Problem Solutions for "Integration on Manifolds"

Problem M.1 Let $\mathcal{A}$ be an atlas for the metric space $\mathcal{M}$. Prove that there is a unique maximal atlas for $\mathcal{M}$ that contains $\mathcal{A}$.

Solution. Define $\mathfrak{A}$ to be the set of all charts $\{\mathcal{V}, \psi\}$ that are compatible with all of the charts of $\mathcal{A}$. If we can show that $\mathfrak{A}$ is an atlas, we are done, because (a) $\mathcal{A}$ is an atlas, so that each chart in $\mathcal{A}$ is compatible with all charts of $\mathcal{A}$ and hence $\mathcal{A} \subset \mathfrak{A}$ and (b) any chart in any atlas that contains $\mathcal{A}$ must be compatible with every chart of $\mathcal{A}$ and hence every atlas that contains $\mathcal{A}$ is contained in $\mathfrak{A}$. Let $\{\mathcal{V}, \psi\}$ and $\{\mathcal{W}, \zeta\}$ be any two charts of $\mathfrak{A}$ with $\mathcal{V} \cap \mathcal{W} \neq \emptyset$. Let $x \in \mathcal{V} \cap \mathcal{W}$. We must show that $\zeta \circ \psi^{-1}$ is C^∞ in some neighbourhood of $\psi(x)$. Since $\mathcal{A}$ is an atlas, it contains a chart $\{\mathcal{U}, \phi\}$ with $x \in \mathcal{U}$. Since $\{\mathcal{V}, \psi\}$ and $\{\mathcal{W}, \zeta\}$ must both be compatible with $\{\mathcal{U}, \phi\}$, $\phi \circ \psi^{-1}$ must be C^∞ in some neighbourhood of $\psi(x)$ and $\zeta \circ \phi^{-1}$ must be C^∞ in some neighbourhood of $\phi(x)$. But then the composition $\zeta \circ \phi^{-1} \circ \phi \circ \psi^{-1}$ is C^∞ in some neighbourhood of $\psi(x)$. As $\zeta \circ \phi^{-1}\phi \circ \psi^{-1} = \zeta \circ \psi^{-1}$ in a neighbourhood of $\psi(x)$, we are done. ■

Problem M.2 Let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of a metric space $\mathcal{M}$. Let φ be a homeomorphism from $\mathcal{U}$ to an open subset of $\mathbb{R}^n$ and ψ be a homeomorphism from $\mathcal{V}$ to an open subset of $\mathbb{R}^m$. Prove that if $\mathcal{U} \cap \mathcal{V}$ is nonempty and

$$\begin{aligned} \psi \circ \varphi^{-1}\colon\ & \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n \to \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m \\ \varphi \circ \psi^{-1}\colon\ & \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m \to \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n \end{aligned}$$

are C^∞, then $m = n$.

Solution. Write $f(x) = \psi \circ \varphi^{-1}(x)$ and $g(y) = \varphi \circ \psi^{-1}(y)$. Fix any $p \in \mathcal{U} \cap \mathcal{V}$. Set

$$A = \left[\tfrac{\partial f_i}{\partial x_j}\big(\varphi(p)\big)\right]_{\substack{1 \le i \le m \\ 1 \le j \le n}} \qquad B = \left[\tfrac{\partial g_i}{\partial y_j}\big(\psi(p)\big)\right]_{\substack{1 \le i \le n \\ 1 \le j \le m}}$$

Since $f\big(g(y)\big) = y$ for all y is a neighbourhood of $\psi(p)$, the chain rule gives that

$$\sum_{k=1}^{n} \tfrac{\partial f_i}{\partial x_k}\big(g(y)\big)\tfrac{\partial g_k}{\partial y_j}(y) = \delta_{i,j}$$

for all y is a neighbourhood of $\psi(p)$ and all $1 \le i, j \le m$. In particular $AB = \mathbb{1}_m$, the $m \times m$ identity matrix. Similarly, since $g\big(f(x)\big) = x$ for all x is a neighbourhood of $\varphi(p)$, $BA = \mathbb{1}_n$. So A is the inverse of the matrix B. But only square matrices have inverses, so $m = n$. ■

Problem M.3 Let $S^n = \left\{ \mathbf{x} = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; |\mathbf{x}| = 1 \right\}$ be the standard n–dimensional sphere.

(a) For each $1 \le i \le n+1$ set

$$\mathcal{U}_i = \left\{ \mathbf{x} \in S^n \;\middle|\; x_i > 0 \right\} \quad \varphi_i(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$$
$$\mathcal{V}_i = \left\{ \mathbf{x} \in S^n \;\middle|\; x_i < 0 \right\} \quad \psi_i(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$$

Prove that $\mathcal{A}_1 = \left\{ (\mathcal{U}_i, \varphi_i),\ (\mathcal{V}_i, \psi_i) \;\middle|\; 1 \le i \le n+1 \right\}$ is an atlas for S^n.

(b) Set

$$\mathcal{U} = S^n \setminus \{(0, \cdots, 0, 1)\} \qquad \mathcal{V} = S^n \setminus \{(0, \cdots, 0, -1)\}$$

and define the stereographic projections, $\varphi : \mathcal{U} \to \mathbb{R}^n$ and $\psi : \mathcal{V} \to \mathbb{R}^n$, by

$$\varphi(x) = \tfrac{2}{1-x_{n+1}}(x_1, \cdots, x_n) \qquad \psi(x) = \tfrac{2}{1+x_{n+1}}(x_1, \cdots, x_n)$$

Prove that $\mathcal{A}_2 = \big\{\{\mathcal{U}, \varphi\},\ \{\mathcal{V}, \psi\}\big\}$ is an atlas for S^n.

Solution. (a) Each $\mathbf{x} \in S^n$ obeys $|\mathbf{x}| = 1$ and so has at least one nonzero component x_j. If $x_j > 0$, then $x \in \mathcal{U}_j$ and if $x_j < 0$, then $x \in \mathcal{V}_j$. So $S^n \subset \bigcup_{1 \le j \le n+1} \big(\mathcal{U}_j \cup \mathcal{V}_j\big)$. We'll now verify that $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_1 \circ \psi_2^{-1}$ are C^∞. The other cases ares virtually identical. First observe that

$$\mathcal{U}_1 \cap \mathcal{U}_2 = \left\{ x \in S^n \;\middle|\; x_1 > 0,\ x_2 > 0 \right\}$$
$$\mathcal{U}_1 \cap \mathcal{V}_2 = \left\{ x \in S^n \;\middle|\; x_1 > 0,\ x_2 < 0 \right\}$$

so that

$$\varphi_1\big(\mathcal{U}_1 \cap \mathcal{U}_2\big) = \left\{ (x_2, x_3, \cdots, x_{n+1}) \in \mathbb{R}^n \;\middle|\; x_2 > 0, x_2^2 + x_3^2 + \cdots + x_{n+1}^2 < 1 \right\}$$
$$\varphi_2\big(\mathcal{U}_1 \cap \mathcal{U}_2\big) = \left\{ (x_1, x_3, \cdots, x_{n+1}) \in \mathbb{R}^n \;\middle|\; x_1 > 0, x_1^2 + x_3^2 + \cdots + x_{n+1}^2 < 1 \right\}$$
$$\varphi_1\big(\mathcal{U}_1 \cap \mathcal{V}_2\big) = \left\{ (x_2, x_3, \cdots, x_{n+1}) \in \mathbb{R}^n \;\middle|\; x_2 < 0, x_2^2 + x_3^2 + \cdots + x_{n+1}^2 < 1 \right\}$$
$$\psi_2\big(\mathcal{U}_1 \cap \mathcal{V}_2\big) = \left\{ (x_1, x_3, \cdots, x_{n+1}) \in \mathbb{R}^n \;\middle|\; x_1 > 0, x_1^2 + x_3^2 + \cdots + x_{n+1}^2 < 1 \right\}$$

are in fact each one of two half disks in $\mathbb{R}^n$. Since

$$\varphi_2^{-1}\big(t_1, t_2, \cdots, t_n\big) = \big(t_1, \sqrt{1-t^2}, t_2, \cdots, t_n\big)$$
$$\psi_2^{-1}\big(t_1, t_2, \cdots, t_n\big) = \big(t_1, -\sqrt{1-t^2}, t_2, \cdots, t_n\big)$$

(where $t^2 = t_1^2 + \cdots + t_n^2$) it is clear that

$$\varphi_1 \circ \varphi_2^{-1}\big(t_1, t_2, \cdots, t_n\big) = \varphi_1\big(t_1, \sqrt{1-t^2}, t_2, \cdots, t_n\big) \qquad = \big(\sqrt{1-t^2}, t_2, \cdots, t_n\big)$$
$$\varphi_1 \circ \psi_2^{-1}\big(t_1, t_2, \cdots, t_n\big) = \varphi_{-1,1}\big(t_1, -\sqrt{1-t^2}, t_2, \cdots, t_n\big) = \big(-\sqrt{1-t^2}, t_2, \cdots, t_n\big)$$

are C^∞.

(b) Observe that

$$\varphi(x)^2 = \tfrac{4}{(1-x_{n+1})^2}\big(x_1^2+\cdots+x_n^2\big) = 4\tfrac{1-x_{n+1}^2}{(1-x_{n+1})^2} = 4\tfrac{1+x_{n+1}}{1-x_{n+1}} \implies x_{n+1} = \tfrac{\varphi(x)^2-4}{\varphi(x)^2+4}$$

Since $\frac{1-x_{n+1}}{2} = \frac{4}{\varphi(x)^2+4}$, we have

$$\varphi^{-1}(t_1,\cdots,t_n) = \tfrac{4}{t^2+4}\big(t_1,\cdots,t_n,\tfrac{t^2-4}{4}\big)$$

and hence

$$\psi\circ\varphi^{-1}(t_1,\cdots,t_n) = 2\big[1+\tfrac{t^2-4}{t^2+4}\big]^{-1}\tfrac{4}{t^2+4}\big(t_1,\cdots,t_n\big) = \tfrac{4}{t^2}\big(t_1,\cdots,t_n\big)$$

which is C^∞ except at $t=0$, which corresponds to $x=(0,\cdots,0,-1)$. Similarly,

$$\psi(x)^2 = 4\tfrac{1-x_{n+1}}{1+x_{n+1}} \implies x_{n+1} = \tfrac{4-\psi(x)^2}{4+\psi(x)^2} \implies \psi^{-1}(t_1,\cdots,t_n) = \tfrac{4}{4+t^2}\big(t_1,\cdots,t_n,\tfrac{4-t^2}{4}\big)$$

so that

$$\varphi\circ\psi^{-1}(t_1,\cdots,t_n) = 2\big[1-\tfrac{4-t^2}{4+t^2}\big]^{-1}\tfrac{4}{4+t^2}\big(t_1,\cdots,t_n\big) = \tfrac{4}{t^2}\big(t_1,\cdots,t_n\big)$$

which is C^∞ except at $t=0$, which this time corresponds to $x=(0,\cdots,0,1)$. ■

Problem M.4 Let $R\in O(3)$.

(a) Prove that if λ is an eigenvalue of R, then $|\lambda|=1$ and $\bar\lambda$ is an eigenvalue of R.
(b) Prove that at least one eigenvalue of R is either $+1$ or -1.
(c) Prove that the columns of R are mutually perpendicular and are each of unit length.
(d) Prove that R is either a rotation, a reflection or a composition of a rotation and a reflection.

Solution. (a) Recall that the inner product on $\mathbb{C}^3$ is $\langle\vec v,\vec w\rangle = \sum_{j=1}^3 v_j\overline{w_j}$. Let $\vec e$ be an eigenvector of R of eigenvalue λ. Then

$$\langle\vec e,\vec e\rangle = \langle\vec e,\mathbb{1}\vec e\rangle = \big\langle\vec e,R^tR\vec e\big\rangle = \langle R\vec e,R\vec e\rangle = \langle\lambda\vec e,\lambda\vec e\rangle = |\lambda|^2\,\langle\vec e,\vec e\rangle$$

As $\langle\vec e,\vec e\rangle\neq 0$, we have $|\lambda|^2=1$ and hence $|\lambda|=1$. Taking the complex conjugate of $R\vec e=\lambda\vec e$ gives $R\overline{\vec e}=\overline{\lambda}\,\overline{\vec e}$, since R has real matrix elements. As $\overline{\vec e}$ is not the zero vector, it is an eigenvector of R with eigenvalue $\overline{\lambda}$.

(b) Let $P(\lambda) = \det(R - \lambda \mathbb{1})$ be the characteristic polynomial of R. It is a polynomial of degree three with real coefficients. So it has exactly three roots, counting multiplicity, and all non–real roots come in complex conjugate pairs. Consequently $P(\lambda)$ has at least one real root. So R has at least one real eigenvalue. As all eigenvalues have modulus one, any real eigenvalue has to be ± 1.

(c) Denote by $C_n^{(\ell)} = R_{n,\ell}$ the nth component of column ℓ. Then, for each $1 \le k, \ell \le 3$,

$$\vec{C}^{(k)} \cdot \vec{C}^{(\ell)} = \sum_{n=1}^{3} C_n^{(k)} \vec{C}_n^{(\ell)} = \sum_{n=1}^{3} R_{n,k} R_{n,\ell} = (R^t R)_{k,\ell} = \mathbb{1}_{k,\ell}$$

So if $k \ne \ell$, $\vec{C}^{(k)} \cdot \vec{C}^{(\ell)} = 0$ and $\vec{C}^{(k)} \perp \vec{C}^{(\ell)}$ and if $k = \ell$, $\|C^{(\ell)}\|^2 = 1$ and $\|C^{(\ell)}\| = 1$.

(d) *Case 1:* If R has exactly one real eigenvalue, we may choose a coordinate system in which the corresponding eigenvector is on the z–axis. (See "Change of Basis", below.) In this coordinate system $(0, 0, 1)$ is an eigenvector of eigenvalue ± 1 and so R is of the form

$$R = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & \pm 1 \end{bmatrix}$$

The columns must be mutually perpendicular and of unit length. This forces $e = f = 0$, $a^2 + c^2 = b^2 + d^2 = 1$ and $(a, c) \cdot (b, d) = 0$. So there is a θ such that $a = \cos\theta$, $c = \sin\theta$ and $(b, d) = \pm(-\sin\theta, \cos\theta)$ and R is one of

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The first two are rotation about the z axis by θ and rotation about the z axis by θ followed by the reflection $z \to -z$. The remaining two actually have three eigenvalues ± 1 and so are included in

Case 2: R has three real eigenvalues. (R cannot have exactly two real eigenvalues, because if λ is an eigenvalue, $\bar{\lambda}$ is too.) If $\vec{e}_1$ is an eigenvector of eigenvalue $+1$ and $\vec{e}_2$ is an eigenvector of eigenvalue -1, then $\vec{e}_1 \perp \vec{e}_2$ because

$$-\langle \vec{e}_1, \vec{e}_2 \rangle = (+1)(-1) \langle \vec{e}_1, \vec{e}_2 \rangle = \langle +1\vec{e}_1, -1\vec{e}_2 \rangle = \langle R\vec{e}_1, R\vec{e}_2 \rangle = \left\langle \vec{e}_1, R^t R \vec{e}_2 \right\rangle = \langle \vec{e}_1, \vec{e}_2 \rangle$$
$$\implies \langle \vec{e}_1, \vec{e}_2 \rangle = 0$$

So we may choose a coordinate system with all three standard basis vectors being eigenvectors. So R is one of (up to permutations of the coordinate axes)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

which are the identity, the reflection $z \to -z$, rotation about the x axis by $180°$ and rotation about the z axis by $180°$ followed by the reflection $z \to -z$.

Change of Basis Let $\vec{e}_1'$, $\vec{e}_2'$ and $\vec{e}_3'$ be three mutually perpendicular unit vectors in $\mathbb{R}^3$. The components, x_1', x_2', x_3', of any vector $\vec{x}$ with respect to the new basis $\{\vec{e}_1', \vec{e}_2', \vec{e}_3'\}$ are determined by

$$x_1'\vec{e}_1' + x_2'\vec{e}_2' + x_3'\vec{e}_3' = \vec{x} \quad \text{or} \quad E\vec{x}\,' = \vec{x} \quad \text{where} \quad E = \begin{bmatrix}\vec{e}_1' & \vec{e}_2' & \vec{e}_3'\end{bmatrix},\ \vec{x}\,' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

Note that $E \in O(3)$, so that $\vec{x}\,' = E^{-1}\vec{x} = E^t\vec{x}$. If we think of a 3×3 matrix R as mapping each $\vec{x} \in \mathbb{R}^3$ to $R\vec{x} \in \mathbb{R}^3$, then in terms of the new coordinate system, $\vec{x}\,' = E^t\vec{x}$ gets mapped to $E^tR\vec{x} = E^tRE\vec{x}\,'$. Thus the matrix of the map $\vec{x} \to R\vec{x}$ in the new coordinate system is E^tRE. Note that if $R \in O(3)$, then E^tRE is again in $O(3)$, because $O(3)$ is closed under multiplication and the taking of transposes.

Problem M.5 Define

$$\begin{aligned}
g_1(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) &= a_1^2 + a_2^2 + a_3^2 \\
g_2(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) &= b_1^2 + b_2^2 + b_3^2 \\
g_3(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) &= c_1^2 + c_2^2 + c_3^2 \\
g_4(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) &= a_1b_1 + a_2b_2 + a_3b_3 \\
g_5(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) &= a_1c_1 + a_2c_2 + a_3c_3 \\
g_6(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) &= b_1c_1 + b_2c_2 + b_3c_3
\end{aligned}$$

Prove that the gradients of $g_1, \cdots, g_6$, evaluated at any

$$R = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \in O(3)$$

are linearly independent.

Solution. Fix any $\big(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\big)$. The six gradients are

$$\begin{aligned}
\nabla g_1 &= 2\big(a_1, a_2, a_3, 0, 0, 0, 0, 0, 0\big) \\
\nabla g_2 &= 2\big(0, 0, 0, b_1, b_2, b_3, 0, 0, 0\big) \\
\nabla g_3 &= 2\big(0, 0, 0, 0, 0, 0, c_1, c_2, c_3\big) \\
\nabla g_4 &= \big(b_1, b_2, b_3, a_1, a_2, a_3, 0, 0, 0\big) \\
\nabla g_5 &= \big(c_1, c_2, c_3, 0, 0, 0, a_1, a_2, a_3\big) \\
\nabla g_6 &= \big(0, 0, 0, c_1, c_2, c_3, b_1, b_2, b_3\big)
\end{aligned}$$

We have to show that the only solution so $\sum_{j=1}^{6} \alpha_j \nabla g_j = 0$ is $\alpha_1 = \cdots = \alpha_6 = 0$. The first three components of $\sum_{j=1}^{6} \alpha_j \nabla g_j$ are (writing them as a column vector)

$$\alpha_1 \begin{bmatrix} 2a_1 \\ 2a_2 \\ 2a_3 \end{bmatrix} + \alpha_4 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \alpha_5 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 2\alpha_1 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = R \begin{bmatrix} 2\alpha_1 \\ \alpha_4 \\ \alpha_5 \end{bmatrix}$$

The matrix R, as an element of $O(3)$ must have determinant ± 1 and hence must be invertible. So the first three components of $\sum_{j=1}^{6} \alpha_j \nabla g_j$ are zero if and only if $2\alpha_1 = \alpha_4 = \alpha_5 = 0$. Similarly components four, five and six of $\sum_{j=1}^{6} \alpha_j \nabla g_j$ are zero if and only if $2\alpha_2 = \alpha_4 = \alpha_6 = 0$ and components seven, eight and nine of $\sum_{j=1}^{6} \alpha_j \nabla g_j$ are zero if and only if $2\alpha_3 = \alpha_5 = \alpha_6 = 0$.

Problem M.6 Use the implicit function theorem to prove that for each $1 \le i, j \le 3$, the (i, j) matrix element, a_{ij}, of matrices $R = \big[a_{ij}\big]_{1 \le i,j \le 3}$ in a neighbourhood of $\mathbb{1}$ in $SO(3)$, is a C^∞ function of the matrix elements a_{21}, a_{31} and a_{32}.

Solution. Define

$$g_1(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, a_{21}, a_{31}, a_{32}) = \sum_{j=1}^{3} a_{1j}^2 - 1$$

$$g_2(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, a_{21}, a_{31}, a_{32}) = \sum_{j=1}^{3} a_{2j}^2 - 1$$

$$g_3(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, a_{21}, a_{31}, a_{32}) = \sum_{j=1}^{3} a_{3j}^2 - 1$$

$$g_4(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, a_{21}, a_{31}, a_{32}) = \sum_{j=1}^{3} a_{1j} a_{2j}$$

$$g_5(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, a_{21}, a_{31}, a_{32}) = \sum_{j=1}^{3} a_{1j} a_{3j}$$

$$g_6(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, a_{21}, a_{31}, a_{32}) = \sum_{j=1}^{3} a_{2j} a_{3j}$$

All six functions are C^∞ functions of their 9 variables. We shall solve for the first six variables as functions of the last three. The gradients of the six functions at the identity matrix are

$$\nabla g_1(1, 0, 0, 1, 0, 1, 0, 0, 0) = \big(2, 0, 0, 0, 0, 0, 0, 0, 0\big)$$

$$\nabla g_2(1,0,0,1,0,1,0,0,0) = \big(0,0,0,2,0,0,0,0,0\big)$$
$$\nabla g_3(1,0,0,1,0,1,0,0,0) = \big(0,0,0,0,0,2,0,0,0\big)$$
$$\nabla g_4(1,0,0,1,0,1,0,0,0) = \big(0,1,0,0,0,0,1,0,0\big)$$
$$\nabla g_5(1,0,0,1,0,1,0,0,0) = \big(0,0,1,0,0,0,0,1,0\big)$$
$$\nabla g_6(1,0,0,1,0,1,0,0,0) = \big(0,0,0,0,1,0,0,0,1\big)$$

(These are linearly independent vectors.) Expanding along the first column and then along the first row and finally along the last column,

$$\det \frac{\partial(g_1\ ,\ g_2\ ,\ g_3\ ,\ g_4\ ,\ g_5\ ,\ g_6)}{\partial(a_{11},a_{12},a_{13},a_{22},a_{23},a_{33})}\Big|_{(0,0,0,1,0,0,1,0,1)}$$
$$= \det \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = 2\det \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$
$$= 4\det \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = -8\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -8 \neq 0$$

the implicit function theorem assures that given any a_{21}, a_{31}, a_{32} sufficiently small, there is a unique element of $SO(3)$ having the given values of a_{21}, a_{31}, a_{32} and being in a neighbourhood (specified by the implicit function theorem) of the identity. Furthermore, each a_{ij} is a C^∞ function of a_{21}, a_{31}, a_{32}. ■

Problem M.7 Prove that the two charts $(\mathcal{U}_2, \varphi_2)$ and $(\mathcal{U}_2, \psi_2)$ of Example M.10 are not compatible.

Solution. The range of φ_2 is $\big[0, \frac{1}{4}\big) \times (-1,1) \cup \big(-\frac{1}{4}, 0\big) \times (-1,1) = \big(-\frac{1}{4}, \frac{1}{4}\big) \times (-1,1)$. On this range

$$\varphi_2^{-1}(x,y) = \begin{cases} (x,y) & \text{if } 0 \le x < \frac{1}{4} \\ (x+1,-y) & \text{if } -\frac{1}{4} < x < 0 \end{cases}$$

so that

$$\psi_2 \circ \varphi_2^{-1}(x,y) = \left\{ \begin{array}{ll} \psi_2(x,y) & \text{if } 0 \le x < \frac{1}{4} \\ \psi_2(x+1,-y) & \text{if } -\frac{1}{4} < x < 0 \end{array} \right\} = \begin{cases} (x,y) & \text{if } 0 \le x < \frac{1}{4} \\ (x,-y) & \text{if } -\frac{1}{4} < x < 0 \end{cases}$$

This is not continuous across $x = 0$, since

$$\lim_{x\to 0+} \psi_2 \circ \varphi_2^{-1}(x,y) = (0,y) \qquad \text{while} \qquad \lim_{x\to 0-} \psi_2 \circ \varphi_2^{-1}(x,y) = (0,-y)$$

■

Problem M.8 Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds. Prove that $f : \mathcal{M} \to \mathcal{N}$ is C^∞ at $m \in \mathcal{M}$ if and only if $\psi \circ f \circ \phi^{-1}$ is C^∞ at $\phi(m)$ for every chart $(\mathcal{U}, \phi)$ for $\mathcal{M}$ with $m \in \mathcal{U}$ and every chart $(\mathcal{V}, \psi)$ for $\mathcal{N}$ with $f(m) \in \mathcal{V}$.

Solution. The "if" part is trivial. To prove the "only if" part, observe that, by definition, there are charts $\{\tilde{\mathcal{U}}, \tilde{\phi}\}$ and $\{\tilde{\mathcal{V}}, \tilde{\psi}\}$ such that $m \in \tilde{\mathcal{U}}$, $f(m) \in \tilde{\mathcal{V}}$ and $\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}$ being C^∞ at $\tilde{\phi}(m)$. Let $\{\mathcal{U}, \phi\}$ and $\{\mathcal{V}, \psi\}$ be any charts with $m \in \mathcal{U}$ and $f(m) \in \mathcal{V}$. Then $m \in \tilde{\mathcal{U}} \cap \mathcal{U}$ and $f(m) \in \tilde{\mathcal{V}} \cap \mathcal{V}$. By compatibility $\tilde{\phi} \circ \phi^{-1}$ and $\psi \circ \tilde{\psi}^{-1}$ are C^∞ at $\phi(m)$ and $\tilde{\psi}(f(m))$ respectively. Consequently,

$$\psi \circ f \circ \phi^{-1} = \left(\psi \circ \tilde{\psi}^{-1}\right) \circ \left(\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}\right) \circ \left(\tilde{\phi} \circ \phi^{-1}\right)$$

is C^∞ at $\phi(m)$. ■

Problem M.9 Prove that $\mathbb{R}^n$ is diffeomorphic to $\left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n x_i^2 < 1 \right\}$.

Solution. The map

$$\Phi(x) = \frac{x}{\sqrt{1 - |x|^2}}$$

is a diffeomorphism from $\left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n x_i^2 < 1 \right\}$ to $\mathbb{R}^n$. The inverse map is

$$\Phi^{-1}(y) = \frac{y}{\sqrt{1 + |y|^2}}$$

■

Problem M.10 Prove that $\mathbb{R}^n$ is not diffeomorphic to S^n.

Solution. Suppose that $\Phi : S^n \to \mathbb{R}^n$ were a diffeomorphism. Let $g : \mathbb{R}^n \to \mathbb{R}$ be defined by $g(x) = x_1$. Then $g \circ \Phi : S^n \to \mathbb{R}$ is C^∞ and is onto $\mathbb{R}$. But that is impossible because S^n is compact so that every C^∞ function on S^n is bounded. ■

Problem M.11 Outline an argument to prove that the disk $\left\{ \mathbf{x} \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 < 2 \right\}$ is not diffeomorphic to the annulus $\left\{ \mathbf{x} \in \mathbb{R}^2 \;\middle|\; 1 < x^2 + y^2 < 2 \right\}$.

Solution. The disk $\mathcal{M} = \left\{ \mathbf{x} \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 < 2 \right\}$ has the property that every C^∞ closed curve may be continuously deformed to a point. To see this parametrize any curve

by a function $f : \mathbb{R} \to \mathcal{M}$ that has period one. Then $f_s(t) = (1-s)f(t)$ implements the deformation, since $f_0(t) = f(t)$ and $f_1(t) = 0$ has range the single point $\{0\}$. If the disk and annulus were diffeomorphic, the annulus would also have this property. It doesn't. For example, the circle $C = \left\{ (x,y) \;\middle|\; x^2+y^2 = 1 \right\}$ cannot be deformed to a point in the annulus. If it could, Green's theorem would yield that $\oint_C \left[-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \right] = 0$, which is false. ■

Problem M.12 In this problem $G = SO(3)$.

a) Fix any $a \in G$. Denote by $I = \left\{ (i,j) \in \mathbb{N}^2 \;\middle|\; 1 \le i \le 3,\ 1 \le j \le 3 \right\}$ the set of indices for the matrix elements of the matrices in G. Prove that there exist $\alpha, \beta, \gamma \in I$ such that every matrix element $g_\delta,\ \delta \in I$ is a C^∞ function of $g_\alpha,\ g_\beta,\ g_\gamma$ for matrices $g \in G$ in a neighbourhood of a.

b) Prove that a curve $q : (c,d) \to G$ is C^∞ if and only if every matrix element $q(t)_{i,j}$ is C^∞.

c) Prove that matrix multiplication $(a,b) \mapsto ab$ is a C^∞ function from $G \times G$ to G.

d) Prove that the inverse function $a \mapsto a^{-1}$ is a C^∞ function from G to G.

Solution. (a) Name the matrix elements of $g \in G$ by

$$\begin{bmatrix} a_1(g) & b_1(g) & c_1(g) \\ a_2(g) & b_2(g) & c_2(g) \\ a_3(g) & b_3(g) & c_3(g) \end{bmatrix}$$

and set

$$\begin{aligned} f_1 &= a_1^2 + a_2^2 + a_3^2 & f_2 &= b_1^2 + b_2^2 + b_3^2 & f_3 &= c_1^2 + c_2^2 + c_3^2 \\ f_4 &= a_1b_1 + a_2b_2 + a_3b_3 & f_5 &= a_1c_1 + a_2c_2 + a_3c_3 & f_6 &= b_1c_1 + b_2c_2 + b_3c_3 \end{aligned}$$

We have to show that, with $\nabla = \left(\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3}, \frac{\partial}{\partial b_1}, \frac{\partial}{\partial b_2}, \frac{\partial}{\partial b_3}, \frac{\partial}{\partial c_1}, \frac{\partial}{\partial c_2}, \frac{\partial}{\partial c_3}, \right)$,

$$\begin{aligned} \nabla f_1 &= \big(2a_1, 2a_2, 2a_3, 0, 0, 0, 0, 0, 0\big) & \nabla f_2 &= \big(0, 0, 0, 2b_1, 2b_2, 2b_3, 0, 0, 0\big) \\ \nabla f_3 &= \big(0, 0, 0, 0, 0, 0, 2c_1, 2c_2, 2c_3\big) & \nabla f_4 &= \big(b_1, b_2, b_3, a_1, a_2, a_3, 0, 0, 0\big) \\ \nabla f_5 &= \big(c_1, c_2, c_3, 0, 0, 0, a_1, a_2, a_3\big) & \nabla f_6 &= \big(0, 0, 0, c_1, c_2, c_3, b_1, b_2, b_3\big) \end{aligned}$$

are linearly independent vectors at any point obeying $f_1 = f_2 = f_3 = 1$ and $f_4 = f_5 = f_6 = 0$. In other words, we have to show that

$$\alpha_1\nabla f_1 + \alpha_2\nabla f_2 + \alpha_3\nabla f_3 + \alpha_4\nabla f_4 + \alpha_5\nabla f_5 + \alpha_6\nabla f_6 = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$$

at every such point. This follows from

$$\begin{aligned}
\alpha_1 &= \tfrac{1}{2}\big(2|a|^2\,\alpha_1 + a\cdot b\,\alpha_4 + a\cdot c\,\alpha_5\big)\\
&= \tfrac{1}{2}\big(a_1,a_2,a_3,0,0,0,0,0,0\big)\cdot\big(\alpha_1\nabla f_1+\alpha_2\nabla f_2+\alpha_3\nabla f_3+\alpha_4\nabla f_4+\alpha_5\nabla f_5+\alpha_6\nabla f_6\big)\\
\alpha_2 &= \tfrac{1}{2}\big(2|b|^2\,\alpha_2 + a\cdot b\,\alpha_4 + b\cdot c\,\alpha_6\big)\\
&= \tfrac{1}{2}\big(b_1,b_2,b_3,0,0,0\big)\cdot\big(\alpha_1\nabla f_1+\alpha_2\nabla f_2+\alpha_3\nabla f_3+\alpha_4\nabla f_4+\alpha_5\nabla f_5+\alpha_6\nabla f_6\big)\\
\alpha_3 &= \tfrac{1}{2}\big(2|c|^2\,\alpha_3 + a\cdot c\,\alpha_5 + b\cdot c\,\alpha_6\big)\\
&= \tfrac{1}{2}\big(0,0,0,0,0,0,c_1,c_2,c_3\big)\cdot\big(\alpha_1\nabla f_1+\alpha_2\nabla f_2+\alpha_3\nabla f_3+\alpha_4\nabla f_4+\alpha_5\nabla f_5+\alpha_6\nabla f_6\big)\\
\alpha_4 &= 2a\cdot b\,\alpha_1 + |b|^2\,\alpha_4 + b\cdot c\,\alpha_5\\
&= \big(b_1,b_2,b_3,0,0,0,0,0,0\big)\cdot\big(\alpha_1\nabla f_1+\alpha_2\nabla f_2+\alpha_3\nabla f_3+\alpha_4\nabla f_4+\alpha_5\nabla f_5+\alpha_6\nabla f_6\big)\\
\alpha_5 &= 2a\cdot c\,\alpha_1 + b\cdot c\,\alpha_4 + |c|^2\,\alpha_5\\
&= \big(c_1,c_2,c_3,0,0,0,0,0,0\big)\cdot\big(\alpha_1\nabla f_1+\alpha_2\nabla f_2+\alpha_3\nabla f_3+\alpha_4\nabla f_4+\alpha_5\nabla f_5+\alpha_6\nabla f_6\big)\\
\alpha_6 &= 2b\cdot c\,\alpha_2 + a\cdot c\,\alpha_4 + |c|^2\,\alpha_6\\
&= \big(0,0,0,c_1,c_2,c_3,0,0,0\big)\cdot\big(\alpha_1\nabla f_1+\alpha_2\nabla f_2+\alpha_3\nabla f_3+\alpha_4\nabla f_4+\alpha_5\nabla f_5+\alpha_6\nabla f_6\big)
\end{aligned}$$

(b) Pick any $c < t_0 < d$. Pick three matrix elements as coordinates in some neighbourhood of $q_0(t)$. By Problem M.8, $q(t)$ is C^∞ at t_0 if and only if those three matrix elements are C^∞ at t_0. If those three matrix elements are C^∞ at t_0, the remaining matrix elements are also C^∞ by part (a).

(c) Every matrix element of ab is a polynomial in the matrix elements of a and the matrix elements of b and hence is C^∞.

(d) Every matrix element of a^{-1} is a polynomial in the matrix elements of a (since $\det a = 1$) and hence is C^∞.

■

Problem M.13 Let $\mathcal{M}$ be a manifold, ω be a 1–form on $\mathcal{M}$ and $c(t) : [0,1] \to \mathcal{M}$ be a path in $\mathcal{M}$. Prove that the definition of $\int_c \omega$ given in part (c) of Definition M.13 is independent of the decomposition of c into finitely many pieces and of the choice of coordinate patches.

Solution. "Decomposing c into finitely many pieces" means picking $0 < t_1 < t_2 < \cdots < t_m$ with $m \in \mathbb{N}$ and $\big\{\ c(t)\ \big|\ t_{\ell-1} \le t \le t_\ell\ \big\}$ contained in a single coordinate patch for each $\ell = 1,\ 2,\ \cdots,\ m$ and then applying the single patch algorithm of Definition M.13 to the part of $c(t)$ with $t_{\ell-1} \le t \le t_\ell$ for each $\ell = 1,\ 2,\ \cdots,\ m$.

Make any two such decompositions and choices of coordinate patches for each piece. Since

$$\int_{t_{\ell-1}}^{t_\ell} h(t)\ dt = \int_{t_{\ell-1}}^{s} h(t)\ dt + \int_{s}^{t_\ell} h(t)\ dt$$

for all $t_{\ell-1} < s < t_\ell$ we are free to add any finite number of decomposition points. So we may assume that the two sets of decomposition times are identical. Thus it suffices to prove that if $0 \le t_\ell < t_{\ell+1} \le 1$ and

- if $\{U, \zeta\}$ and $\{\tilde{U}, \tilde{\zeta}\}$ are two patches with $c(t) \in U \cap \tilde{U}$ for all $t_\ell \le t \le t_{\ell+1}$ and
- if ω assigns $\{U, \zeta\}$ the pair of functions (f, g) and assigns $\{\tilde{U}, \tilde{\zeta}\}$ the pair of functions $(\tilde{f}, \tilde{g})$

then

$$\int_0^1 \Big[f\big(\zeta(c(t))\big) \tfrac{dx(c(t))}{dt} + g\big(\zeta(c(t))\big) \tfrac{dy(c(t))}{dt} \Big]\ dt = \int_0^1 \Big[\tilde{f}\big(\tilde{\zeta}(c(t))\big) \tfrac{d\tilde{x}(c(t))}{dt} + \tilde{g}\big(\tilde{\zeta}(c(t))\big) \tfrac{d\tilde{y}(c(t))}{dt} \Big]\ dt$$

But this is the computation of part (c) of Remark M.14. ■

Problem M.14 Let $\mathcal{M}$ be a manifold of dimension $n \in \mathbb{N}$ (not necessarily 2) and suppose that we have defined a wedge product for $\mathcal{M}$ that is bilinear, graded anticommutative and associative (i.e. is satisfies properties (a), (b) and (c) above). Let, for each $\le j \le n$, ω_j be a 1–form on $\mathcal{M}$ and, for each $1 \le i, j \le n$, f_{ij} be a function on $\mathcal{M}$. Prove that

$$\Big(\sum_{j=1}^n f_{1j}\omega_j\Big) \bigwedge \Big(\sum_{j=1}^n f_{2j}\omega_j\Big) \bigwedge \cdots \bigwedge \Big(\sum_{j=1}^n f_{nj}\omega_j\Big) = \det\big[f_{ij}\big]_{1\le i,j\le n}\ \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$$

Solution. By linearity

$$\Big(\sum_{j=1}^n f_{1j}\omega_j\Big) \bigwedge \Big(\sum_{j=1}^n f_{2j}\omega_j\Big) \bigwedge \cdots \bigwedge \Big(\sum_{j=1}^n f_{nj}\omega_j\Big) = \sum_{j_1,\cdots,j_n=1}^n f_{1j_1}\cdots f_{nj_n}\ \omega_{j_1} \wedge \cdots \wedge \omega_{j_n}$$

By anticummutativity

$$\omega_{j_\ell} \wedge \omega_{j_{\ell+1}} = \begin{cases} -\omega_{j_{\ell+1}} \wedge \omega_{j_\ell} & \text{if } j_\ell \ne j_{\ell+1} \\ 0 & \text{if } j_\ell = j_{\ell+1} \end{cases}$$

so that

$$\omega_{j_1} \wedge \cdots \wedge \omega_{j_\ell} \wedge \omega_{j_{\ell+1}} \wedge \cdots \wedge \omega_{j_n} = \begin{cases} -\omega_{j_1} \wedge \cdots \wedge \omega_{j_{\ell+1}} \wedge \omega_{j_\ell} \wedge \cdots \wedge \omega_{j_n} & \text{if } j_\ell \ne j_{\ell+1} \\ 0 & \text{if } j_\ell = j_{\ell+1} \end{cases} \quad (1)$$

Repeatedly applying this, we have that $\omega_{j_1} \wedge \cdots \wedge \omega_{j_n}$ is zero unless $j_1, \cdots, j_n$ are all different, that is unless $j_1 = \pi(1)$, $\cdots$, $j_n = \pi(n)$ for some permutation of $(1, \cdots, n)$. Repeatedly applying (1) also gives that, for any permutation[(1)] π,

$$\omega_{\pi(1)} \wedge \cdots \wedge \omega_{\pi(n)} = \operatorname{sgn} \pi \; \omega_1 \wedge \cdots \wedge \omega_n$$

where $\operatorname{sgn} \pi$ is the sign[(1)] of the permutation. Denoting by S_n the set of all permutations of $(1, \cdots, n)$, we have

$$\begin{aligned}
\Big(\sum_{j=1}^n f_{1j}\omega_j\Big) \bigwedge \Big(\sum_{j=1}^n f_{2j}\omega_j\Big) \bigwedge \cdots \bigwedge \Big(\sum_{j=1}^n f_{nj}\omega_j\Big) &= \sum_{j_1,\cdots,j_n=1}^n f_{1j_1} \cdots f_{nj_n} \; \omega_{j_1} \wedge \cdots \wedge \omega_{j_n} \\
&= \sum_{\pi \in S_n}^n f_{1\pi(1)} \cdots f_{n\pi(n)} \; \omega_{\pi(1)} \wedge \cdots \wedge \omega_{\pi(n)} \\
&= \sum_{\pi \in S_n} \operatorname{sgn} \pi \; f_{1\pi(1)} \cdots f_{n\pi(n)} \; \omega_1 \wedge \cdots \wedge \omega_n \\
&= \det \big[f_{ij}\big]_{1 \le i,j \le n} \; \omega_1 \wedge \cdots \wedge \omega_n
\end{aligned}$$

Problem M.15 Prove that Definition M.18 is independent of the choice of coordinate patch.

Solution. Assume that

- $\{U, \zeta\}$ and $\{\tilde{U}, \tilde{\zeta}\}$ are two charts with $U \cap \tilde{U} \neq \emptyset$ and
- the transition function $\tilde{\zeta} \circ \zeta^{-1}$ $\big($from $\zeta(U \cap \tilde{U}) \subset \mathbb{R}^2$ to $\tilde{\zeta}(U \cap \tilde{U}) \subset \mathbb{R}^2\big)$ is $\big(\tilde{x}(x,y), \tilde{y}(x,y)\big)$

(a) Let $F : \mathcal{M} \to \mathbb{C}$ be a C^1 0–form and set $\varphi = F \circ \zeta^{-1}$ and $\tilde{\varphi} = F \circ \tilde{\zeta}^{-1}$. Then $\varphi(x,y) = \tilde{\varphi}\big(\tilde{x}(x,y), \tilde{y}(x,y)\big)$ and

$$\begin{aligned}
dF\big|_{\{\tilde{U},\tilde{\zeta}\}} &= \tilde{f}(\tilde{x},\tilde{y})\, d\tilde{x} + \tilde{g}(\tilde{x},\tilde{y})\, d\tilde{y} \text{ with } \tilde{f}(\tilde{x},\tilde{y}) = \tfrac{\partial\tilde{\varphi}}{\partial\tilde{x}}(\tilde{x},\tilde{y}),\ \tilde{g}(\tilde{x},\tilde{y}) = \tfrac{\partial\tilde{\varphi}}{\partial\tilde{y}}(\tilde{x},\tilde{y}) \\
dF\big|_{\{U,\zeta\}} &= \tfrac{\partial\varphi}{\partial x}(x,y)\, dx + \tfrac{\partial\varphi}{\partial y}(x,y)\, dy \\
&= \Big\{\tfrac{\partial\tilde{\varphi}}{\partial\tilde{x}}\big(\tilde{x}(x,y),\tilde{y}(x,y)\big)\tfrac{\partial\tilde{x}}{\partial x}(x,y) + \tfrac{\partial\tilde{\varphi}}{\partial\tilde{y}}\big(\tilde{x}(x,y),\tilde{y}(x,y)\big)\tfrac{\partial\tilde{y}}{\partial x}(x,y)\Big\}\, dx \\
&\quad + \Big\{\tfrac{\partial\tilde{\varphi}}{\partial\tilde{x}}\big(\tilde{x}(x,y),\tilde{y}(x,y)\big)\tfrac{\partial\tilde{x}}{\partial y}(x,y) + \tfrac{\partial\tilde{\varphi}}{\partial\tilde{y}}\big(\tilde{x}(x,y),\tilde{y}(x,y)\big)\tfrac{\partial\tilde{y}}{\partial y}(x,y)\Big\}\, dy \\
&= f(x,y)\, dx + g(x,y)\, dy
\end{aligned}$$

(1) A permutation of $(1, \cdots, n)$ is a 1–1 map from $\{1, \cdots, n\}$ onto $\{1, \cdots, n\}$. A transposition is a permutation that just exchanges two neighbouring elements. Any permutation may be expressed as a product (i.e. composition) of transpositions. The sign of a permutation is $+1$ if it is the product of an even number of transpositions and -1 if it is the product of an odd number of transpositions.

with

$$f(x,y) = \tilde{f}\big(\tilde{x}(x,y),\tilde{y}(x,y)\big)\tfrac{\partial\tilde{x}}{\partial x}(x,y) + \tilde{g}\big(\tilde{x}(x,y),\tilde{y}(x,y)\big)\tfrac{\partial\tilde{y}}{\partial x}(x,y)$$
$$g(x,y) = \tilde{f}\big(\tilde{x}(x,y),\tilde{y}(x,y)\big)\tfrac{\partial\tilde{x}}{\partial y}(x,y) + \tilde{g}\big(\tilde{x}(x,y),\tilde{y}(x,y)\big)\tfrac{\partial\tilde{y}}{\partial y}(x,y)$$

This agrees with the coordinate transformation rule of Definition M.13.

(b) If ω is a C^1 1–form with

$$\omega\big|_{\{U,\zeta\}} = f(x,y)\,dx + g(x,y)\,dy$$
$$\omega\big|_{\{\tilde{U},\tilde{\zeta}\}} = \tilde{f}(\tilde{x},\tilde{y})\,d\tilde{x} + \tilde{g}(\tilde{x},\tilde{y})\,d\tilde{y}$$

then

$$f(x,y) = \tilde{f}\big(\tilde{x}(x,y),\tilde{y}(x,y)\big)\tfrac{\partial\tilde{x}}{\partial x}(x,y) + \tilde{g}\big(\tilde{x}(x,y),\tilde{y}(x,y)\big)\tfrac{\partial\tilde{y}}{\partial x}(x,y)$$
$$g(x,y) = \tilde{f}\big(\tilde{x}(x,y),\tilde{y}(x,y)\big)\tfrac{\partial\tilde{x}}{\partial y}(x,y) + \tilde{g}\big(\tilde{x}(x,y),\tilde{y}(x,y)\big)\tfrac{\partial\tilde{y}}{\partial y}(x,y)$$

Since

$$\tfrac{\partial g}{\partial x}(x,y) = \tfrac{\partial\tilde{f}}{\partial\tilde{x}}\tfrac{\partial\tilde{x}}{\partial x}\tfrac{\partial\tilde{x}}{\partial y} + \tfrac{\partial\tilde{f}}{\partial\tilde{y}}\tfrac{\partial\tilde{y}}{\partial x}\tfrac{\partial\tilde{x}}{\partial y} + \tilde{f}\tfrac{\partial^2\tilde{x}}{\partial x\partial y} + \tfrac{\partial\tilde{g}}{\partial\tilde{x}}\tfrac{\partial\tilde{x}}{\partial x}\tfrac{\partial\tilde{y}}{\partial y} + \tfrac{\partial\tilde{g}}{\partial\tilde{y}}\tfrac{\partial\tilde{y}}{\partial x}\tfrac{\partial\tilde{y}}{\partial y} + \tilde{g}\tfrac{\partial^2\tilde{y}}{\partial x\partial y}$$
$$\tfrac{\partial f}{\partial y}(x,y) = \tfrac{\partial\tilde{f}}{\partial\tilde{x}}\tfrac{\partial\tilde{x}}{\partial y}\tfrac{\partial\tilde{x}}{\partial x} + \tfrac{\partial\tilde{f}}{\partial\tilde{y}}\tfrac{\partial\tilde{y}}{\partial y}\tfrac{\partial\tilde{x}}{\partial x} + \tilde{f}\tfrac{\partial^2\tilde{x}}{\partial x\partial y} + \tfrac{\partial\tilde{g}}{\partial\tilde{x}}\tfrac{\partial\tilde{x}}{\partial y}\tfrac{\partial\tilde{y}}{\partial x} + \tfrac{\partial\tilde{g}}{\partial\tilde{y}}\tfrac{\partial\tilde{y}}{\partial y}\tfrac{\partial\tilde{y}}{\partial x} + \tilde{g}\tfrac{\partial^2\tilde{y}}{\partial x\partial y}$$

we have that

$$\begin{aligned} d\omega\big|_{\{U,\zeta\}} &= \big[\tfrac{\partial g}{\partial x}(x,y) - \tfrac{\partial f}{\partial y}(x,y)\big]\,dx\wedge dy \\ &= \big[\tfrac{\partial\tilde{g}}{\partial\tilde{x}} - \tfrac{\partial\tilde{f}}{\partial y}\big]\,\big[\tfrac{\partial\tilde{x}}{\partial x}\tfrac{\partial\tilde{y}}{\partial y} - \tfrac{\partial\tilde{x}}{\partial y}\tfrac{\partial\tilde{y}}{\partial x}\big]\,dx\wedge dy \end{aligned}$$

which agrees with the coordinate transformation rule of Definition M.15.

(c) The 2–form case is trivial. ■

Problem M.16 Prove the graded product rule that if ω is a k–form and ω' is a k'–form, then

$$d(\omega\wedge\omega') = (d\omega)\wedge\omega' + (-1)^k\omega\wedge(d\omega')$$

Solution. It suffices to consider $k+k'\le 1$, since all of $d(\omega\wedge\omega')$, $(d\omega)\wedge\omega'$ and $\omega\wedge(d\omega')$ vanish if $k+k'\ge 2$ because the manifold $\mathcal{M}$ is assumed to be two dimensional. It suffices to verify the specified formula in any chart $\{U,\zeta\}$,

(a) If $k=k'=0$, $\omega\circ\zeta^{-1}(x,y) = f(x,y)$ and $\omega'\circ\zeta^{-1}(x,y) = g(x,y)$, then

$$\begin{aligned} d\omega &= \tfrac{\partial f}{\partial x}(x,y)\,dx + \tfrac{\partial f}{\partial y}(x,y)\,dy \\ d\omega' &= \tfrac{\partial g}{\partial x}(x,y)\,dx + \tfrac{\partial g}{\partial y}(x,y)\,dy \\ (d\omega)\wedge\omega' + (-1)^k\omega\wedge(d\omega') &= \tfrac{\partial f}{\partial x}(x,y)g(x,y)\,dx + \tfrac{\partial f}{\partial y}(x,y)g(x,y)\,dy \\ &\quad + f(x,y)\tfrac{\partial g}{\partial x}(x,y)\,dx + f(x,y)\tfrac{\partial g}{\partial y}(x,y)\,dy \end{aligned}$$

and

$$d(\omega \wedge \omega') = \tfrac{\partial}{\partial x}(fg)\,dx + \tfrac{\partial}{\partial y}(fg)\,dy$$

agree by the product rule.

(b) If $k = 0$, $k' = 1$, $\omega \circ \zeta^{-1}(x,y) = f(x,y)$ and $\omega'\big|_{\{\mathcal{U},\zeta\}} = g(x,y)\,dx + h(x,y)\,dy$, then

$$\begin{aligned}
d\omega &= \tfrac{\partial f}{\partial x}(x,y)\,dx + \tfrac{\partial f}{\partial y}(x,y)\,dy \\
d\omega' &= \big[\tfrac{\partial h}{\partial x}(x,y) - \tfrac{\partial g}{\partial y}(x,y)\big]\,dx \wedge dy \\
(d\omega) \wedge \omega' + (-1)^k \omega \wedge (d\omega') &= \big[\tfrac{\partial f}{\partial x}(x,y)\,dx + \tfrac{\partial f}{\partial y}(x,y)\,dy\big] \wedge \big[g(x,y)\,dx + h(x,y)\,dy\big] \\
&\qquad + f(x,y)\big[\tfrac{\partial h}{\partial x}(x,y) - \tfrac{\partial g}{\partial y}(x,y)\big]\,dx \wedge dy \\
&= \big[h\tfrac{\partial f}{\partial x} - g\tfrac{\partial f}{\partial y} + f\tfrac{\partial h}{\partial x} - f\tfrac{\partial g}{\partial y}\big]\,dx \wedge dy
\end{aligned}$$

and

$$\begin{aligned}
d(\omega \wedge \omega') &= d\big(f(x,y)g(x,y)\,dx + f(x,y)h(x,y)\,dy\big) \\
&= \big[-\tfrac{\partial}{\partial y}\big(f(x,y)g(x,y)\big) + \tfrac{\partial}{\partial x}\big(f(x,y)h(x,y)\big)\big]\,dx \wedge dy
\end{aligned}$$

agree by the product rule.

(c) If $k = 1$ and $k' = 0$, then, by the previous case

$$d(\omega \wedge \omega') = d(\omega' \wedge \omega) = (d\omega') \wedge \omega + \omega' \wedge (d\omega) = (d\omega) \wedge \omega' - \omega \wedge (d\omega')$$

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Problem M.17 (Vector analysis in $\mathbb{R}^3$) Let $\mathcal{M}$ be $\mathbb{R}^3$ with atlas $\big(U = \mathbb{R}^3,\ \zeta(\vec{x}) = \vec{x} = (x,y,z)\big)$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be any C^∞ function on $\mathbb{R}^3$ and $\vec{a}(\vec{x}) = \big(a^1(\vec{x})\ ,\ a^2(\vec{x})\ ,\ a^3(\vec{x})\big)$ and $\vec{b}(\vec{x}) = \big(b^1(\vec{x})\ ,\ b^2(\vec{x})\ ,\ b^3(\vec{x})\big)$ be any two vector fields (i.e. vector valued functions) on $\mathbb{R}^3$. We can associate to $\vec{a}(\vec{x})$ a 1–form $\omega^1_{\vec{a}}$ and a 2–form $\omega^2_{\vec{a}}$ by

$$\begin{aligned}
\omega^1_{\vec{a}} &= a^1(\vec{x})\,dx + a^2(\vec{x})\,dy + a^3(\vec{x})\,dz \\
\omega^2_{\vec{a}} &= a^1(\vec{x})\,dy \wedge dz + a^2(\vec{x})\,dz \wedge dx + a^3(\vec{x})\,dx \wedge dy
\end{aligned}$$

Prove that

(a) $\omega^1_{\vec{a}} \wedge \omega^1_{\vec{b}} = \omega^2_{\vec{a}\times\vec{b}}$

(b) $\omega^1_{\vec{a}} \wedge \omega^2_{\vec{b}} = \vec{a}(\vec{x}) \cdot \vec{b}(\vec{x})\ dx \wedge dy \wedge dz$

(c) $df = \omega^1_{\vec{\nabla} f}$

(d) $d\omega^1_{\vec{a}} = \omega^2_{\vec{\nabla}\times\vec{a}}$

(e) $d\omega^2_{\vec{a}} = \vec{\nabla} \cdot \vec{a}(\vec{x})\ dx \wedge dy \wedge dz$

Observe that

- $d^2 f = 0$ says that $0 = d\omega^1_{\vec{\nabla} f} = \omega^2_{\vec{\nabla}\times\vec{\nabla} f}$ i.e. that $\vec{\nabla}\times\vec{\nabla} f = 0$.
- $d^2\omega^1_{\vec{a}} = 0$ says that $0 = d\omega^2_{\vec{\nabla}\times\vec{a}} = \vec{\nabla}\cdot\vec{\nabla}\times\vec{a}(\vec{x})\ dx\wedge dy\wedge dz$ i.e. that $\vec{\nabla}\cdot\vec{\nabla}\times\vec{a} = 0$.

Solution. (a)

$$\begin{aligned}
\omega^1_{\vec{a}}\wedge\omega^1_{\vec{b}} &= \big(a^1(\vec{x})\,dx + a^2(\vec{x})\,dy + a^3(\vec{x})\,dz\big)\wedge\big(b^1(\vec{x})\,dx + b^2(\vec{x})\,dy + b^3(\vec{x})\,dz\big)\\
&= a^1(\vec{x})b^2(\vec{x})\,dx\wedge dy + a^1(\vec{x})b^3(\vec{x})\,dx\wedge dz + a^2(\vec{x})b^1(\vec{x})\,dy\wedge dx\\
&\qquad + a^2(\vec{x})b^3(\vec{x})\,dy\wedge dz + a^3(\vec{x})b^1(\vec{x})\,dz\wedge dx + a^3(\vec{x})b^2(\vec{x})\,dz\wedge dy\\
&= \big(a^1(\vec{x})b^2(\vec{x}) - a^2(\vec{x})b^1(\vec{x})\big)\,dx\wedge dy + \big(a^3(\vec{x})b^1(\vec{x}) - a^1(\vec{x})b^3(\vec{x})\big)\,dz\wedge dx\\
&\qquad + \big(a^2(\vec{x})b^3(\vec{x}) - a^3(\vec{x})b^2(\vec{x})\big)\,dy\wedge dz\\
&= \omega^2_{\vec{a}\times\vec{b}}
\end{aligned}$$

(b)

$$\begin{aligned}
\omega^1_{\vec{a}}\wedge\omega^2_{\vec{b}} &= \big(a^1(\vec{x})\,dx + a^2(\vec{x})\,dy + a^3(\vec{x})\,dz\big)\wedge\big(b^1(\vec{x})\,dy\wedge dz + b^2(\vec{x})\,dz\wedge dx + b^3(\vec{x})\,dx\wedge dy\big)\\
&= a^1(\vec{x})b^1(\vec{x})\,dx\wedge dy\wedge dz + a^2(\vec{x})b^2(\vec{x})\,dy\wedge dz\wedge dx + a^3(\vec{x})b^3(\vec{x})\,dz\wedge dx\wedge dy\\
&= \vec{a}(\vec{x})\cdot\vec{b}(\vec{x})\ dx\wedge dy\wedge dz
\end{aligned}$$

(c)

$$df = \tfrac{\partial f}{\partial x}\,dx + \tfrac{\partial f}{\partial y}\,dy + \tfrac{\partial f}{\partial z}\,dz = \omega^1_{\vec{\nabla} f}$$

(d)

$$\begin{aligned}
d\omega^1_{\vec{a}} &= d\big(a^1(\vec{x})\,dx + a^2(\vec{x})\,dy + a^3(\vec{x})\,dz\big)\\
&= \tfrac{\partial a^1}{\partial y}\,dy\wedge dx + \tfrac{\partial a^1}{\partial z}\,dz\wedge dx + \tfrac{\partial a^2}{\partial x}\,dx\wedge dy + \tfrac{\partial a^2}{\partial z}\,dz\wedge dy + \tfrac{\partial a^3}{\partial x}\,dx\wedge dz + \tfrac{\partial a^3}{\partial y}\,dy\wedge dz\\
&= \big(\tfrac{\partial a^3}{\partial y} - \tfrac{\partial a^2}{\partial z}\big)\,dy\wedge dz + \big(\tfrac{\partial a^1}{\partial z} - \tfrac{\partial a^3}{\partial x}\big)\,dz\wedge dx + \big(\tfrac{\partial a^2}{\partial x} - \tfrac{\partial a^1}{\partial y}\big)\,dx\wedge dy\\
&= \omega^2_{\vec{\nabla}\times\vec{a}}
\end{aligned}$$

(e)

$$\begin{aligned}
d\omega^2_{\vec{a}} &= d\big(a^1(\vec{x})\,dy\wedge dz + a^2(\vec{x})\,dz\wedge dx + a^3(\vec{x})\,dx\wedge dy\big)\\
&= \tfrac{\partial a^1}{\partial z}\,dz\wedge dx\wedge dy + \tfrac{\partial a^2}{\partial y}\,dy\wedge dz\wedge dx + \tfrac{\partial a^3}{\partial z}\,dz\wedge dx\wedge dy\\
&= \vec{\nabla}\cdot\vec{a}(\vec{x})\ dx\wedge dy\wedge dz
\end{aligned}$$

Problem M.18 Let Ω be an open connected, simply connected subset of $\mathbb{R}^2$. Think of Ω as a two dimensional manifold as in Example M.2. Let $F_1(x,y),\ F_2(x,y) \in C^\infty(\Omega)$ obey the compatibility condition that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. The goal of this problem is to prove that there exists a function $\varphi(x,y) \in C^\infty(\Omega)$ such that

$$F_1(x,y) = \tfrac{\partial \varphi}{\partial x}(x,y) \qquad \text{and} \qquad F_2(x,y) = \tfrac{\partial \varphi}{\partial y}(x,y)$$

This is the analog in two dimensions of the statement that, if Ω is a simply connected region in $\mathbb{R}^3$ and $\vec{F}(\vec{x})$ is a vector field in Ω that obeys $\vec{\nabla} \times \vec{F}(\vec{x}) = \vec{0}$, then there is a "potential" $\varphi(\vec{x})$ such that $\vec{F}(\vec{x}) = \vec{\nabla}\varphi(\vec{x})$.

(a) Define the 1–form $\omega = F_1(x,y)\,dx + F_2(x,y)\,dy$. Prove that ω is closed.

(b) Let $C_1(t), C_2(t) : [0,1] \to \Omega$ be any two paths in Ω with $C_1(0) = C_2(0)$ and $C_1(1) = C_2(1)$. That is, the two paths have the same initial and final points. Prove that $\int_{C_1} \omega = \int_{C_2} \omega$.

(c) Fix any point $(x_0, y_0) \in \Omega$. For each point $(x,y) \in \Omega$, select a path $C_{x,y}(t) : [0,1] \to \Omega$ such that $C_{x,y}(0) = (x_0, y_0)$ and $C_{x,y}(1) = (x,y)$. Define $\varphi(x,y) = \int_{C_{x,y}} \omega$. Prove that

$$\tfrac{\partial \varphi}{\partial x}(x,y) = F_1(x,y) \qquad \text{and} \qquad \tfrac{\partial \varphi}{\partial y}(x,y) = F_2(x,y)$$

(d) Let $\phi(x,y)$ and $\psi(x,y)$ be any two functions on Ω that obey

$$\tfrac{\partial \phi}{\partial x}(x,y) = \tfrac{\partial \psi}{\partial x}(x,y) = F_1(x,y) \qquad \text{and} \qquad \tfrac{\partial \phi}{\partial y}(x,y) = \tfrac{\partial \psi}{\partial y}(x,y) = F_2(x,y)$$

Prove that $\phi(x,y) - \psi(x,y)$ is a constant independent of x and y.

Solution. (a) By the hypothesised compatibility,

$$d\omega = \tfrac{\partial F_1}{\partial y}\, dy \wedge dx + \tfrac{\partial F_2}{\partial x}\, dx \wedge dy = \left(\tfrac{\partial F_2}{\partial x} - \tfrac{\partial F_1}{\partial y}\right) dx \wedge dy = 0$$

(b) Since Ω is simply connected, it is possible to deform C_1 continouously to C_2. That is, there is a smooth function $D : [0,1] \times [0,1] \to \Omega$ with

$$\begin{aligned}
D(t,0) &= C_1(t) && \text{for all } 0 \le t \le 1 \\
D(t,1) &= C_2(t) && \text{for all } 0 \le t \le 1 \\
D(0,s) &= C_1(0) = C_2(0) && \text{for all } 0 \le s \le 1 \\
D(1,s) &= C_1(1) = C_2(1) && \text{for all } 0 \le s \le 1
\end{aligned}$$

The boundary $\partial D = C_1 + E_1 - C_2 - E_2$ of D consists of $C_1 - C_2$ and the two degenerate curves

$$E_1(s) = C_1(1) = C_2(1) \qquad E_2(s) = C_1(0) = C_2(0) \qquad \text{for all } 0 \le s \le 1$$

So $\int_{E_1} \omega = \int_{E_2} \omega = 0$ and, by Stoke's theorem,

$$\int_{C_1} \omega - \int_{C_2} \omega = \int_{\partial D} \omega = \int_D d\omega = 0$$

(c) Fix any point (x, y) in the open set Ω. We prove that $\frac{\partial \varphi}{\partial x}(x, y) = F_1(x, y)$. The proof of other case is similar. Fix another point (x_1, y) in Ω with the property that $(x'', y) \in \Omega$ for all x'' between x_1 and x. Choose, for each (x', y) with x' sufficiently close to x, the curve $C_{x',y}$ to obey

- $C_{x',y}(0) = (x_0, y_0)$
- $C_{x',y}(\frac{1}{2}) = (x_1, y)$
- $C_{x',y}(t)$ is independent of x' for $0 \le t \le \frac{1}{2}$
- $C_{x',y}(t) = \big(x_1 + 2(x' - x_1)(t - \frac{1}{2}), y\big)$ for $\frac{1}{2} \le t \le 1$.

That is, $C_{x'y}$ starts at (x_0, y_0) at $t = 0$, moves to (x_1, y), along a path independent of x', at $t = \frac{1}{2}$ and then follows a horizontal straight line to (x', y) at $t = 1$. The contribution to $\varphi(x', y) = \int_{C_{x',y}} \omega$ arising from $0 \le t \le \frac{1}{2}$ is independent of x' and will give contribution zero to $\frac{\partial \varphi}{\partial x}(x, y)$. The contribution to $\varphi(x', y) = \int_{C_{x',y}} \omega$ arising from $\frac{1}{2} \le t \le 1$ is precisely

$$\int_{\frac{1}{2}}^{1} F_1\big(x_1 + 2(x' - x_1)(t - \tfrac{1}{2}), y\big)\, 2(x' - x_1)\, dt = \int_{x_1}^{x'} F_1(\xi, y)\, d\xi \qquad \text{where } \xi = x_1 + 2(x' - x_1)(t - \tfrac{1}{2})$$

The derivative of this with respect to x' is $F_1(x', y)$. Hence $\frac{\partial \varphi}{\partial x}(x, y) = F_1(x, y)$ as desired.

(d) The gradient of $\theta = \phi - \psi$ vanishes identically. So $\theta = \phi - \psi$ is a constant. Here is another proof in the language of forms. If (x_0, y_0) and (x, y) are any two points in Ω and C is a curve from (x_0, y_0) to (x, y) in Ω, then

$$\theta(x, y) - \theta(x_0, y_0) = \int_{\partial C} \theta = \int_C d\theta = 0$$

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