Problem Solutions for “Integration on Manifolds”

Problem M.1 Let $\mathcal{A}$ be an atlas for the metric space $\mathcal{M}$. Prove that there is a unique maximal atlas for $\mathcal{M}$ that contains $\mathcal{A}$.

Solution. Define $\mathfrak{A}$ to be the set of all charts $\{V, \psi\}$ that are compatible with all of the charts of $\mathcal{A}$. If we can show that $\mathfrak{A}$ is an atlas, we are done, because (a) $\mathfrak{A}$ is an atlas, so that each chart in $\mathfrak{A}$ is compatible with all charts of $\mathcal{A}$ and hence $\mathfrak{A} \subset \mathcal{A}$ and (b) any chart in any atlas that contains $\mathcal{A}$ must be compatible with every chart of $\mathcal{A}$ and hence every atlas that contains $\mathcal{A}$ is contained in $\mathfrak{A}$. Let $\{V, \psi\}$ and $\{W, \zeta\}$ be any two charts of $\mathfrak{A}$ with $V \cap W \neq \emptyset$. Let $x \in V \cap W$. We must show that $\zeta \circ \psi^{-1}$ is $C^\infty$ in some neighbourhood of $\psi(x)$. Since $\mathfrak{A}$ is an atlas, it contains a chart $\{U, \phi\}$ with $x \in U$. Since $\{V, \psi\}$ and $\{W, \zeta\}$ must both be compatible with $\{U, \phi\}$, $\phi \circ \psi^{-1}$ must be $C^\infty$ in some neighbourhood of $\psi(x)$ and $\zeta \circ \phi^{-1}$ must be $C^\infty$ in some neighbourhood of $\phi(x)$. But then the composition $\zeta \circ \phi^{-1} \circ \phi \circ \psi^{-1}$ is $C^\infty$ in some neighbourhood of $\psi(x)$. As $\zeta \circ \phi^{-1} \circ \phi \circ \psi^{-1} = \zeta \circ \psi^{-1}$ in a neighbourhood of $\psi(x)$, we are done.

Problem M.2 Let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of a metric space $\mathcal{M}$. Let $\varphi$ be a homeomorphism from $\mathcal{U}$ to an open subset of $\mathbb{R}^n$ and $\psi$ be a homeomorphism from $\mathcal{V}$ to an open subset of $\mathbb{R}^m$. Prove that if $\mathcal{U} \cap \mathcal{V}$ is nonempty and

$$
\psi \circ \varphi^{-1} : \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n \to \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m
$$

$$
\varphi \circ \psi^{-1} : \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m \to \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n
$$

are $C^\infty$, then $m = n$.

Solution. Write $f(x) = \psi \circ \varphi^{-1}(x)$ and $g(y) = \varphi \circ \psi^{-1}(y)$. Fix any $p \in \mathcal{U} \cap \mathcal{V}$. Set

$$
A = \begin{bmatrix}
\frac{\partial f_i}{\partial x_j}(\varphi(p)) & & \\
& \ddots & \\
& & \frac{\partial f_n}{\partial x_j}(\varphi(p))
\end{bmatrix}_{1 \leq i \leq m, 1 \leq j \leq n}
$$

$$
B = \begin{bmatrix}
\frac{\partial g_i}{\partial y_j}(\psi(p)) & & \\
& \ddots & \\
& & \frac{\partial g_n}{\partial y_j}(\psi(p))
\end{bmatrix}_{1 \leq i \leq n, 1 \leq j \leq m}
$$

Since $f(g(y)) = y$ for all $y$ is a neighbourhood of $\psi(p)$, the chain rule gives that

$$
\sum_{k=1}^n \frac{\partial f_k}{\partial x_k}(g(y)) \frac{\partial g_k}{\partial y_j}(y) = \delta_{i,j}
$$

for all $y$ is a neighbourhood of $\psi(p)$ and all $1 \leq i, j \leq m$. In particular $AB = I_m$, the $m \times m$ identity matrix. Similarly, since $g(f(x)) = x$ for all $x$ is a neighbourhood of $\varphi(p)$, $BA = I_n$. So $A$ is the inverse of the matrix $B$. But only square matrices have inverses, so $m = n$. 

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Problem M.3 Let $S^n = \{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid |x| = 1 \}$ be the standard $n$-dimensional sphere.

(a) For each $1 \leq i \leq n+1$ set

$$\mathcal{U}_i = \{ x \in S^n \mid x_i > 0 \} \quad \varphi_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$$

$$\mathcal{V}_i = \{ x \in S^n \mid x_i < 0 \} \quad \psi_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$$

Prove that $\mathcal{A}_1 = \{ (\mathcal{U}_i, \varphi_i), (\mathcal{V}_i, \psi_i) \mid 1 \leq i \leq n + 1 \}$ is an atlas for $S^n$.

(b) Set

$$\mathcal{U} = S^n \setminus \{(0, \ldots, 0, 1)\} \quad \mathcal{V} = S^n \setminus \{(0, \ldots, 0, -1)\}$$

and define the stereographic projections, $\varphi : \mathcal{U} \to \mathbb{R}^n$ and $\psi : \mathcal{V} \to \mathbb{R}^n$, by

$$\varphi(x) = \frac{2}{1-x_{n+1}}(x_1, \ldots, x_n) \quad \psi(x) = \frac{2}{1+x_{n+1}}(x_1, \ldots, x_n)$$

Prove that $\mathcal{A}_2 = \{ (\mathcal{U}, \varphi), (\mathcal{V}, \psi) \}$ is an atlas for $S^n$.

Solution. (a) Each $x \in S^n$ obeys $|x| = 1$ and so has at least one nonzero component $x_j$. If $x_j > 0$, then $x \in \mathcal{U}_j$ and if $x_j < 0$, then $x \in \mathcal{V}_j$. So $S^n \subset \bigcup_{1 \leq j \leq n+1} (\mathcal{U}_j \cup \mathcal{V}_j)$. We’ll now verify that $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_1 \circ \psi_2^{-1}$ are $C^\infty$. The other cases are virtually identical. First observe that

$$\mathcal{U}_1 \cap \mathcal{U}_2 = \{ x \in S^n \mid x_1 > 0, \ x_2 > 0 \}$$

$$\mathcal{U}_1 \cap \mathcal{V}_2 = \{ x \in S^n \mid x_1 > 0, \ x_2 < 0 \}$$

so that

$$\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) = \{ (x_2, x_3, \ldots, x_{n+1}) \in \mathbb{R}^n \mid x_2 > 0, x_2^2 + x_3^2 + \cdots + x_{n+1}^2 < 1 \}$$

$$\varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2) = \{ (x_1, x_3, \ldots, x_{n+1}) \in \mathbb{R}^n \mid x_1 > 0, x_1^2 + x_3^2 + \cdots + x_{n+1}^2 < 1 \}$$

$$\varphi_1(\mathcal{U}_1 \cap \mathcal{V}_2) = \{ (x_2, x_3, \ldots, x_{n+1}) \in \mathbb{R}^n \mid x_2 < 0, x_2^2 + x_3^2 + \cdots + x_{n+1}^2 < 1 \}$$

$$\psi_2(\mathcal{U}_1 \cap \mathcal{V}_2) = \{ (x_1, x_3, \ldots, x_{n+1}) \in \mathbb{R}^n \mid x_1 > 0, x_1^2 + x_3^2 + \cdots + x_{n+1}^2 < 1 \}$$

are in fact each one of two half disks in $\mathbb{R}^n$. Since

$$\varphi_2^{-1}(t_1, t_2, \ldots, t_n) = (t_1, \sqrt{1-t_2^2}, t_2, \ldots, t_n)$$

$$\psi_2^{-1}(t_1, t_2, \ldots, t_n) = (t_1, -\sqrt{1-t_2^2}, t_2, \ldots, t_n)$$

(where $t^2 = t_1^2 + \cdots + t_n^2$) it is clear that

$$\varphi_1 \circ \varphi_2^{-1}(t_1, t_2, \ldots, t_n) = \varphi_1(t_1, \sqrt{1-t_2^2}, t_2, \ldots, t_n) = (\sqrt{1-t_2^2}, t_2, \ldots, t_n)$$

$$\varphi_1 \circ \psi_2^{-1}(t_1, t_2, \ldots, t_n) = \varphi_{-1,1}(t_1, -\sqrt{1-t_2^2}, t_2, \ldots, t_n) = (-\sqrt{1-t_2^2}, t_2, \ldots, t_n)$$
are $C^\infty$.

(b) Observe that
\[ \varphi(x)^2 = \frac{4}{(1-x_{n+1})^2} (x_1^2 + \cdots + x_n^2) = 4 \frac{1-x_{n+1}^2}{(1-x_{n+1})^2} = 4 \frac{1+x_{n+1}}{1-x_{n+1}} \implies x_{n+1} = \frac{\varphi(x)^2 - 4}{\varphi(x)^2 + 4} \]

Since \( \frac{1-x_{n+1}}{2} = \frac{4}{\varphi(x)^2 + 4} \), we have
\[ \varphi^{-1}(t_1, \cdots, t_n) = \frac{4}{t_2^2 + 4} (t_1, \cdots, t_n, \frac{t_2^2 - 4}{4}) \]

and hence
\[ \psi \circ \varphi^{-1}(t_1, \cdots, t_n) = 2 [1 + \frac{t_2^2 - 4}{4 + \psi(x)^2}]^{-1} \frac{4}{t_2^2 + 4} (t_1, \cdots, t_n) = \frac{4}{t_2^4} (t_1, \cdots, t_n) \]

which is $C^\infty$ except at $t = 0$, which corresponds to $x = (0, \cdots, 0, -1)$. Similarly,
\[ \psi(x)^2 = 4 \frac{1-x_{n+1}}{1+x_{n+1}} \implies x_{n+1} = \frac{4 - \psi(x)^2}{4 + \psi(x)^2} \implies \psi^{-1}(t_1, \cdots, t_n) = \frac{4}{4+t^2} (t_1, \cdots, t_n, \frac{4-t^2}{4}) \]

so that
\[ \varphi \circ \psi^{-1}(t_1, \cdots, t_n) = 2 [1 - \frac{4-t^2}{4+t^2}]^{-1} \frac{4}{4+t^2} (t_1, \cdots, t_n) = \frac{1}{t_2^2} (t_1, \cdots, t_n) \]

which is $C^\infty$ except at $t = 0$, which this time corresponds to $x = (0, \cdots, 0, 1)$. □

**Problem M.4** Let $R \in O(3)$.

(a) Prove that if $\lambda$ is an eigenvalue of $R$, then $|\lambda| = 1$ and $\bar{\lambda}$ is an eigenvalue of $R$.

(b) Prove that at least one eigenvalue of $R$ is either $+1$ or $-1$.

(c) Prove that the columns of $R$ are mutually perpendicular and are each of unit length.

(d) Prove that $R$ is either a rotation, a reflection or a composition of a rotation and a reflection.

**Solution.** (a) Recall that the inner product on $\mathbb{C}^3$ is $\langle \vec{v}, \vec{w} \rangle = \sum_{j=1}^{3} v_j \overline{w_j}$. Let $\vec{e}$ be an eigenvector of $R$ of eigenvalue $\lambda$. Then
\[ \langle \vec{e}, \vec{e} \rangle = \langle \vec{e}, \vec{e} \rangle = \langle \vec{e}, R^t R \vec{e} \rangle = \langle R \vec{e}, \vec{e} \rangle = \langle \lambda \vec{e}, \lambda \vec{e} \rangle = |\lambda|^2 \langle \vec{e}, \vec{e} \rangle \]

As $\langle \vec{e}, \vec{e} \rangle \neq 0$, we have $|\lambda|^2 = 1$ and hence $|\lambda| = 1$. Taking the complex conjugate of $R \vec{e} = \lambda \vec{e}$ gives $R \overline{\vec{e}} = \overline{\lambda} \overline{\vec{e}}$, since $R$ has real matrix elements. As $\overline{\vec{e}}$ is not the zero vector, it is an eigenvector of $R$ with eigenvalue $\overline{\lambda}$. 

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(b) Let \( P(\lambda) = \det(R - \lambda I) \) be the characteristic polynomial of \( R \). It is a polynomial of degree three with real coefficients. So it has exactly three roots, counting multiplicity, and all non-real roots come in complex conjugate pairs. Consequently \( P(\lambda) \) has at least one real root. So \( R \) has at least one real eigenvalue. As all eigenvalues have modulus one, any real eigenvalue has to be \( \pm 1 \).

(c) Denote by \( C_n^{(\ell)} = R_{n,\ell} \) the \( n \)th component of column \( \ell \). Then, for each \( 1 \leq k, \ell \leq 3 \),

\[
\tilde{C}^{(k)} \cdot \tilde{C}^{(\ell)} = \sum_{n=1}^{3} C_n^{(k)} \tilde{C}_n^{(\ell)} = \sum_{n=1}^{3} R_{n,k} R_{n,\ell} = (R^t R)_{k,\ell} = \mathbb{I}_{k,\ell}
\]

So if \( k \neq \ell \), \( \tilde{C}^{(k)} \cdot \tilde{C}^{(\ell)} = 0 \) and \( \tilde{C}^{(k)} \perp \tilde{C}^{(\ell)} \) and if \( k = \ell \), \( \|C^{(\ell)}\|^2 = 1 \) and \( \|C^{(\ell)}\| = 1 \).

(d) Case 1: If \( R \) has exactly one real eigenvalue, we may choose a coordinate system in which the corresponding eigenvector is on the \( z \)-axis. (See “Change of Basis”, below.) In this coordinate system \((0,0,1)\) is an eigenvector of eigenvalue \( \pm 1 \) and so \( R \) is of the form

\[
R = \begin{bmatrix}
a & b & 0 \\
c & d & 0 \\
e & f & \pm 1
\end{bmatrix}
\]

The columns must be mutually perpendicular and of unit length. This forces \( e = f = 0 \), \( a^2 + c^2 = b^2 + d^2 = 1 \) and \((a,c) \cdot (b,d) = 0\). So there is a \( \theta \) such that \( a = \cos \theta \), \( c = \sin \theta \) and \((b,d) = \pm(\sin \theta, \cos \theta)\) and \( R \) is one of

\[
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\quad
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & -1
\end{bmatrix}
\quad
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
\sin \theta & -\cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\quad
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
\sin \theta & -\cos \theta & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

The first two are rotation about the \( z \) axis by \( \theta \) and rotation about the \( z \) axis by \( \theta \) followed by the reflection \( z \to -z \). The remaining two actually have three eigenvalues \( \pm 1 \) and so are included in

Case 2: \( R \) has three real eigenvalues. (\( R \) cannot have exactly two real eigenvalues, because if \( \lambda \) is an eigenvalue, \( \overline{\lambda} \) is too.) If \( \vec{e}_1 \) is an eigenvector of eigenvalue \( \pm 1 \) and \( \vec{e}_2 \) is an eigenvector of eigenvalue \( -1 \), then \( \vec{e}_1 \perp \vec{e}_2 \) because

\[
-\langle \vec{e}_1, \vec{e}_2 \rangle = (+1)(-1) \langle \vec{e}_1, \vec{e}_2 \rangle = \langle +1 \vec{e}_1, -1 \vec{e}_2 \rangle = \langle R\vec{e}_1, R\vec{e}_2 \rangle = \langle \vec{e}_1, R^t R \vec{e}_2 \rangle = \langle \vec{e}_1, \vec{e}_2 \rangle
\Rightarrow \langle \vec{e}_1, \vec{e}_2 \rangle = 0
\]

So we may choose a coordinate system with all three standard basis vectors being eigenvectors. So \( R \) is one of (up to permutations of the coordinate axes)

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\quad
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

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which are the identity, the reflection \( z \to -z \), rotation about the \( x \) axis by 180° and rotation about the \( z \) axis by 180° followed by the reflection \( z \to -z \).

**Change of Basis** Let \( \vec{e}_1', \vec{e}_2' \) and \( \vec{e}_3' \) be three mutually perpendicular unit vectors in \( \mathbb{R}^3 \). The components, \( x_1', x_2', x_3' \), of any vector \( \vec{x} \) with respect to the new basis \( \{ \vec{e}_1', \vec{e}_2', \vec{e}_3' \} \) are determined by

\[
x_1' \vec{e}_1' + x_2' \vec{e}_2' + x_3' \vec{e}_3' = \vec{x} \quad \text{or} \quad E \vec{x}' = \vec{x} \quad \text{where} \quad E = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}
\]

Note that \( E \in O(3) \), so that \( \vec{x}' = E^{-1} \vec{x} = E^t \vec{x} \). If we think of a 3 \times 3 matrix \( R \) as mapping each \( \vec{x} \in \mathbb{R}^3 \) to \( R\vec{x} \in \mathbb{R}^3 \), then in terms of the new coordinate system, \( \vec{x}' = E^t \vec{x} \) gets mapped to \( E^t R \vec{x} = E^t RE \vec{x}' \). Thus the matrix of the map \( \vec{x} \to R\vec{x} \) in the new coordinate system is \( E^t RE \). Note that if \( R \in O(3) \), then \( E^t RE \) is again in \( O(3) \), because \( O(3) \) is closed under multiplication and the taking of transposes.

**Problem M.5** Define

\[
\begin{align*}
g_1(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) &= a_1^2 + a_2^2 + a_3^2 \\
g_2(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) &= b_1^2 + b_2^2 + b_3^2 \\
g_3(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) &= c_1^2 + c_2^2 + c_3^2 \\
g_4(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\
g_5(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) &= a_1 c_1 + a_2 c_2 + a_3 c_3 \\
g_6(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) &= b_1 c_1 + b_2 c_2 + b_3 c_3
\end{align*}
\]

Prove that the gradients of \( g_1, \ldots, g_6 \), evaluated at any

\[
R = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \in O(3)
\]

are linearly independent.

**Solution.** Fix any \( (a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) \). The six gradients are

\[
\begin{align*}
\nabla g_1 &= 2(a_1, a_2, a_3, 0, 0, 0, 0, 0) \\
\nabla g_2 &= 2(0, 0, 0, b_1, b_2, b_3, 0, 0) \\
\nabla g_3 &= 2(0, 0, 0, 0, 0, c_1, c_2, c_3) \\
\nabla g_4 &= (b_1, b_2, b_3, a_1, a_2, a_3, 0, 0) \\
\nabla g_5 &= (c_1, c_2, c_3, 0, 0, a_1, a_2, a_3) \\
\nabla g_6 &= (0, 0, 0, c_1, c_2, c_3, b_1, b_2, b_3)
\end{align*}
\]
We have to show that the only solution so \( \sum_{j=1}^{6} \alpha_j \nabla g_j = 0 \) is \( \alpha_1 = \cdots = \alpha_6 = 0 \). The first three components of \( \sum_{j=1}^{6} \alpha_j \nabla g_j \) are (writing them as a column vector)

\[
\begin{bmatrix}
2a_1 \\
2a_2 \\
2a_3 \\
\end{bmatrix} + \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
\end{bmatrix} + \begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
\end{bmatrix} = \begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
\end{bmatrix} \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\end{bmatrix} = \begin{bmatrix}
2\alpha_1 \\
2\alpha_2 \\
2\alpha_3 \\
\end{bmatrix}
\]

The matrix \( R \), as an element of \( O(3) \) must have determinant \( \pm 1 \) and hence must be invertible. So the first three components of \( \sum_{j=1}^{6} \alpha_j \nabla g_j \) are zero if and only if \( 2\alpha_1 = 2\alpha_4 = \alpha_5 = 0 \). Similarly components four, five and six of \( \sum_{j=1}^{6} \alpha_j \nabla g_j \) are zero if and only if \( 2\alpha_2 = \alpha_4 = \alpha_6 = 0 \) and components seven, eight and nine of \( \sum_{j=1}^{6} \alpha_j \nabla g_j \) are zero if and only if \( 2\alpha_3 = \alpha_5 = \alpha_6 = 0 \).

**Problem M.6** Use the implicit function theorem to prove that for each \( 1 \leq i, j \leq 3 \), the \( (i, j) \) matrix element, \( a_{ij} \), of matrices \( R = [a_{ij}]_{1 \leq i, j \leq 3} \) in a neighbourhood of \( 1 \) in \( SO(3) \), is a \( C^\infty \) function of the matrix elements \( a_{21}, a_{31} \) and \( a_{32} \).

**Solution.** Define

\[
g_1(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, a_{21}, a_{31}, a_{32}) = \sum_{j=1}^{3} a_{1j}^2 - 1
\]
\[
g_2(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, a_{21}, a_{31}, a_{32}) = \sum_{j=1}^{3} a_{2j}^2 - 1
\]
\[
g_3(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, a_{21}, a_{31}, a_{32}) = \sum_{j=1}^{3} a_{3j}^2 - 1
\]
\[
g_4(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, a_{21}, a_{31}, a_{32}) = \sum_{j=1}^{3} a_{1j}a_{2j}
\]
\[
g_5(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, a_{21}, a_{31}, a_{32}) = \sum_{j=1}^{3} a_{1j}a_{3j}
\]
\[
g_6(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, a_{21}, a_{31}, a_{32}) = \sum_{j=1}^{3} a_{2j}a_{3j}
\]

All six functions are \( C^\infty \) functions of their 9 variables. We shall solve for the first six variables as functions of the last three. The gradients of the six functions at the identity matrix are

\[
\nabla g_1(1, 0, 0, 1, 0, 1, 0, 0, 0) = (2, 0, 0, 0, 0, 0, 0, 0, 0)
\]

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\[ \nabla g_2(1, 0, 0, 1, 0, 1, 0, 0, 0) = (0, 0, 0, 2, 0, 0, 0, 0, 0) \]
\[ \nabla g_3(1, 0, 0, 1, 0, 1, 0, 0, 0) = (0, 0, 0, 0, 2, 0, 0, 0) \]
\[ \nabla g_4(1, 0, 0, 1, 0, 1, 0, 0, 0) = (0, 1, 0, 0, 0, 0, 1, 0, 0) \]
\[ \nabla g_5(1, 0, 0, 1, 0, 1, 0, 0, 0) = (0, 0, 1, 0, 0, 0, 0, 1, 0) \]
\[ \nabla g_6(1, 0, 0, 1, 0, 1, 0, 0, 0) = (0, 0, 0, 1, 0, 0, 0, 1, 0) \]

(These are linearly independent vectors.) Expanding along the first column and then along the first row and finally along the last column,

\[
\det \left. \frac{\partial(g_1, g_2, g_3, g_4, g_5, g_6)}{\partial(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33})} \right|_{(0,0,0,1,0,1,0,1)} = \det \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = 2 \det \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]

\[ = 4 \det \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = -8 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -8 \neq 0 \]

the implicit function theorem assures that given any \( a_{21}, a_{31}, a_{32} \) sufficiently small, there is a unique element of \( SO(3) \) having the given values of \( a_{21}, a_{31}, a_{32} \) and being in a neighbourhood (specified by the implicit function theorem) of the identity. Furthermore, each \( a_{ij} \) is a \( C^\infty \) function of \( a_{21}, a_{31}, a_{32} \).

Problem M.7 Prove that the two charts \((\mathcal{U}_2, \varphi_2)\) and \((\mathcal{U}_2, \psi_2)\) of Example M.10 are not compatible.

Solution. The range of \( \varphi_2 \) is \([0, \frac{1}{4}) \times (-1, 1) \cup (-\frac{1}{4}, 0) \times (-1, 1) = (-\frac{1}{4}, \frac{1}{4}) \times (-1, 1)\). On this range

\[ \varphi_2^{-1}(x, y) = \begin{cases} (x, y) & \text{if } 0 \leq x < \frac{1}{4} \\ (x + 1, -y) & \text{if } -\frac{1}{4} < x < 0 \end{cases} \]

so that

\[ \psi_2 \circ \varphi_2^{-1}(x, y) = \begin{cases} \psi_2(x, y) & \text{if } 0 \leq x < \frac{1}{4} \\ \psi_2(x + 1, -y) & \text{if } -\frac{1}{4} < x < 0 \end{cases} = \begin{cases} (x, y) & \text{if } 0 \leq x < \frac{1}{4} \\ (x, -y) & \text{if } -\frac{1}{4} < x < 0 \end{cases} \]

This is not continuous across \( x = 0 \), since

\[ \lim_{x \to 0^+} \psi_2 \circ \varphi_2^{-1}(x, y) = (0, y) \]

while

\[ \lim_{x \to 0^-} \psi_2 \circ \varphi_2^{-1}(x, y) = (0, -y) \]

© Joel Feldman. 2008. All rights reserved. April 9, 2008 Problem Solutions for “Integration on Manifolds” 7
Problem M.8 Let $M$ and $N$ be manifolds. Prove that $f : M \to N$ is $C^\infty$ at $m \in M$ if and only if $\psi \circ f \circ \phi^{-1}$ is $C^\infty$ at $\phi(m)$ for every chart $(U, \phi)$ for $M$ with $m \in U$ and every chart $(V, \psi)$ for $N$ with $f(m) \in V$.

Solution. The “if” part is trivial. To prove the “only if” part, observe that, by definition, there are charts $\{\tilde{U}, \tilde{\phi}\}$ and $\{V, \psi\}$ such that $m \in \tilde{U}$, $f(m) \in V$ and $\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}$ being $C^\infty$ at $\tilde{\phi}(m)$. Let $\{U, \phi\}$ and $\{V, \psi\}$ be any charts with $m \in U$ and $f(m) \in V$. Then $m \in \tilde{U} \cap U$ and $f(m) \in V \cap V$. By compatibility $\tilde{\phi} \circ \phi^{-1}$ and $\psi \circ \tilde{\psi}^{-1}$ are $C^\infty$ at $\tilde{\phi}(m)$ and $\tilde{\psi}(f(m))$ respectively. Consequently,

$$\psi \circ f \circ \phi^{-1} = (\psi \circ \tilde{\psi}^{-1}) \circ (\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}) \circ (\tilde{\phi} \circ \phi^{-1})$$

is $C^\infty$ at $\phi(m)$.

Problem M.9 Prove that $\mathbb{R}^n$ is diffeomorphic to $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 < 1\}$.

Solution. The map

$$\Phi(x) = \frac{x}{\sqrt{1 - |x|^2}}$$

is a diffeomorphism from $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 < 1\}$ to $\mathbb{R}^n$. The inverse map is

$$\Phi^{-1}(y) = \frac{y}{\sqrt{1 + |y|^2}}$$

Problem M.10 Prove that $\mathbb{R}^n$ is not diffeomorphic to $S^n$.

Solution. Suppose that $\Phi : S^n \to \mathbb{R}^n$ were a diffeomorphism. Let $g : \mathbb{R}^n \to \mathbb{R}$ be defined by $g(x) = x_1$. Then $g \circ \Phi : S^n \to \mathbb{R}$ is $C^\infty$ and is onto $\mathbb{R}$. But that is impossible because $S^n$ is compact so that every $C^\infty$ function on $S^n$ is bounded.

Problem M.11 Outline an argument to prove that the disk $\{x \in \mathbb{R}^2 \mid x^2 + y^2 < 2\}$ is not diffeomorphic to the annulus $\{x \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$.

Solution. The disk $M = \{x \in \mathbb{R}^2 \mid x^2 + y^2 < 2\}$ has the property that every $C^\infty$ closed curve may be continuously deformed to a point. To see this parametrize any curve
by a function $f : \mathbb{R} \to \mathcal{M}$ that has period one. Then $f_s(t) = (1 - s)f(t)$ implements the deformation, since $f_0(t) = f(t)$ and $f_1(t) = 0$ has range the single point $\{0\}$. If the disk and annulus were diffeomorphic, the annulus would also have this property. It doesn’t. For example, the circle $C = \{ (x, y) \mid x^2 + y^2 = 1 \}$ cannot be deformed to a point in the annulus. If it could, Green’s theorem would yield that $\oint_C \left[ -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \right] = 0$, which is false.

Problem M.12 In this problem $G = \text{SO}(3)$.

a) Fix any $a \in G$. Denote by $I = \{ (i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq 3, 1 \leq j \leq 3 \}$ the set of indices for the matrix elements of the matrices in $G$. Prove that there exist $\alpha, \beta, \gamma \in I$ such that every matrix element $g_\delta$, $\delta \in I$ is a $C^\infty$ function of $g_\alpha$, $g_\beta$, $g_\gamma$ for matrices $g \in G$ in a neighbourhood of $a$.

b) Prove that a curve $q : (c, d) \to G$ is $C^\infty$ if and only if every matrix element $q(t)_{i,j}$ is $C^\infty$.

c) Prove that matrix multiplication $(a, b) \mapsto ab$ is a $C^\infty$ function from $G \times G$ to $G$.

d) Prove that the inverse function $a \mapsto a^{-1}$ is a $C^\infty$ function from $G$ to $G$.

Solution. (a) Name the matrix elements of $g \in G$ by

$$
\begin{bmatrix}
  a_1(g) & b_1(g) & c_1(g) \\
  a_2(g) & b_2(g) & c_2(g) \\
  a_3(g) & b_3(g) & c_3(g)
\end{bmatrix}
$$

and set

$$
\begin{align*}
  f_1 &= a_1^2 + a_2^2 + a_3^2 \\
  f_2 &= b_1^2 + b_2^2 + b_3^2 \\
  f_3 &= c_1^2 + c_2^2 + c_3^2 \\
  f_4 &= a_1b_1 + a_2b_2 + a_3b_3 \\
  f_5 &= a_1c_1 + a_2c_2 + a_3c_3 \\
  f_6 &= b_1c_1 + b_2c_2 + b_3c_3
\end{align*}
$$

We have to show that, with $\nabla = \left( \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3}, \frac{\partial}{\partial b_1}, \frac{\partial}{\partial b_2}, \frac{\partial}{\partial b_3}, \frac{\partial}{\partial c_1}, \frac{\partial}{\partial c_2}, \frac{\partial}{\partial c_3} \right)$,

$$
\begin{align*}
  \nabla f_1 &= (2a_1, 2a_2, 2a_3, 0, 0, 0, 0, 0, 0) \\
  \nabla f_2 &= (0, 0, 0, 2b_1, 2b_2, 2b_3, 0, 0, 0) \\
  \nabla f_3 &= (0, 0, 0, 0, 0, 0, 2c_1, 2c_2, 2c_3) \\
  \nabla f_4 &= (b_1, b_2, b_3, a_1, a_2, a_3, 0, 0, 0) \\
  \nabla f_5 &= (c_1, c_2, c_3, 0, 0, 0, a_1, a_2, a_3) \\
  \nabla f_6 &= (0, 0, 0, c_1, c_2, c_3, b_1, b_2, b_3)
\end{align*}
$$

are linearly independent vectors at any point obeying $f_1 = f_2 = f_3 = 1$ and $f_4 = f_5 = f_6 = 0$. In other words, we have to show that

$$
\alpha_1 \nabla f_1 + \alpha_2 \nabla f_2 + \alpha_3 \nabla f_3 + \alpha_4 \nabla f_4 + \alpha_5 \nabla f_5 + \alpha_6 \nabla f_6 = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0
$$
at every such point. This follows from

\[ \alpha_1 = \frac{1}{2} (2a|b|^2 \alpha_1 + a \cdot b \alpha_4 + c \cdot \alpha_5) \]
\[ = \frac{1}{2} (a_1, a_2, a_3, 0, 0, 0, 0, 0, 0) \cdot (\alpha_1 \nabla f_1 + \alpha_2 \nabla f_2 + \alpha_3 \nabla f_3 + \alpha_4 \nabla f_4 + \alpha_5 \nabla f_5 + \alpha_6 \nabla f_6) \]
\[ \alpha_2 = \frac{1}{2} (2|b|^2 \alpha_2 + a \cdot b \alpha_4 + b \cdot c \alpha_6) \]
\[ = \frac{1}{2} (b_1, b_2, b_3, 0, 0, 0) \cdot (\alpha_1 \nabla f_1 + \alpha_2 \nabla f_2 + \alpha_3 \nabla f_3 + \alpha_4 \nabla f_4 + \alpha_5 \nabla f_5 + \alpha_6 \nabla f_6) \]
\[ \alpha_3 = \frac{1}{2} (2|c|^2 \alpha_3 + a \cdot c \alpha_5 + b \cdot c \alpha_6) \]
\[ = \frac{1}{2} (0, 0, 0, 0, 0, 0, c_1, c_2, c_3) \cdot (\alpha_1 \nabla f_1 + \alpha_2 \nabla f_2 + \alpha_3 \nabla f_3 + \alpha_4 \nabla f_4 + \alpha_5 \nabla f_5 + \alpha_6 \nabla f_6) \]
\[ \alpha_4 = 2a \cdot b \alpha_1 + |b|^2 \alpha_4 + b \cdot c \alpha_5 \]
\[ = (b_1, b_2, b_3, 0, 0, 0, 0, 0, 0) \cdot (\alpha_1 \nabla f_1 + \alpha_2 \nabla f_2 + \alpha_3 \nabla f_3 + \alpha_4 \nabla f_4 + \alpha_5 \nabla f_5 + \alpha_6 \nabla f_6) \]
\[ \alpha_5 = 2a \cdot c \alpha_1 + b \cdot c \alpha_4 + |c|^2 \alpha_5 \]
\[ = (c_1, c_2, c_3, 0, 0, 0, 0, 0, 0) \cdot (\alpha_1 \nabla f_1 + \alpha_2 \nabla f_2 + \alpha_3 \nabla f_3 + \alpha_4 \nabla f_4 + \alpha_5 \nabla f_5 + \alpha_6 \nabla f_6) \]
\[ \alpha_6 = 2b \cdot c \alpha_2 + a \cdot c \alpha_4 + |c|^2 \alpha_6 \]
\[ = (0, 0, 0, c_1, c_2, c_3, 0, 0, 0) \cdot (\alpha_1 \nabla f_1 + \alpha_2 \nabla f_2 + \alpha_3 \nabla f_3 + \alpha_4 \nabla f_4 + \alpha_5 \nabla f_5 + \alpha_6 \nabla f_6) \]

(b) Pick any \( c < t_0 < d \). Pick three matrix elements as coordinates in some neighbourhood of \( q_0(t) \). By Problem M.8, \( q(t) \) is \( C^\infty \) at \( t_0 \) if and only if those three matrix elements are \( C^\infty \) at \( t_0 \). If those three matrix elements are \( C^\infty \) at \( t_0 \), the remaining matrix elements are also \( C^\infty \) by part (a).

(c) Every matrix element of \( ab \) is a polynomial in the matrix elements of \( a \) and the matrix elements of \( b \) and hence is \( C^\infty \).

(d) Every matrix element of \( a^{-1} \) is a polynomial in the matrix elements of \( a \) (since \( \det a = 1 \)) and hence is \( C^\infty \).

Problem M.13 Let \( M \) be a manifold, \( \omega \) be a 1–form on \( M \) and \( c(t) : [0, 1] \to M \) be a path in \( M \). Prove that the definition of \( \int_c \omega \) given in part (c) of Definition M.13 is independent of the decomposition of \( c \) into finitely many pieces and of the choice of coordinate patches.

Solution. “Decomposing \( c \) into finitely many pieces” means picking \( 0 < t_1 < t_2 < \cdots < t_m \) with \( m \in \mathbb{N} \) and \( \{ c(t) \mid t_{\ell-1} \leq t \leq t_\ell \} \) contained in a single coordinate patch for each \( \ell = 1, 2, \cdots, m \) and then applying the single patch algorithm of Definition M.13 to the part of \( c(t) \) with \( t_{\ell-1} \leq t \leq t_\ell \) for each \( \ell = 1, 2, \cdots, m \).
Make any two such decompositions and choices of coordinate patches for each piece. Since
\[ \int_{t_{\ell-1}}^{t_{\ell}} h(t) \, dt = \int_{t_{\ell-1}}^{s} h(t) \, dt + \int_{s}^{t_{\ell}} h(t) \, dt \]
for all \( t_{\ell-1} < s < t_{\ell} \) we are free to add any finite number of decomposition points. So we may assume that the two sets of decomposition times are identical. Thus it suffices to prove that if \( 0 \leq t_{\ell} < t_{\ell+1} \leq 1 \) and
- if \( \{U, \zeta\} \) and \( \{\tilde{U}, \tilde{\zeta}\} \) are two patches with \( c(t) \in U \cap \tilde{U} \) for all \( t_{\ell} \leq t \leq t_{\ell+1} \) and
- if \( \omega \) assigns \( \{U, \zeta\} \) the pair of functions \((f, g)\) and assigns \( \{\tilde{U}, \tilde{\zeta}\} \) the pair of functions \((\tilde{f}, \tilde{g})\)
then
\[ \int_{0}^{1} \left[ f(\zeta(c(t))) \frac{dx(c(t))}{dt} + g(\zeta(c(t))) \frac{dy(c(t))}{dt} \right] \, dt = \int_{0}^{1} \left[ \tilde{f}(\tilde{\zeta}(c(t))) \frac{d\tilde{x}(c(t))}{dt} + \tilde{g}(\tilde{\zeta}(c(t))) \frac{d\tilde{y}(c(t))}{dt} \right] \, dt \]
But this is the computation of part (c) of Remark M.14.

**Problem M.14** Let \( M \) be a manifold of dimension \( n \in \mathbb{N} \) (not necessarily 2) and suppose that we have defined a wedge product for \( M \) that is bilinear, graded anticommutative and associative (i.e. satisfies properties (a), (b) and (c) above). Let, for each \( \leq j \leq n \), \( \omega_j \) be a 1–form on \( M \) and, for each \( 1 \leq i, j \leq n \), \( f_{ij} \) be a function on \( M \). Prove that
\[
\left( \sum_{j=1}^{n} f_{1j} \omega_j \right) \wedge \left( \sum_{j=1}^{n} f_{2j} \omega_j \right) \wedge \cdots \wedge \left( \sum_{j=1}^{n} f_{nj} \omega_j \right) = \det \left[ f_{ij} \right]_{1 \leq i, j \leq n} \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n
\]

**Solution.** By linearity
\[
\left( \sum_{j=1}^{n} f_{1j} \omega_j \right) \wedge \left( \sum_{j=1}^{n} f_{2j} \omega_j \right) \wedge \cdots \wedge \left( \sum_{j=1}^{n} f_{nj} \omega_j \right) = \sum_{j_1, \ldots, j_n=1} f_{1j_1} \cdots f_{nj_n} \omega_{j_1} \wedge \cdots \wedge \omega_{j_n}
\]
By anticummutativity
\[
\omega_{j_{\ell}} \wedge \omega_{j_{\ell+1}} = \begin{cases} -\omega_{j_{\ell+1}} \wedge \omega_{j_{\ell}} & \text{if } j_{\ell} \neq j_{\ell+1} \\ 0 & \text{if } j_{\ell} = j_{\ell+1} \end{cases}
\]
so that
\[
\omega_{j_1} \wedge \cdots \wedge \omega_{j_{\ell}} \wedge \omega_{j_{\ell+1}} \wedge \cdots \wedge \omega_{j_n} = \begin{cases} -\omega_{j_1} \wedge \cdots \wedge \omega_{j_{\ell+1}} \wedge \omega_{j_{\ell}} \wedge \cdots \wedge \omega_{j_n} & \text{if } j_{\ell} \neq j_{\ell+1} \\ 0 & \text{if } j_{\ell} = j_{\ell+1} \end{cases} \tag{1}
\]
Repeatedly applying this, we have that \( \omega_{j_1} \wedge \cdots \wedge \omega_{j_n} \) is zero unless \( j_1, \ldots, j_n \) are all different, that is unless \( j_1 = \pi(1), \ldots, j_n = \pi(n) \) for some permutation of \( (1, \ldots, n) \). Repeatedly applying (1) also gives that, for any permutation(1) \( \pi \),

\[
\omega_{\pi(1)} \wedge \cdots \wedge \omega_{\pi(n)} = \text{sgn} \ \pi \ \omega_1 \wedge \cdots \wedge \omega_n
\]

where \( \text{sgn} \ \pi \) is the sign(1) of the permutation. Denoting by \( S_n \) the set of all permutations of \( (1, \ldots, n) \), we have

\[
\left( \sum_{j=1}^{n} f_{1j} \omega_j \right) \wedge \left( \sum_{j=1}^{n} f_{2j} \omega_j \right) \wedge \cdots \wedge \left( \sum_{j=1}^{n} f_{nj} \omega_j \right) = \sum_{j_1, \ldots, j_n=1}^{n} f_{1j_1} \cdots f_{nj_n} \ \omega_{j_1} \wedge \cdots \wedge \omega_{j_n}
\]

\[
= \sum_{\pi \in S_n} \text{sgn} \ \pi \ f_{1\pi(1)} \cdots f_{n\pi(n)} \ \omega_{\pi(1)} \wedge \cdots \wedge \omega_{\pi(n)}
\]

\[
= \det \left[ f_{ij} \right]_{1 \leq i, j \leq n} \ \omega_1 \wedge \cdots \wedge \omega_n
\]

Problem M.15 Prove that Definition M.18 is independent of the choice of coordinate patch.

Solution. Assume that

- \( \{U, \zeta\} \) and \( \{\tilde{U}, \tilde{\zeta}\} \) are two charts with \( U \cap \tilde{U} \neq \emptyset \) and
- the transition function \( \tilde{\zeta} \circ \zeta^{-1} \) (from \( \zeta(U \cap \tilde{U}) \subset \mathbb{R}^2 \) to \( \tilde{\zeta}(U \cap \tilde{U}) \subset \mathbb{R}^2 \)) is

\((\tilde{x}(x,y), \tilde{y}(x,y))\)

(a) Let \( F : M \rightarrow \mathbb{R} \) be a \( C^1 \) 0–form and set \( \varphi = F \circ \zeta^{-1} \) and \( \tilde{\varphi} = F \circ \tilde{\zeta}^{-1} \). Then \( \varphi(x,y) = \tilde{\varphi}(\tilde{x}(x,y), \tilde{y}(x,y)) \) and

\[
dF|_{\{U, \zeta\}} = \tilde{f}(x,\tilde{y}) \, d\tilde{x} + \tilde{g}(x,\tilde{y}) \, d\tilde{y}
\]

\[
dF|_{\{U, \zeta\}} = \left. \frac{\partial \tilde{\varphi}}{\partial x}(x, y) \right| dx + \left. \frac{\partial \tilde{\varphi}}{\partial y}(x, y) \right| dy
\]

\[
= \left\{ \frac{\partial \tilde{\varphi}}{\partial x}(\tilde{x}(x,y), \tilde{y}(x,y)) \frac{\partial \varphi}{\partial x}(x,y) + \frac{\partial \tilde{\varphi}}{\partial y}(\tilde{x}(x,y), \tilde{y}(x,y)) \frac{\partial \varphi}{\partial y}(x,y) \right\} \, dx
\]

\[
+ \left\{ \frac{\partial \tilde{\varphi}}{\partial x}(\tilde{x}(x,y), \tilde{y}(x,y)) \frac{\partial \varphi}{\partial y}(x,y) + \frac{\partial \tilde{\varphi}}{\partial y}(\tilde{x}(x,y), \tilde{y}(x,y)) \frac{\partial \varphi}{\partial x}(x,y) \right\} \, dy
\]

\[
= f(x,y) \, dx + g(x,y) \, dy
\]

(1) A permutation of \( (1, \ldots, n) \) is a \( 1–1 \) map from \( \{1, \ldots, n\} \) onto \( \{1, \ldots, n\} \). A transposition is a permutation that just exchanges two neighbouring elements. Any permutation may be expressed as a product (i.e. composition) of transpositions. The sign of a permutation is +1 if it is the product of an even number of transpositions and −1 if it is the product of an odd number of transpositions.
with
\[
\begin{align*}
  f(x,y) &= \tilde{f}(\tilde{x}(x,y), \tilde{y}(x,y)) \frac{\partial \tilde{x}}{\partial x}(x,y) + \tilde{g}(\tilde{x}(x,y), \tilde{y}(x,y)) \frac{\partial \tilde{y}}{\partial x}(x,y) \\
  g(x,y) &= \tilde{f}(\tilde{x}(x,y), \tilde{y}(x,y)) \frac{\partial \tilde{x}}{\partial y}(x,y) + \tilde{g}(\tilde{x}(x,y), \tilde{y}(x,y)) \frac{\partial \tilde{y}}{\partial y}(x,y)
\end{align*}
\]

This agrees with the coordinate transformation rule of Definition M.13.

(b) If \( \omega \) is a \( C^1 \) 1-form with
\[
\omega|_{\{U, \zeta\}} = f(x,y) \, dx + g(x,y) \, dy
\]
then
\[
\omega|_{\{O, \xi\}} = \tilde{f}(\tilde{x}, \tilde{y}) \, d\tilde{x} + \tilde{g}(\tilde{x}, \tilde{y}) \, d\tilde{y}
\]

Since
\[
\begin{align*}
  \frac{\partial g}{\partial x}(x,y) &= \frac{\partial f}{\partial x}(x,y) \frac{\partial \tilde{x}}{\partial x} + \frac{\partial f}{\partial y}(x,y) \frac{\partial \tilde{x}}{\partial y} + \frac{\partial \tilde{g}}{\partial x}(x,y) - \frac{\partial \tilde{g}}{\partial y}(x,y) \\
  \frac{\partial f}{\partial y}(x,y) &= \frac{\partial f}{\partial x}(x,y) \frac{\partial \tilde{y}}{\partial x} + \frac{\partial f}{\partial y}(x,y) \frac{\partial \tilde{y}}{\partial y} + \frac{\partial \tilde{g}}{\partial x}(x,y) - \frac{\partial \tilde{g}}{\partial y}(x,y)
\end{align*}
\]
we have that
\[
d\omega|_{\{U, \zeta\}} = \left[ \frac{\partial g}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y) \right] \, dx \wedge dy
\]
\[
= \left[ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] \left[ \frac{\partial \tilde{g}}{\partial x} - \frac{\partial \tilde{g}}{\partial y} \right] \, dx \wedge dy
\]
which agrees with the coordinate transformation rule of Definition M.15.

(c) The 2-form case is trivial.

\[\square\]

**Problem M.16** Prove the graded product rule that if \( \omega \) is a \( k \)-form and \( \omega' \) is a \( k' \)-form, then
\[
d(\omega \wedge \omega') = (d\omega) \wedge \omega' + (-1)^k \omega \wedge (d\omega')
\]

**Solution.** It suffices to consider \( k + k' \leq 1 \), since all of \( d(\omega \wedge \omega') \), \( (d\omega) \wedge \omega' \) and \( \omega \wedge (d\omega') \) vanish if \( k + k' \geq 2 \) because the manifold \( M \) is assumed to be two dimensional. It suffices to verify the specified formula in any chart \( \{U, \zeta\} \),

(a) If \( k = k' = 0 \), \( \omega \circ \zeta^{-1}(x,y) = f(x,y) \) and \( \omega' \circ \zeta^{-1}(x,y) = g(x,y) \), then
\[
\begin{align*}
  d\omega &= \frac{\partial f}{\partial x}(x,y) \, dx + \frac{\partial f}{\partial y}(x,y) \, dy \\
  d\omega' &= \frac{\partial g}{\partial x}(x,y) \, dx + \frac{\partial g}{\partial y}(x,y) \, dy \\
  (d\omega) \wedge \omega' + (-1)^k \omega \wedge (d\omega') &= \frac{\partial f}{\partial x}(x,y)g(x,y) \, dx + \frac{\partial f}{\partial y}(x,y)g(x,y) \, dy \\
  &\quad + f(x,y)\frac{\partial g}{\partial x}(x,y) \, dx + f(x,y)\frac{\partial g}{\partial y}(x,y) \, dy
\end{align*}
\]
and 
\[ d(\omega \wedge \omega') = \frac{\partial}{\partial x} (fg) \, dx + \frac{\partial}{\partial y} (fg) \, dy \]
agree by the product rule.

(b) If \( k = 0 \), \( k' = 1 \), \( \omega \circ \zeta^{-1}(x, y) = f(x, y) \) and \( \omega'\bigr|_{\mathcal{U}, \zeta} = g(x, y) \, dx + h(x, y) \, dy \), then

\[
    d\omega = \frac{\partial f}{\partial x}(x, y) \, dx + \frac{\partial f}{\partial y}(x, y) \, dy \\
    d\omega' = \left[ \frac{\partial h}{\partial x}(x, y) - \frac{\partial g}{\partial y}(x, y) \right] \, dx \wedge dy
\]

\[
    (d\omega) \wedge \omega' + (-1)^k \omega \wedge (d\omega') = \left[ \frac{\partial f}{\partial x}(x, y) \, dx + \frac{\partial f}{\partial y}(x, y) \, dy \right] \wedge \left[ g(x, y) \, dx + h(x, y) \, dy \right] \\
    \hspace{1cm} + f(x, y) \left[ \frac{\partial h}{\partial x}(x, y) - \frac{\partial g}{\partial y}(x, y) \right] \, dx \wedge dy \\
    \hspace{1cm} = \left[ h \frac{\partial f}{\partial x} - g \frac{\partial f}{\partial y} + f \frac{\partial h}{\partial x} - f \frac{\partial g}{\partial y} \right] \, dx \wedge dy
\]

and

\[
    d(\omega \wedge \omega') = d(\omega' \wedge \omega) = (d\omega') \wedge \omega + \omega' \wedge (d\omega) = (d\omega) \wedge \omega' - \omega \wedge (d\omega')
\]

agree by the product rule.

(c) If \( k = 1 \) and \( k' = 0 \), then, by the previous case

\[
    d(\omega \wedge \omega') = d(\omega' \wedge \omega) = (d\omega') \wedge \omega + \omega' \wedge (d\omega) = (d\omega) \wedge \omega' - \omega \wedge (d\omega')
\]

\[ \blacksquare \]

**Problem M.17** (Vector analysis in \( \mathbb{R}^3 \)) Let \( \mathcal{M} \) be \( \mathbb{R}^3 \) with atlas \( (U = \mathbb{R}^3, \zeta(\vec{x}) = \vec{x} = (x, y, z)) \). Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be any \( C^\infty \) function on \( \mathbb{R}^3 \) and \( \vec{a}(\vec{x}) = (a^1(\vec{x}), a^2(\vec{x}), a^3(\vec{x})) \) and \( \vec{b}(\vec{x}) = (b^1(\vec{x}), b^2(\vec{x}), b^3(\vec{x})) \) be any two vector fields (i.e. vector valued functions) on \( \mathbb{R}^3 \). We can associate to \( \vec{a}(\vec{x}) \) a 1–form \( \omega^1_a \) and a 2–form \( \omega^2_a \) by

\[
    \omega^1_a = a^1(\vec{x}) \, dx + a^2(\vec{x}) \, dy + a^3(\vec{x}) \, dz \\
    \omega^2_a = a^1(\vec{x}) \, dy \wedge dz + a^2(\vec{x}) \, dz \wedge dx + a^3(\vec{x}) \, dx \wedge dy
\]

Prove that

(a) \( \omega^1_a \wedge \omega^1_b = \omega^2_{a \times b} \)

(b) \( \omega^1_a \wedge \omega^2_b = \vec{a}(\vec{x}) \cdot \vec{b}(\vec{x}) \, dx \wedge dy \wedge dz \)

(c) \( df = \omega^1_{\vec{V}f} \)

(d) \( d\omega^1_a = \omega^2_{\vec{V} \times a} \)

(e) \( d\omega^2_a = \nabla \cdot \vec{a}(\vec{x}) \, dx \wedge dy \wedge dz \)

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Observe that

- $d^2 f = 0$ says that $0 = d\omega^1_f = \omega^2_{\nabla \times \nabla f}$ i.e. that $\nabla \times \nabla f = 0$.
- $d^2 \omega^1_{\vec{a}} = 0$ says that $0 = d\omega^2_{\nabla \times \vec{a}} = \nabla \cdot \nabla \times \vec{a}(\vec{x})\, dx \land dy \land dz$ i.e. that $\nabla \cdot \nabla \times \vec{a} = 0$.

**Solution. (a)**

\[
\omega^1_a \land \omega^1_b = (a^1(\vec{x})\, dx + a^2(\vec{x})\, dy + a^3(\vec{x})\, dz) \land (b^1(\vec{x})\, dx + b^2(\vec{x})\, dy + b^3(\vec{x})\, dz)
\]
\[
= a^1(\vec{x})b^2(\vec{x})\, dx \land dy + a^1(\vec{x})b^3(\vec{x})\, dx \land dz + a^2(\vec{x})b^1(\vec{x})\, dy \land dx
\]
\[
+ a^2(\vec{x})b^3(\vec{x})\, dy \land dz + a^3(\vec{x})b^1(\vec{x})\, dz \land dx + a^3(\vec{x})b^2(\vec{x})\, dz \land dy
\]
\[
= (a^1(\vec{x})b^2(\vec{x}) - a^2(\vec{x})b^1(\vec{x}))\, dx \land dy + (a^3(\vec{x})b^1(\vec{x}) - a^1(\vec{x})b^3(\vec{x}))\, dz \land dx
\]
\[
+ (a^2(\vec{x})b^3(\vec{x}) - a^3(\vec{x})b^2(\vec{x}))\, dy \land dz
\]
\[
= \omega^2_{\vec{a} \times \vec{b}}
\]

(b)

\[
\omega^1_a \land \omega^2_b = (a^1(\vec{x})\, dx + a^2(\vec{x})\, dy + a^3(\vec{x})\, dz) \land (b^1(\vec{x})\, dy \land dz + b^2(\vec{x})\, dz \land dx + b^3(\vec{x})\, dx \land dy)
\]
\[
= a^1(\vec{x})b^1(\vec{x})\, dx \land dy \land dz + a^2(\vec{x})b^2(\vec{x})\, dy \land dz \land dx + a^3(\vec{x})b^3(\vec{x})\, dz \land dx \land dy
\]
\[
= \vec{a}(\vec{x}) \cdot \vec{b}(\vec{x})\, dx \land dy \land dz
\]

(c)

\[
df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz = \omega^1_{\nabla f}
\]

(d)

\[
d\omega^1_a = d(a^1(\vec{x})\, dx + a^2(\vec{x})\, dy + a^3(\vec{x})\, dz)
\]
\[
= \frac{\partial a^1}{\partial y}\, dy \land dx + \frac{\partial a^1}{\partial z}\, dz \land dx + \frac{\partial a^2}{\partial x}\, dx \land dy + \frac{\partial a^2}{\partial z}\, dz \land dy + \frac{\partial a^3}{\partial x}\, dx \land dz + \frac{\partial a^3}{\partial y}\, dy \land dz
\]
\[
= \omega^2_{\nabla \times a}
\]

(e)

\[
d\omega^2_a = d(a^1(\vec{x})\, dy \land dz + a^2(\vec{x})\, dz \land dx + a^3(\vec{x})\, dx \land dy)
\]
\[
= \frac{\partial a^1}{\partial z}\, dz \land dx \land dy + \frac{\partial a^2}{\partial y}\, dy \land dz \land dx + \frac{\partial a^3}{\partial z}\, dz \land dx \land dy
\]
\[
= \nabla \cdot \vec{a}(\vec{x})\, dx \land dy \land dz
\]
Problem M.18 Let \( \Omega \) be an open connected, simply connected subset of \( \mathbb{R}^2 \). Think of \( \Omega \) as a two dimensional manifold as in Example M.2. Let \( F_1(x, y), F_2(x, y) \in C^\infty(\Omega) \) obey the compatibility condition that \( \partial F_1 / \partial y = \partial F_2 / \partial x \). The goal of this problem is to prove that there exists a function \( \varphi(x, y) \in C^\infty(\Omega) \) such that

\[
F_1(x, y) = \frac{\partial \varphi}{\partial x}(x, y) \quad \text{and} \quad F_2(x, y) = \frac{\partial \varphi}{\partial y}(x, y)
\]

This is the analog in two dimensions of the statement that, if \( \Omega \) is a simply connected region in \( \mathbb{R}^3 \) and \( \vec{F}(\vec{x}) \) is a vector field in \( \Omega \) that obeys \( \vec{\nabla} \times \vec{F}(\vec{x}) = \vec{0} \), then there is a “potential” \( \varphi(\vec{x}) \) such that \( \vec{F}(\vec{x}) = \vec{\nabla} \varphi(\vec{x}) \).

(a) Define the 1–form \( \omega = F_1(x, y) \, dx + F_2(x, y) \, dy \). Prove that \( \omega \) is closed.

(b) Let \( C_1(t), C_2(t) : [0, 1] \to \Omega \) be any two paths in \( \Omega \) with \( C_1(0) = C_2(0) \) and \( C_1(1) = C_2(1) \). That is, the two paths have the same initial and final points. Prove that \( \int_{C_1} \omega = \int_{C_2} \omega \).

(c) Fix any point \((x_0, y_0) \in \Omega \). For each point \((x, y) \in \Omega \), select a path \( C_{x,y}(t) : [0, 1] \to \Omega \) such that \( C_{x,y}(0) = (x_0, y_0) \) and \( C_{x,y}(1) = (x, y) \). Define \( \varphi(x, y) = \int_{C_{x,y}} \omega \). Prove that

\[
\frac{\partial \varphi}{\partial x}(x, y) = F_1(x, y) \quad \text{and} \quad \frac{\partial \varphi}{\partial y}(x, y) = F_2(x, y)
\]

(d) Let \( \phi(x, y) \) and \( \psi(x, y) \) be any two functions on \( \Omega \) that obey

\[
\frac{\partial \phi}{\partial x}(x, y) = \frac{\partial \psi}{\partial x}(x, y) = F_1(x, y) \quad \text{and} \quad \frac{\partial \phi}{\partial y}(x, y) = \frac{\partial \psi}{\partial y}(x, y) = F_2(x, y)
\]

Prove that \( \phi(x, y) - \psi(x, y) \) is a constant independent of \( x \) and \( y \).

Solution. (a) By the hypothesised compatibility,

\[
d\omega = \frac{\partial F_1}{\partial y} \, dy \wedge dx + \frac{\partial F_2}{\partial x} \, dx \wedge dy = \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \wedge dy = 0
\]

(b) Since \( \Omega \) is simply connected, it is possible to deform \( C_1 \) continuously to \( C_2 \). That is, there is a smooth function \( D : [0, 1] \times [0, 1] \to \Omega \) with

\[
D(t, 0) = C_1(t) \quad \text{for all } 0 \leq t \leq 1
\]

\[
D(t, 1) = C_2(t) \quad \text{for all } 0 \leq t \leq 1
\]

\[
D(0, s) = C_1(0) = C_2(0) \quad \text{for all } 0 \leq s \leq 1
\]

\[
D(1, s) = C_1(1) = C_2(1) \quad \text{for all } 0 \leq s \leq 1
\]
The boundary $\partial D = C_1 + E_1 - C_2 - E_2$ of $D$ consists of $C_1 - C_2$ and the two degenerate curves

$$E_1(s) = C_1(1) = C_2(1) \quad E_2(s) = C_1(0) = C_2(0) \quad \text{for all } 0 \leq s \leq 1$$

So $\int_{E_1} \omega = \int_{E_2} \omega = 0$ and, by Stoke’s theorem,

$$\int_{C_1} \omega - \int_{C_2} \omega = \int_{\partial D} \omega = \int_D d\omega = 0$$

(c) Fix any point $(x, y)$ in the open set $\Omega$. We prove that $\frac{\partial \varphi}{\partial x}(x, y) = F_1(x, y)$. The proof of other case is similar. Fix another point $(x_1, y)$ in $\Omega$ with the property that $(x'', y) \in \Omega$ for all $x''$ between $x_1$ and $x$. Choose, for each $(x', y)$ with $x'$ sufficiently close to $x$, the curve $C_{x', y}$ to obey

- $C_{x', y}(0) = (x_0, y_0)$
- $C_{x', y}(\frac{1}{2}) = (x_1, y)$
- $C_{x', y}(t)$ is independent of $x'$ for $0 \leq t \leq \frac{1}{2}$
- $C_{x', y}(t) = (x_1 + 2(x' - x_1)(t - \frac{1}{2}), y)$ for $\frac{1}{2} \leq t \leq 1$.

That is, $C_{x', y}$ starts at $(x_0, y_0)$ at $t = 0$, moves to $(x_1, y)$, along a path independent of $x'$, at $t = \frac{1}{2}$ and then follows a horizontal straight line to $(x', y)$ at $t = 1$. The contribution to $\varphi(x', y) = \int_{C_{x', y}} \omega$ arising from $0 \leq t \leq \frac{1}{2}$ is independent of $x'$ and will give contribution zero to $\frac{\partial \varphi}{\partial x}(x, y)$. The contribution to $\varphi(x', y) = \int_{C_{x', y}} \omega$ arising from $\frac{1}{2} \leq t \leq 1$ is precisely

$$\int_{\frac{1}{2}}^{1} F_1(x_1 + 2(x' - x_1)(t - \frac{1}{2}), y) \ 2(x' - x_1) \ dt = \int_{x_1}^{x'} F_1(\xi, y) \ d\xi \quad \text{where } \xi = x_1 + 2(x' - x_1)(t - \frac{1}{2})$$

The derivative of this with respect to $x'$ is $F_1(x', y)$. Hence $\frac{\partial \varphi}{\partial x}(x, y) = F_1(x, y)$ as desired.

(d) The gradient of $\theta = \phi - \psi$ vanishes identically. So $\theta = \phi - \psi$ is a constant. Here is another proof in the language of forms. If $(x_0, y_0)$ and $(x, y)$ are any two points in $\Omega$ and $C$ is a curve from $(x_0, y_0)$ to $(x, y)$ in $\Omega$, then

$$\theta(x, y) - \theta(x_0, y_0) = \int_{\partial C} \theta = \int_{C} d\theta = 0$$