Alternate Proof of Integrability

Theorem. *Let* $\alpha : [a,b] \to \mathbb{R}$ *be monotone and* $f : [a,b] \to \mathbb{R}$ *be continuous. Then* $f \in \mathcal{R}(\alpha)$ *on* $[a,b]$*. That is, the integral* $\int_a^b f d\alpha$ *exists.*

Proof: We use the Cauchy criterion, which says

> Let $\alpha, f : [a,b] \to \mathbb{R}$. Then f is integrable with respect to α on $[a,b]$ (i.e. $f \in \mathcal{R}(\alpha)$ on $[a,b]$) if and only if for every $\varepsilon > 0$ there is a partition P_ε of $[a,b]$ such that
>
> $$\big|S(P_1,T_1,f,\alpha) - S(P_2,T_2,f,\alpha)\big| < \varepsilon$$
>
> for all partitions $P_1, P_2 \supset P_\varepsilon$ and all choices T_1, T_2 for P_1, P_2, respectively.

Your verified this criterion in Problem Set 2, #1. So we let $\varepsilon > 0$ and must find a partition P_ε of $[a,b]$ such that $\big|S(P,T,f,\alpha) - S(P',T',f,\alpha)\big| \le \varepsilon$ for all partitions $P \supset P_\varepsilon$ and $P' \supset P_\varepsilon$ and all choices T, T' for P, P', respectively. Since f is continuous on the compact set $[a,b]$, it is uniformly continuous. So there is a $\delta > 0$ such that $\big|f(t) - f(s)\big| \le \frac{\varepsilon}{2|\alpha(b)-\alpha(a)|}$ for all $s,t \in [a,b]$ with $|s-t| < \delta$. We choose for P_ε any partition of $[a,b]$ with $\|P_\varepsilon\| \le \delta$.

Let $P \supset P_\varepsilon$ and $P' \supset P_\varepsilon$ be partitions of $[a,b]$ and T and T' be choices for P and P' respectively. Set $Q = P \cup P'$ and let S be any choice for Q. It suffices to prove that $\big|S(P,T,f,\alpha) - S(Q,S,f,\alpha)\big| \le \frac{\varepsilon}{2}$ and $\big|S(P',T',f,\alpha) - S(Q,S,f,\alpha)\big| \le \frac{\varepsilon}{2}$. We'll prove the first inequality. To prove the second, just add primes. Suppose that $P = \{x_0, x_1, \cdots, x_n\}$. Concentrate on the contributions to $S(P,T,f,\alpha)$ and $S(Q,S,f,\alpha)$ from $[x_{i-1}, x_i]$, for some $1 \le i \le n$. For $S(P,T,f,\alpha)$, that contribution is

$$C_{P,i} = f(t_i)\big[\alpha(x_i) - \alpha(x_{i-1})\big]$$

If $Q \cap [x_{i-1}, x_i] = \big\{x_{i-1}, y_1, \cdots, y_{m-1}, x_i\big\}$, the corresponding contribution for $S(Q,S,f,\alpha)$ is

$$C_{Q,i} = \sum_{j=1}^{m} f(s_j)\big[\alpha(y_j) - \alpha(y_{j-1})\big]$$

where, for notational convenience, we have set $y_0 = x_{i-1}$ and $y_m = x_i$. The difference between these two contributions is

$$\begin{aligned} C_{P,i} - C_{Q,i} &= f(t_i)\big[\alpha(x_i) - \alpha(x_{i-1})\big] - \sum_{j=1}^{m} f(s_j)\big[\alpha(y_j) - \alpha(y_{j-1})\big] \\ &= \sum_{j=1}^{m} f(t_i)\big[\alpha(y_j) - \alpha(y_{j-1})\big] - \sum_{j=1}^{m} f(s_j)\big[\alpha(y_j) - \alpha(y_{j-1})\big] \\ &= \sum_{j=1}^{m} \big[f(t_i) - f(s_j)\big]\big[\alpha(y_j) - \alpha(y_{j-1})\big] \end{aligned}$$

Since $t_i, s_1, \cdots, s_m \in [x_{i-1}, x_i]$ and $|x_i - x_{i-1}| \le \delta$, we have $|s_j - t_i| \le \delta$ and hence $\big|f(t_i) - f(s_j)\big| \le \frac{\varepsilon}{2|\alpha(b)-\alpha(a)|}$ for every $1 \le j \le m$. Hence

$$\big|C_{P,i} - C_{Q,i}\big| \le \sum_{j=1}^{m} \tfrac{\varepsilon}{2|\alpha(b)-\alpha(a)|}\big|\alpha(y_j) - \alpha(y_{j-1})\big|$$

Since α is monotonic, the sign of $\alpha(y_j) - \alpha(y_{j-1})$ is independent of j so that $\sum_{j=1}^{m} \big|\alpha(y_j) - \alpha(y_{j-1})\big| = \big|\alpha(x_i) - \alpha(x_{i-1})\big|$ and $\big|C_{P,i} - C_{Q,i}\big| \le \frac{\varepsilon}{2|\alpha(b)-\alpha(a)|}\big|\alpha(x_i) - \alpha(x_{i-1})\big|$. Adding up the contributions from $[x_{i-1}, x_i]$ for $1 \le i \le n$,

$$\begin{aligned} \big|S(P,T,f,\alpha) - S(Q,S,f,\alpha)\big| &\le \sum_{i=1}^{n} \big|C_{P,i} - C_{Q,i}\big| \le \sum_{i=1}^{n} \tfrac{\varepsilon}{2|\alpha(b)-\alpha(a)|}\big|\alpha(x_i) - \alpha(x_{i-1})\big| \\ &= \tfrac{\varepsilon}{2|\alpha(b)-\alpha(a)|}|\alpha(b) - \alpha(a)| = \tfrac{\varepsilon}{2} \end{aligned}$$

as desired. ■