Alternate Proof of Integrability

Theorem. Let \( \alpha : [a, b] \to \mathbb{R} \) be monotone and \( f : [a, b] \to \mathbb{R} \) be continuous. Then \( f \in \mathcal{R}(\alpha) \) on \([a, b] \). That is, the integral \( \int_{a}^{b} f \, d\alpha \) exists.

Proof: We use the Cauchy criterion, which says

"Your verified this criterion in Problem Set 2, #1. So we let \( \varepsilon > 0 \) such that suppose that for every \( 1 \leq m \leq n \) for every \( \varepsilon > 0 \) for every \( \alpha \in [a, b] \) such that for every \( \varepsilon > 0 \) for every \( \alpha \in [a, b] \) such that \( |S(P_1, T_1, f, \alpha) - S(P_2, T_2, f, \alpha)| < \varepsilon \)

for all partitions \( P_1, P_2 \supset P_\varepsilon \) and all choices \( T_1, T_2 \) for \( P_1, P_2 \), respectively.

Your verified this criterion in Problem Set 2, #1. So we let \( \varepsilon > 0 \) and must find a partition \( P_\varepsilon \) of \([a, b] \) such that \( |S(P, T, f, \alpha) - S(P', T', f, \alpha)| \leq \varepsilon \) for all partitions \( P \supset P_\varepsilon \) and \( P' \supset P_\varepsilon \) and all choices \( T, T' \) for \( P, P' \), respectively. Since \( f \) is continuous on the compact set \([a, b] \), it is uniformly continuous. So there is a \( \delta > 0 \) such that \( |f(t) - f(s)| \leq \frac{\varepsilon}{2|\alpha(b) - \alpha(a)|} \) for all \( s, t \in [a, b] \) with \( |s - t| < \delta \). We choose for \( P_\varepsilon \) any partition of \([a, b] \) with \(|P_\varepsilon| \leq \delta \).

Let \( P \supset P_\varepsilon \) and \( P' \supset P_\varepsilon \) be partitions of \([a, b] \) and \( T \) and \( T' \) be choices for \( P \) and \( P' \), respectively. Set \( Q = P \cup P' \) and let \( S \) be any choice for \( Q \). It suffices to prove that \( |S(P, T, f, \alpha) - S(Q, S, f, \alpha)| \leq \frac{\varepsilon}{2} \) and \( |S(P', T', f, \alpha) - S(Q, S, f, \alpha)| \leq \frac{\varepsilon}{2} \). We'll prove the first inequality. To prove the second, just add primes. Suppose that \( P = \{x_0, x_1, \ldots, x_n\} \). Concentrate on the contributions to \( S(P, T, f, \alpha) \) and \( S(Q, S, f, \alpha) \) from \([x_{i-1}, x_i] \), for some \( 1 \leq i \leq n \). For \( S(P, T, f, \alpha) \), the contribution is

\[
C_{P,i} = f(t_i) [\alpha(x_i) - \alpha(x_{i-1})]
\]

If \( Q \cap [x_{i-1}, x_i] = \{x_{i-1}, y_1, \ldots, y_{m-1}, x_i\} \), the corresponding contribution for \( S(Q, S, f, \alpha) \) is

\[
C_{Q,i} = \sum_{j=1}^{m} f(s_j) [\alpha(y_j) - \alpha(y_{j-1})]
\]

where, for notational convenience, we have set \( y_0 = x_{i-1} \) and \( y_m = x_i \). The difference between these two contributions is

\[
C_{P,i} - C_{Q,i} = f(t_i) [\alpha(x_i) - \alpha(x_{i-1})] - \sum_{j=1}^{m} f(s_j) [\alpha(y_j) - \alpha(y_{j-1})]
\]

\[
= \sum_{j=1}^{m} f(t_i) [\alpha(y_j) - \alpha(y_{j-1})] - \sum_{j=1}^{m} f(s_j) [\alpha(y_j) - \alpha(y_{j-1})]
\]

\[
= \sum_{j=1}^{m} [f(t_i) - f(s_j)] [\alpha(y_j) - \alpha(y_{j-1})]
\]

Since \( t_i, s_j, \ldots, s_m \in [x_{i-1}, x_i] \) and \( |x_i - x_{i-1}| \leq \delta \), we have \( |s_j - t_i| \leq \delta \) and hence \( |f(t_i) - f(s_j)| \leq \frac{\varepsilon}{2|\alpha(b) - \alpha(a)|} \) for every \( 1 \leq j \leq m \). Hence

\[
|C_{P,i} - C_{Q,i}| \leq \sum_{j=1}^{m} \frac{\varepsilon}{2|\alpha(b) - \alpha(a)|} |\alpha(y_j) - \alpha(y_{j-1})|
\]
Since $\alpha$ is monotonic, the sign of $\alpha(y_j) - \alpha(y_{j-1})$ is independent of $j$ so that $\sum_{j=1}^{m} |\alpha(y_j) - \alpha(y_{j-1})| = |\alpha(x_i) - \alpha(x_{i-1})|$ and $|C_{P,i} - C_{Q,i}| \leq \frac{\varepsilon}{2|\alpha(b) - \alpha(a)|} |\alpha(x_i) - \alpha(x_{i-1})|$. Adding up the contributions from $[x_{i-1}, x_i]$ for $1 \leq i \leq n$,

$$|S(P, T, f, \alpha) - S(Q, S, f, \alpha)| \leq \sum_{i=1}^{n} |C_{P,i} - C_{Q,i}| \leq \sum_{i=1}^{n} \frac{\varepsilon}{2|\alpha(b) - \alpha(a)|} |\alpha(x_i) - \alpha(x_{i-1})|$$

$$= \frac{\varepsilon}{2|\alpha(b) - \alpha(a)|} |\alpha(b) - \alpha(a)| = \frac{\varepsilon}{2}$$

as desired.