the Riemann–Stieltjes integral $\int_{a}^{b} f(x) \, d\alpha(x)$ we simply replace $x_i - x_{i-1}$ with $\alpha(x_i) - \alpha(x_{i-1})$. That is, Steps 1, 2 and 4 are left as is and Steps 3 and 5 are replaced with

**Step 3:** *Calculate an approximate value for the integral.*

The Riemann–Stieltjes sum with partition $P$ and choice $T$ is
\[
S(P, T, f, \alpha) = \sum_{i=1}^{n} f(t_i) \left[ \alpha(x_i) - \alpha(x_{i-1}) \right]
\]

**Step 5:** *Hope the Riemann–Stieltjes sums converge.*

A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable with respect to $\alpha$ on $[a, b]$, denoted “$f \in \mathcal{R}(\alpha)$ on $[a, b]$”, if there exists an $I \in \mathbb{R}$ such that
\[
\forall \varepsilon > 0 \, \exists \text{ a partition } P_{\varepsilon} \text{ of } [a, b] \text{ s.t. } |S(P_{\varepsilon}, f, \alpha) - I| < \varepsilon
\]
for all partitions $P$ of $[a, b]$ finer than $P_{\varepsilon}$ and all choices $T$ for $P$.

If so, $I$ is denoted $\int_{a}^{b} f(x) \, d\alpha(x)$ or $\int_{a}^{b} f \, d\alpha$ and we write
\[
\lim_{P \to \infty} S(P, T, f, \alpha) = \int_{a}^{b} f(x) \, d\alpha(x)
\]

**Remark.**

1. If $\alpha(x) = x$, then the Riemann–Stieltjes integral $\int_{a}^{b} f(x) \, d\alpha(x)$ reduces to the Riemann integral $\int_{a}^{b} f(x) \, dx$.

2. We shall eventually prove that
   (a) If $\alpha$ is monotonic and $f$ is continuous, then $f \in \mathcal{R}(\alpha)$.
   (b) If $\alpha$ is continuous and $f$ is monotonic, then $f \in \mathcal{R}(\alpha)$.
   (c) If $\alpha$ is strictly monotonic and $f$ is unbounded, then $f \not\in \mathcal{R}(\alpha)$.

3. Observe that
\[
S(P, T, f, \alpha) = \sum_{i=1}^{n} f(t_i) \left[ \frac{\alpha(x_i) - \alpha(x_{i-1})}{x_i - x_{i-1}} \right] [x_i - x_{i-1}]
\]
\[
\approx \sum_{i=1}^{n} f(t_i) \alpha'(t_i) [x_i - x_{i-1}]
\]
We shall eventually prove that if $\alpha$ has a continuous derivative, then
\[
\int_{a}^{b} f(x) \, d\alpha(x) = \int_{a}^{b} f(x) \alpha'(x) \, dx
\]
Example.

1. If \( \alpha(b) \neq \alpha(a) \) and
\[
f(x) = \begin{cases} 
1 & \text{if } x \text{ is rational} \\
0 & \text{if } x \text{ is irrational}
\end{cases}
\]
then \( f \notin \mathcal{R}(\alpha) \) on \([a, b]\) because, for any choice \( T \) consisting only of rational points,
\[
S(\mathcal{P}, T, f, \alpha) = \sum_{i=1}^{n} f(t_i) \left[ \alpha(x_i) - \alpha(x_{i-1}) \right] = \sum_{i=1}^{n} 1 \times \left[ \alpha(x_i) - \alpha(x_{i-1}) \right] = \alpha(b) - \alpha(a)
\]
while, for any choice \( T \) consisting only of irrational points,
\[
S(\mathcal{P}, T, f, \alpha) = \sum_{i=1}^{n} f(t_i) \left[ \alpha(x_i) - \alpha(x_{i-1}) \right] = \sum_{i=1}^{n} 0 \times \left[ \alpha(x_i) - \alpha(x_{i-1}) \right] = 0
\]

2. If \( \alpha(x) = x \) and
\[
f(x) = \begin{cases} 
\frac{1}{n} & \text{if } x \text{ is rational with } x = \frac{m}{n} \text{ in lowest terms} \\
0 & \text{if } x \text{ is irrational}
\end{cases}
\]
then \( f \in \mathcal{R} \) on \([a, b]\) and \( \int_{a}^{b} f(x) \, dx = 0 \).

Proof: Let \( \varepsilon > 0 \). There are only finitely many, say \( N \), points in \([a, b]\) with \( f(x) > \frac{\varepsilon}{2(b-a)} \). Let \( \mathcal{P} \) be any partition with mesh \( ||\mathcal{P}|| \leq \frac{\varepsilon}{4N} \). Any \( x \in [a, b] \) can appear in at most two subintervals, \( x_{i-1} \leq x \leq x_i \), of \( \mathcal{P} \). So most \( 2N \) subintervals \( x_{i-1} \leq x \leq x_i \) of \( \mathcal{P} \) can contain a point \( x \) where \( f(x) > \frac{\varepsilon}{2} \).

\( \circ \) For any subinterval that does contain a point \( x \) where \( f(x) > \frac{\varepsilon}{2} \)
\[
|f(t_i)| |x_i - x_{i-1}| \leq 1 \cdot ||\mathcal{P}|| < \frac{\varepsilon}{4N}
\]

\( \circ \) and for any subinterval that does not contain a point \( x \) where \( f(x) > \frac{\varepsilon}{2(b-a)} \)
\[
|f(t_i)| |x_i - x_{i-1}| \leq \frac{\varepsilon}{2(b-a)} |x_i - x_{i-1}|
\]
so that
\[
|S(\mathcal{P}, T, f)| < 2N \cdot \frac{\varepsilon}{4N} + \sum_{i} \frac{\varepsilon}{2(b-a)} |x_i - x_{i-1}| \leq \varepsilon
\]