Alternate Proof of Integrability

**Theorem.** Let $\alpha : [a, b] \to \mathbb{R}$ be monotone and $f : [a, b] \to \mathbb{R}$ be continuous. Then $f \in \mathcal{R}(\alpha)$ on $[a, b]$. That is, the integral $\int_a^b f \, d\alpha$ exists.

**Proof:** If $\alpha(b) = \alpha(a)$, i.e. $\alpha$ is constant, $\int_a^b f \, d\alpha$ exists and is zero trivially. So assume that $\alpha(b) \neq \alpha(a)$. We use the Cauchy criterion, which says

Let $\alpha, f : [a, b] \to \mathbb{R}$. Then $f$ is integrable with respect to $\alpha$ on $[a, b]$ (i.e. $f \in \mathcal{R}(\alpha)$ on $[a, b]$) if and only if for every $\varepsilon > 0$ there is a partition $I_P$ of $[a, b]$ such that

$$\left| S(I_P, \mathbb{T}, f, \alpha) - S(I_P', \mathbb{T}', f, \alpha) \right| < \varepsilon$$

for all partitions $I_P, I_P' \supset I_\varepsilon$ and all choices $\mathbb{T}, \mathbb{T}'$ for $I_P, I_P'$, respectively.

Your verified this criterion in Problem Set 2, #2. So we let $\varepsilon > 0$ and must find a partition $I_\varepsilon$ of $[a, b]$ such that $\left| S(I_P, \mathbb{T}, f, \alpha) - S(I_P', \mathbb{T}', f, \alpha) \right| < \varepsilon$ for all partitions $I_P \supset I_\varepsilon$ and $I_P' \supset I_\varepsilon$ and all choices $\mathbb{T}, \mathbb{T}'$ for $I_P, I_P'$, respectively. Since $f$ is continuous on the compact set $[a, b]$, it is uniformly continuous. So there is a $\delta > 0$ such that $|f(t) - f(s)| < \frac{\varepsilon}{2|\alpha(b) - \alpha(a)|}$ for all $s, t \in [a, b]$ with $|s - t| < \delta$. We choose for $I_\varepsilon$ any partition of $[a, b]$ with mesh $\|I_\varepsilon\| < \delta$.

Let $I_P \supset I_\varepsilon$ and $I_P' \supset I_\varepsilon$ be partitions of $[a, b]$ and $\mathbb{T}$ and $\mathbb{T}'$ be choices for $I_P$ and $I_P'$ respectively. Set $I_P = I_P \cup I_P'$ and let $\mathbb{T}$ be any choice for $I_P$. It suffices to prove that $|S(I_P, \mathbb{T}, f, \alpha) - S(I_P', \mathbb{T}', f, \alpha)| < \frac{\varepsilon}{2}$ and $|S(I_P', \mathbb{T}', f, \alpha) - S(I_P, \mathbb{T}, f, \alpha)| < \frac{\varepsilon}{2}$. We’ll prove the first inequality. To prove the second, just add primes.

Suppose that $I_P = \{x_0, x_1, \ldots, x_n\}$. Concentrate on the contributions to $S(I_P, \mathbb{T}, f, \alpha)$ and $S(I_P', \mathbb{T}, f, \alpha)$ from $[x_{i-1}, x_i]$, for some $1 \leq i \leq n$. For $S(I_P, \mathbb{T}, f, \alpha)$, that contribution is

$$C_{I_P, i} = f(t_i) [\alpha(x_i) - \alpha(x_{i-1})]$$

If $I_P \cap [x_{i-1}, x_i] = \{x_{i-1}, y_1, \ldots, y_{m-1}, x_i\}$, the corresponding contribution for $S(I_P', \mathbb{T}, f, \alpha)$ is

$$C_{I_P', i} = \sum_{j=1}^{m} f(s_j) [\alpha(y_j) - \alpha(y_{j-1})]$$

where, for notational convenience, we have set $y_0 = x_{i-1}$ and $y_m = x_i$. Of course $s_j$ is the element of the choice $\mathbb{T}$ for the interval $[y_{j-1}, y_j]$. If $\alpha(x_i) = \alpha(x_{i-1})$, i.e. $\alpha$ is constant on the interval $[x_{i-1}, x_i]$, then $C_{I_P, i} = C_{I_P', i} = 0$. So assume that $\alpha(x_i) \neq \alpha(x_{i-1})$. The difference between these two contributions is

$$C_{I_P, i} - C_{I_P', i} = f(t_i) [\alpha(x_i) - \alpha(x_{i-1})] - \sum_{j=1}^{m} f(s_j) [\alpha(y_j) - \alpha(y_{j-1})]$$

$$= \sum_{j=1}^{m} f(t_i) [\alpha(y_j) - \alpha(y_{j-1})] - \sum_{j=1}^{m} f(s_j) [\alpha(y_j) - \alpha(y_{j-1})]$$

$$= \sum_{j=1}^{m} [f(t_i) - f(s_j)] [\alpha(y_j) - \alpha(y_{j-1})]$$

Since $t_i, s_1, \ldots, s_m \in [x_{i-1}, x_i]$ and $|x_i - x_{i-1}| < \delta$, we have $|s_j - t_i| < \delta$ and hence $|f(t_i) - f(s_j)| < \frac{\varepsilon}{2|\alpha(b) - \alpha(a)|}$ for every $1 \leq j \leq m$. Hence

$$|C_{I_P, i} - C_{I_P', i}| < \sum_{j=1}^{m} \frac{\varepsilon}{2|\alpha(b) - \alpha(a)|} |\alpha(y_j) - \alpha(y_{j-1})|$$

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Since $\alpha$ is monotonic, the sign of $\alpha(y_j) - \alpha(y_{j-1})$ is independent of $j$ so that $\sum_{j=1}^{n} |\alpha(y_j) - \alpha(y_{j-1})| = |\alpha(x_i) - \alpha(x_{i-1})|$ and $|C_{\mathcal{P},i} - C_{\mathcal{P}^\top,i}| < \frac{\varepsilon}{2|\alpha(b) - \alpha(a)|} |\alpha(x_i) - \alpha(x_{i-1})|$. Adding up the contributions from $[x_{i-1}, x_i]$ for $1 \leq i \leq n$,

$$
|S(\mathcal{P}, T, f, \alpha) - S(\mathcal{P}^\top, T, f, \alpha)| \leq \sum_{i=1}^{n} |C_{\mathcal{P},i} - C_{\mathcal{P}^\top,i}| < \sum_{i=1}^{n} \frac{\varepsilon}{2|\alpha(b) - \alpha(a)|} |\alpha(x_i) - \alpha(x_{i-1})|
$$

$$
= \frac{\varepsilon}{2|\alpha(b) - \alpha(a)|} |\alpha(b) - \alpha(a)| = \frac{\varepsilon}{2}
$$

as desired.