Alternate Proof of Integrability

Theorem. *Let $\alpha : [a,b] \to \mathbb{R}$ be monotone and $f : [a,b] \to \mathbb{R}$ be continuous. Then $f \in \mathcal{R}(\alpha)$ on $[a,b]$. That is, the integral $\int_a^b f d\alpha$ exists.*

Proof: If $\alpha(b) = \alpha(a)$, i.e. α is constant, $\int_a^b f d\alpha$ exists and is zero trivially. So assume that $\alpha(b) \neq \alpha(a)$. We use the Cauchy criterion, which says

> Let $\alpha, f : [a,b] \to \mathbb{R}$. Then f is integrable with respect to α on $[a,b]$ (i.e. $f \in \mathcal{R}(\alpha)$ on $[a,b]$) if and only if for every $\varepsilon > 0$ there is a partition $\mathbb{P}_\varepsilon$ of $[a,b]$ such that
>
> $$\big|S(\mathbb{P},\mathbb{T},f,\alpha) - S(\mathbb{P}',\mathbb{T}',f,\alpha)\big| < \varepsilon$$
>
> for all partitions $\mathbb{P},\mathbb{P}' \supset \mathbb{P}_\varepsilon$ and all choices $\mathbb{T},\mathbb{T}'$ for $\mathbb{P},\mathbb{P}'$, respectively.

Your verified this criterion in Problem Set 2, #2. So we let $\varepsilon > 0$ and must find a partition $\mathbb{P}_\varepsilon$ of $[a,b]$ such that $\big|S(\mathbb{P},\mathbb{T},f,\alpha) - S(\mathbb{P}',\mathbb{T}',f,\alpha)\big| < \varepsilon$ for all partitions $\mathbb{P} \supset \mathbb{P}_\varepsilon$ and $\mathbb{P}' \supset \mathbb{P}_\varepsilon$ and all choices $\mathbb{T}$, $\mathbb{T}'$ for $\mathbb{P}$, $\mathbb{P}'$, respectively. Since f is continuous on the compact set $[a,b]$, it is uniformly continuous. So there is a $\delta > 0$ such that $\big|f(t) - f(s)\big| < \frac{\varepsilon}{2|\alpha(b)-\alpha(a)|}$ for all $s,t \in [a,b]$ with $|s-t| < \delta$. We choose for $\mathbb{P}_\varepsilon$ any partition of $[a,b]$ with mesh $\|\mathbb{P}_\varepsilon\| < \delta$.

Let $\mathbb{P} \supset \mathbb{P}_\varepsilon$ and $\mathbb{P}' \supset \mathbb{P}_\varepsilon$ be partitions of $[a,b]$ and $\mathbb{T}$ and $\mathbb{T}'$ be choices for $\mathbb{P}$ and $\mathbb{P}'$ respectively. Set $\bar{\mathbb{P}} = \mathbb{P} \cup \mathbb{P}'$ and let $\bar{\mathbb{T}}$ be any choice for $\bar{\mathbb{P}}$. It suffices to prove that $\big|S(\mathbb{P},\mathbb{T},f,\alpha) - S(\bar{\mathbb{P}},\bar{\mathbb{T}},f,\alpha)\big| < \frac{\varepsilon}{2}$ and $\big|S(\mathbb{P}',\mathbb{T}',f,\alpha) - S(\bar{\mathbb{P}},\bar{\mathbb{T}},f,\alpha)\big| < \frac{\varepsilon}{2}$. We'll prove the first inequality. To prove the second, just add primes. Suppose that $\mathbb{P} = \{x_0, x_1, \cdots, x_n\}$. Concentrate on the contributions to $S(\mathbb{P},\mathbb{T},f,\alpha)$ and $S(\bar{\mathbb{P}},\bar{\mathbb{T}},f,\alpha)$ from $[x_{i-1},x_i]$, for some $1 \le i \le n$. For $S(\mathbb{P},\mathbb{T},f,\alpha)$, that contribution is

$$C_{\mathbb{P},i} = f(t_i)\big[\alpha(x_i) - \alpha(x_{i-1})\big]$$

If $\bar{\mathbb{P}} \cap [x_{i-1},x_i] = \big\{x_{i-1}, y_1, \cdots, y_{m-1}, x_i\big\}$, the corresponding contribution for $S(\bar{\mathbb{P}},\bar{\mathbb{T}},f,\alpha)$ is

$$C_{\bar{\mathbb{P}},i} = \sum_{j=1}^{m} f(s_j)\big[\alpha(y_j) - \alpha(y_{j-1})\big]$$

where, for notational convenience, we have set $y_0 = x_{i-1}$ and $y_m = x_i$. Of course s_j is the element of the choice $\bar{T}$ for the interval $[y_{j-1},y_j]$. If $\alpha(x_i) = \alpha(x_{i-1})$, i.e. α is constant on the interval $[x_{i-1},x_i]$, then $C_{\mathbb{P},i} = C_{\bar{\mathbb{P}},i} = 0$. So assume that $\alpha(x_i) \neq \alpha(x_{i-1})$. The difference between these two contributions is

$$\begin{aligned}
C_{\mathbb{P},i} - C_{\bar{\mathbb{P}},i} &= f(t_i)\big[\alpha(x_i) - \alpha(x_{i-1})\big] - \sum_{j=1}^{m} f(s_j)\big[\alpha(y_j) - \alpha(y_{j-1})\big] \\
&= \sum_{j=1}^{m} f(t_i)\big[\alpha(y_j) - \alpha(y_{j-1})\big] - \sum_{j=1}^{m} f(s_j)\big[\alpha(y_j) - \alpha(y_{j-1})\big] \\
&= \sum_{j=1}^{m} \big[f(t_i) - f(s_j)\big]\big[\alpha(y_j) - \alpha(y_{j-1})\big]
\end{aligned}$$

Since $t_i, s_1, \cdots, s_m \in [x_{i-1},x_i]$ and $|x_i - x_{i-1}| < \delta$, we have $|s_j - t_i| < \delta$ and hence $\big|f(t_i) - f(s_j)\big| < \frac{\varepsilon}{2|\alpha(b)-\alpha(a)|}$ for every $1 \le j \le m$. Hence

$$\big|C_{\mathbb{P},i} - C_{\bar{\mathbb{P}},i}\big| < \sum_{j=1}^{m} \tfrac{\varepsilon}{2|\alpha(b)-\alpha(a)|}\big|\alpha(y_j) - \alpha(y_{j-1})\big|$$

Since α is monotonic, the sign of $\alpha(y_j) - \alpha(y_{j-1})$ is independent of j so that $\sum_{j=1}^{m} \big|\alpha(y_j) - \alpha(y_{j-1})\big| = \big|\alpha(x_i) - \alpha(x_{i-1})\big|$ and $\big|C_{\mathbb{P},i} - C_{\bar{\mathbb{P}},i}\big| < \frac{\varepsilon}{2|\alpha(b)-\alpha(a)|}\big|\alpha(x_i) - \alpha(x_{i-1})\big|$. Adding up the contributions from $[x_{i-1}, x_i]$ for $1 \le i \le n$,

$$\begin{aligned}\big|S(\mathbb{P},\mathbb{T},f,\alpha) - S(\bar{\mathbb{P}},\bar{\mathbb{T}},f,\alpha)\big| &\le \sum_{i=1}^{n} \big|C_{\mathbb{P},i} - C_{\bar{\mathbb{P}},i}\big| < \sum_{i=1}^{n} \tfrac{\varepsilon}{2|\alpha(b)-\alpha(a)|}\big|\alpha(x_i) - \alpha(x_{i-1})\big| \\ &= \tfrac{\varepsilon}{2|\alpha(b)-\alpha(a)|}|\alpha(b) - \alpha(a)| = \tfrac{\varepsilon}{2}\end{aligned}$$

as desired. ■