Problem Solutions for "Integration on Manifolds"

Problem M.1 Let $\mathcal{A}$ be an atlas for the metric space $\mathcal{M}$. Prove that there is a unique maximal atlas for $\mathcal{M}$ that contains $\mathcal{A}$.

Solution. Define $\mathfrak{A}$ to be the set of all charts $\{\mathcal{V},\psi\}$ that are compatible with all of the charts of $\mathcal{A}$. If we can show that $\mathfrak{A}$ is an atlas, we are done, because (a) $\mathcal{A}$ is an atlas, so that each chart in $\mathcal{A}$ is compatible with all charts of $\mathcal{A}$ and hence $\mathcal{A}\subset\mathfrak{A}$ and (b) any chart in any atlas that contains $\mathcal{A}$ must be compatible with every chart of $\mathcal{A}$ and hence every atlas that contains $\mathcal{A}$ is contained in $\mathfrak{A}$. Let $\{\mathcal{V},\psi\}$ and $\{\mathcal{W},\zeta\}$ be any two charts of $\mathfrak{A}$ with $\mathcal{V}\cap\mathcal{W}\neq\emptyset$. Let $x\in\mathcal{V}\cap\mathcal{W}$. We must show that $\zeta\circ\psi^{-1}$ is C^∞ in some neighbourhood of $\psi(x)$. Since $\mathcal{A}$ is an atlas, it contains a chart $\{\mathcal{U},\phi\}$ with $x\in\mathcal{U}$. Since $\{\mathcal{V},\psi\}$ and $\{\mathcal{W},\zeta\}$ must both be compatible with $\{\mathcal{U},\phi\}$, $\phi\circ\psi^{-1}$ must be C^∞ in some neighbourhood of $\psi(x)$ and $\zeta\circ\phi^{-1}$ must be C^∞ in some neighbourhood of $\phi(x)$. But then the composition $\zeta\circ\phi^{-1}\circ\phi\circ\psi^{-1}$ is C^∞ in some neighbourhood of $\psi(x)$. As $\zeta\circ\phi^{-1}\phi\circ\psi^{-1}=\zeta\circ\psi^{-1}$ in a neighbourhood of $\psi(x)$, we are done. ■

Problem M.2 Let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of a metric space $\mathcal{M}$. Let φ be a homeomorphism from $\mathcal{U}$ to an open subset of $\mathbb{R}^n$ and ψ be a homeomorphism from $\mathcal{V}$ to an open subset of $\mathbb{R}^m$. Prove that if $\mathcal{U}\cap\mathcal{V}$ is nonempty and

$$\begin{aligned}\psi\circ\varphi^{-1}&:\ \varphi(\mathcal{U}\cap\mathcal{V})\subset\mathbb{R}^n\to\psi(\mathcal{U}\cap\mathcal{V})\subset\mathbb{R}^m\\ \varphi\circ\psi^{-1}&:\ \psi(\mathcal{U}\cap\mathcal{V})\subset\mathbb{R}^m\to\varphi(\mathcal{U}\cap\mathcal{V})\subset\mathbb{R}^n\end{aligned}$$

are C^∞, then $m=n$.

Solution. Write $f(x)=\psi\circ\varphi^{-1}(x)$ and $g(y)=\varphi\circ\psi^{-1}(y)$. Fix any $p\in\mathcal{U}\cap\mathcal{V}$. Set

$$A=\Big[\tfrac{\partial f_i}{\partial x_j}\big(\varphi(p)\big)\Big]_{\substack{1\le i\le m\\1\le j\le n}}\qquad B=\Big[\tfrac{\partial g_i}{\partial y_j}\big(\psi(p)\big)\Big]_{\substack{1\le i\le n\\1\le j\le m}}$$

Since $f\big(g(y)\big)=y$ for all y is a neighbourhood of $\psi(p)$, the chain rule gives that

$$\sum_{k=1}^{n}\tfrac{\partial f_i}{\partial x_k}\big(g(y)\big)\tfrac{\partial g_k}{\partial y_j}(y)=\delta_{i,j}$$

for all y is a neighbourhood of $\psi(p)$ and all $1\le i,j\le m$. In particular $AB=\mathbb{1}_m$, the $m\times m$ identity matrix. Similarly, since $g\big(f(x)\big)=x$ for all x is a neighbourhood of $\varphi(p)$, $BA=\mathbb{1}_n$. So A is the inverse of the matrix B. But only square matrices have inverses, so $m=n$. ■

Problem M.3 Let $S^n = \big\{\ \mathbf{x} = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}\ \big|\ |\mathbf{x}| = 1\ \big\}$ be the standard n–dimensional sphere.

(a) For each $1 \le i \le n+1$ set

$$\begin{aligned} \mathcal{U}_i &= \big\{\ \mathbf{x} \in S^n\ \big|\ x_i > 0\ \big\} \quad \varphi_i(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1}) \\ \mathcal{V}_i &= \big\{\ \mathbf{x} \in S^n\ \big|\ x_i < 0\ \big\} \quad \psi_i(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1}) \end{aligned}$$

Prove that $\mathcal{A}_1 = \big\{\ (\mathcal{U}_i, \varphi_i),\ (\mathcal{V}_i, \psi_i)\ \big|\ 1 \le i \le n+1\ \big\}$ is an atlas for S^n.

(b) Set

$$\mathcal{U} = S^n \setminus \{(0, \cdots, 0, 1)\} \qquad \mathcal{V} = S^n \setminus \{(0, \cdots, 0, -1)\}$$

and define the stereographic projections, $\varphi : \mathcal{U} \to \mathbb{R}^n$ and $\psi : \mathcal{V} \to \mathbb{R}^n$, by

$$\varphi(x) = \tfrac{2}{1-x_{n+1}}(x_1, \cdots, x_n) \qquad \psi(x) = \tfrac{2}{1+x_{n+1}}(x_1, \cdots, x_n)$$

Prove that $\mathcal{A}_2 = \big\{\{\mathcal{U}, \varphi\},\ \{\mathcal{V}, \psi\}\big\}$ is an atlas for S^n.

Solution. (a) Each $\mathbf{x} \in S^n$ obeys $|\mathbf{x}| = 1$ and so has at least one nonzero component x_j. If $x_j > 0$, then $x \in \mathcal{U}_j$ and if $x_j < 0$, then $x \in \mathcal{V}_j$. So $S^n \subset \bigcup_{1 \le j \le n+1} \big(\mathcal{U}_j \cup \mathcal{V}_j\big)$. We'll now verify that $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_1 \circ \psi_2^{-1}$ are C^∞. The other cases ares virtually identical. First observe that

$$\begin{aligned} \mathcal{U}_1 \cap \mathcal{U}_2 &= \big\{\ x \in S^n\ \big|\ x_1 > 0,\ x_2 > 0\ \big\} \\ \mathcal{U}_1 \cap \mathcal{V}_2 &= \big\{\ x \in S^n\ \big|\ x_1 > 0,\ x_2 < 0\ \big\} \end{aligned}$$

so that

$$\begin{aligned} \varphi_1\big(\mathcal{U}_1 \cap \mathcal{U}_2\big) &= \big\{\ (x_2, x_3, \cdots, x_{n+1}) \in \mathbb{R}^n\ \big|\ x_2 > 0, x_2^2 + x_3^2 + \cdots + x_{n+1}^2 < 1\ \big\} \\ \varphi_2\big(\mathcal{U}_1 \cap \mathcal{U}_2\big) &= \big\{\ (x_1, x_3, \cdots, x_{n+1}) \in \mathbb{R}^n\ \big|\ x_1 > 0, x_1^2 + x_3^2 + \cdots + x_{n+1}^2 < 1\ \big\} \\ \varphi_1\big(\mathcal{U}_1 \cap \mathcal{V}_2\big) &= \big\{\ (x_2, x_3, \cdots, x_{n+1}) \in \mathbb{R}^n\ \big|\ x_2 < 0, x_2^2 + x_3^2 + \cdots + x_{n+1}^2 < 1\ \big\} \\ \psi_2\big(\mathcal{U}_1 \cap \mathcal{V}_2\big) &= \big\{\ (x_1, x_3, \cdots, x_{n+1}) \in \mathbb{R}^n\ \big|\ x_1 > 0, x_1^2 + x_3^2 + \cdots + x_{n+1}^2 < 1\ \big\} \end{aligned}$$

are in fact each one of two half disks in $\mathbb{R}^n$. Since

$$\begin{aligned} \varphi_2^{-1}\big(t_1, t_2, \cdots, t_n\big) &= \big(t_1, \sqrt{1-t^2}, t_2, \cdots, t_n\big) \\ \psi_2^{-1}\big(t_1, t_2, \cdots, t_n\big) &= \big(t_1, -\sqrt{1-t^2}, t_2, \cdots, t_n\big) \end{aligned}$$

(where $t^2 = t_1^2 + \cdots + t_n^2$) it is clear that

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1}\big(t_1, t_2, \cdots, t_n\big) &= \varphi_1\big(t_1, \sqrt{1-t^2}, t_2, \cdots, t_n\big) &&= \big(\sqrt{1-t^2}, t_2, \cdots, t_n\big) \\ \varphi_1 \circ \psi_2^{-1}\big(t_1, t_2, \cdots, t_n\big) &= \varphi_{-1,1}\big(t_1, -\sqrt{1-t^2}, t_2, \cdots, t_n\big) &&= \big(-\sqrt{1-t^2}, t_2, \cdots, t_n\big) \end{aligned}$$

are C^∞.

(b) Observe that

$$\varphi(x)^2 = \tfrac{4}{(1-x_{n+1})^2}\big(x_1^2+\cdots+x_n^2\big) = 4\tfrac{1-x_{n+1}^2}{(1-x_{n+1})^2} = 4\tfrac{1+x_{n+1}}{1-x_{n+1}} \implies x_{n+1} = \tfrac{\varphi(x)^2-4}{\varphi(x)^2+4}$$

Since $\frac{1-x_{n+1}}{2} = \frac{4}{\varphi(x)^2+4}$, we have

$$\varphi^{-1}(t_1,\cdots,t_n) = \tfrac{4}{t^2+4}\big(t_1,\cdots,t_n,\tfrac{t^2-4}{4}\big)$$

and hence

$$\psi\circ\varphi^{-1}(t_1,\cdots,t_n) = 2\big[1+\tfrac{t^2-4}{t^2+4}\big]^{-1}\tfrac{4}{t^2+4}\big(t_1,\cdots,t_n\big) = \tfrac{4}{t^2}\big(t_1,\cdots,t_n\big)$$

which is C^∞ except at $t=0$, which corresponds to $x=(0,\cdots,0,-1)$. Similarly,

$$\psi(x)^2 = 4\tfrac{1-x_{n+1}}{1+x_{n+1}} \implies x_{n+1} = \tfrac{4-\psi(x)^2}{4+\psi(x)^2} \implies \psi^{-1}(t_1,\cdots,t_n) = \tfrac{4}{4+t^2}\big(t_1,\cdots,t_n,\tfrac{4-t^2}{4}\big)$$

so that

$$\varphi\circ\psi^{-1}(t_1,\cdots,t_n) = 2\big[1-\tfrac{4-t^2}{4+t^2}\big]^{-1}\tfrac{4}{4+t^2}\big(t_1,\cdots,t_n\big) = \tfrac{4}{t^2}\big(t_1,\cdots,t_n\big)$$

which is C^∞ except at $t=0$, which this time corresponds to $x=(0,\cdots,0,1)$. ■

Problem M.4 Let $R\in O(3)$.

(a) Prove that if λ is an eigenvalue of R, then $|\lambda|=1$ and $\bar\lambda$ is an eigenvalue of R.
(b) Prove that at least one eigenvalue of R is either $+1$ or -1.
(c) Prove that the columns of R are mutually perpendicular and are each of unit length.
(d) Prove that R is either a rotation, a reflection or a composition of a rotation and a reflection.

Solution. (a) Recall that the inner product on $\mathbb{C}^3$ is $\langle\vec v,\vec w\rangle = \sum_{j=1}^3 v_j\overline{w_j}$. Let $\vec e$ be an eigenvector of R of eigenvalue λ. Then

$$\langle\vec e,\vec e\rangle = \langle\vec e,\mathbb{1}\vec e\rangle = \big\langle\vec e,R^tR\vec e\big\rangle = \langle R\vec e,R\vec e\rangle = \langle\lambda\vec e,\lambda\vec e\rangle = |\lambda|^2\,\langle\vec e,\vec e\rangle$$

As $\langle\vec e,\vec e\rangle\neq 0$, we have $|\lambda|^2=1$ and hence $|\lambda|=1$. Taking the complex conjugate of $R\vec e=\lambda\vec e$ gives $R\overline{\vec e}=\overline{\lambda}\,\overline{\vec e}$, since R has real matrix elements. As $\overline{\vec e}$ is not the zero vector, it is an eigenvector of R with eigenvalue $\overline{\lambda}$.

(b) Let $P(\lambda) = \det(R - \lambda\mathbb{1})$ be the characteristic polynomial of R. It is a polynomial of degree three with real coefficients. So it has exactly three roots, counting multiplicity, and all non–real roots come in complex conjugate pairs. Consequently $P(\lambda)$ has at least one real root. So R has at least one real eigenvalue. As all eigenvalues have modulus one, any real eigenvalue has to be ±1.

(c) Denote by $C_n^{(\ell)} = R_{n,\ell}$ the $n^{\rm th}$ component of column ℓ. Then, for each $1 \le k, \ell \le 3$,

$$\vec{C}^{(k)} \cdot \vec{C}^{(\ell)} = \sum_{n=1}^{3} C_n^{(k)} \vec{C}_n^{(\ell)} = \sum_{n=1}^{3} R_{n,k} R_{n,\ell} = (R^t R)_{k,\ell} = \mathbb{1}_{k,\ell}$$

So if $k \ne \ell$, $\vec{C}^{(k)} \cdot \vec{C}^{(\ell)} = 0$ and $\vec{C}^{(k)} \perp \vec{C}^{(\ell)}$ and if $k = \ell$, $\|C^{(\ell)}\|^2 = 1$ and $\|C^{(\ell)}\| = 1$.

(d) *Case 1:* If R has exactly one real eigenvalue, we may choose a coordinate system in which the corresponding eigenvector is on the z–axis. (See "Change of Basis", below.) In this coordinate system $(0, 0, 1)$ is an eigenvector of eigenvalue ± 1 and so R is of the form

$$R = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & \pm 1 \end{bmatrix}$$

The columns must be mutually perpendicular and of unit length. This forces $e = f = 0$, $a^2 + c^2 = b^2 + d^2 = 1$ and $(a, c) \cdot (b, d) = 0$. So there is a θ such that $a = \cos\theta$, $c = \sin\theta$ and $(b, d) = \pm(-\sin\theta, \cos\theta)$ and R is one of

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The first two are rotation about the z axis by θ and rotation about the z axis by θ followed by the reflection $z \to -z$. The remaining two actually have three eigenvalues ±1 and so are included in

Case 2: R has three real eigenvalues. (R cannot have exactly two real eigenvalues, because if λ is an eigenvalue, $\bar{\lambda}$ is too.) If $\vec{e}_1$ is an eigenvector of eigenvalue $+1$ and $\vec{e}_2$ is an eigenvector of eigenvalue -1, then $\vec{e}_1 \perp \vec{e}_2$ because

$$\begin{aligned} &-\langle \vec{e}_1, \vec{e}_2 \rangle = (+1)(-1)\langle \vec{e}_1, \vec{e}_2 \rangle = \langle +1\vec{e}_1, -1\vec{e}_2 \rangle = \langle R\vec{e}_1, R\vec{e}_2 \rangle = \langle \vec{e}_1, R^t R\vec{e}_2 \rangle = \langle \vec{e}_1, \vec{e}_2 \rangle \\ &\implies \langle \vec{e}_1, \vec{e}_2 \rangle = 0 \end{aligned}$$

So we may choose a coordinate system with all three standard basis vectors being eigenvectors. So R is one of (up to permutations of the coordinate axes)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

which are the identity, the reflection $z \to -z$, rotation about the x axis by 180° and rotation about the z axis by 180° followed by the reflection $z \to -z$.

Change of Basis Let $\vec{e}_1'$, $\vec{e}_2'$ and $\vec{e}_3'$ be three mutually perpendicular unit vectors in $\mathbb{R}^3$. The components, x_1', x_2', x_3', of any vector $\vec{x}$ with respect to the new basis $\{\vec{e}_1', \vec{e}_2', \vec{e}_3'\}$ are determined by

$$x_1'\vec{e}_1' + x_2'\vec{e}_2' + x_3'\vec{e}_3' = \vec{x} \quad \text{or} \quad E\vec{x}\,' = \vec{x} \quad \text{where} \quad E = \big[\vec{e}_1' \ \vec{e}_2' \ \vec{e}_3'\big],\ \vec{x}\,' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

Note that $E \in O(3)$, so that $\vec{x}\,' = E^{-1}\vec{x} = E^t\vec{x}$. If we think of a 3×3 matrix R as mapping each $\vec{x} \in \mathbb{R}^3$ to $R\vec{x} \in \mathbb{R}^3$, then in terms of the new coordinate system, $\vec{x}\,' = E^t\vec{x}$ gets mapped to $E^tR\vec{x} = E^tRE\vec{x}\,'$. Thus the matrix of the map $\vec{x} \to R\vec{x}$ in the new coordinate system is E^tRE. Note that if $R \in O(3)$, then E^tRE is again in $O(3)$, because $O(3)$ is closed under multiplication and the taking of transposes.

Problem M.5 Define

$$\begin{aligned}
g_1(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3) &= a_1^2 + a_2^2 + a_3^2 \\
g_2(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3) &= b_1^2 + b_2^2 + b_3^2 \\
g_3(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3) &= c_1^2 + c_2^2 + c_3^2 \\
g_4(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3) &= a_1b_1 + a_2b_2 + a_3b_3 \\
g_5(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3) &= a_1c_1 + a_2c_2 + a_3c_3 \\
g_6(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3) &= b_1c_1 + b_2c_2 + b_3c_3
\end{aligned}$$

Prove that the gradients of $g_1, \cdots, g_6$, evaluated at any

$$R = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \in O(3)$$

are linearly independent.

Solution. Fix any $\big(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3\big)$. The six gradients are

$$\begin{aligned}
\nabla g_1 &= 2\big(a_1,a_2,a_3,0,0,0,0,0,0\big) \\
\nabla g_2 &= 2\big(0,0,0,b_1,b_2,b_3,0,0,0\big) \\
\nabla g_3 &= 2\big(0,0,0,0,0,0,c_1,c_2,c_3\big) \\
\nabla g_4 &= \big(b_1,b_2,b_3,a_1,a_2,a_3,0,0,0\big) \\
\nabla g_5 &= \big(c_1,c_2,c_3,0,0,0,a_1,a_2,a_3\big) \\
\nabla g_6 &= \big(0,0,0,c_1,c_2,c_3,b_1,b_2,b_3\big)
\end{aligned}$$

We have to show that the only solution so $\sum_{j=1}^{6}\alpha_j\nabla g_j = 0$ is $\alpha_1 = \cdots = \alpha_6 = 0$. The first three components of $\sum_{j=1}^{6}\alpha_j\nabla g_j$ are (writing them as a column vector)

$$\alpha_1\begin{bmatrix}2a_1\\2a_2\\2a_3\end{bmatrix}+\alpha_4\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}+\alpha_5\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{bmatrix}\begin{bmatrix}2\alpha_1\\\alpha_4\\\alpha_5\end{bmatrix}=R\begin{bmatrix}2\alpha_1\\\alpha_4\\\alpha_5\end{bmatrix}$$

The matrix R, as an element of $O(3)$ must have determinant ±1 and hence must be invertible. So the first three components of $\sum_{j=1}^{6}\alpha_j\nabla g_j$ are zero if and only if $2\alpha_1 = \alpha_4 = \alpha_5 = 0$. Similarly components four, five and six of $\sum_{j=1}^{6}\alpha_j\nabla g_j$ are zero if and only if $2\alpha_2 = \alpha_4 = \alpha_6 = 0$ and components seven, eight and nine of $\sum_{j=1}^{6}\alpha_j\nabla g_j$ are zero if and only if $2\alpha_3 = \alpha_5 = \alpha_6 = 0$.

Problem M.6 Use the implicit function theorem to prove that for each $1 \le i, j \le 3$, the (i, j) matrix element, a_{ij}, of matrices $R = \big[a_{ij}\big]_{1\le i,j\le 3}$ in a neighbourhood of $\mathbb{1}$ in $SO(3)$, is a C^∞ function of the matrix elements a_{21}, a_{31} and a_{32}.

Solution. Define

$$g_1(a_{11},a_{12},a_{13},a_{22},a_{23},a_{33},a_{21},a_{31},a_{32}) = \sum_{j=1}^{3} a_{1j}^2 - 1$$

$$g_2(a_{11},a_{12},a_{13},a_{22},a_{23},a_{33},a_{21},a_{31},a_{32}) = \sum_{j=1}^{3} a_{2j}^2 - 1$$

$$g_3(a_{11},a_{12},a_{13},a_{22},a_{23},a_{33},a_{21},a_{31},a_{32}) = \sum_{j=1}^{3} a_{3j}^2 - 1$$

$$g_4(a_{11},a_{12},a_{13},a_{22},a_{23},a_{33},a_{21},a_{31},a_{32}) = \sum_{j=1}^{3} a_{1j}a_{2j}$$

$$g_5(a_{11},a_{12},a_{13},a_{22},a_{23},a_{33},a_{21},a_{31},a_{32}) = \sum_{j=1}^{3} a_{1j}a_{3j}$$

$$g_6(a_{11},a_{12},a_{13},a_{22},a_{23},a_{33},a_{21},a_{31},a_{32}) = \sum_{j=1}^{3} a_{2j}a_{3j}$$

All six functions are C^∞ functions of their 9 variables. We shall solve for the first six variables as functions of the last three. The gradients of the six functions at the identity matrix are

$$\nabla g_1(1,0,0,1,0,1,0,0,0) = \big(2,0,0,0,0,0,0,0,0\big)$$

$$\begin{aligned}
\nabla g_2(1,0,0,1,0,1,0,0,0) &= \big(0,0,0,2,0,0,0,0,0\big)\\
\nabla g_3(1,0,0,1,0,1,0,0,0) &= \big(0,0,0,0,0,2,0,0,0\big)\\
\nabla g_4(1,0,0,1,0,1,0,0,0) &= \big(0,1,0,0,0,0,1,0,0\big)\\
\nabla g_5(1,0,0,1,0,1,0,0,0) &= \big(0,0,1,0,0,0,0,1,0\big)\\
\nabla g_6(1,0,0,1,0,1,0,0,0) &= \big(0,0,0,0,1,0,0,0,1\big)
\end{aligned}$$

(These are linearly independent vectors.) Expanding along the first column and then along the first row and finally along the last column,

$$\begin{aligned}
&\det \tfrac{\partial(g_1\ ,\ g_2\ ,\ g_3\ ,\ g_4\ ,\ g_5\ ,\ g_6)}{\partial(a_{11},a_{12},a_{13},a_{22},a_{23},a_{33})}\Big|_{(0,0,0,1,0,0,1,0,1)}\\
&\qquad = \det \begin{bmatrix} 2&0&0&0&0&0\\ 0&0&0&2&0&0\\ 0&0&0&0&0&2\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0 \end{bmatrix} = 2\det \begin{bmatrix} 0&0&2&0&0\\ 0&0&0&0&2\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0 \end{bmatrix}\\
&\qquad = 4\det \begin{bmatrix} 0&0&0&2\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{bmatrix} = -8\det \begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&0&1 \end{bmatrix} = -8 \neq 0
\end{aligned}$$

the implicit function theorem assures that given any a_{21}, a_{31}, a_{32} sufficiently small, there is a unique element of $SO(3)$ having the given values of a_{21}, a_{31}, a_{32} and being in a neighbourhood (specified by the implicit function theorem) of the identity. Furthermore, each a_{ij} is a C^∞ function of a_{21}, a_{31}, a_{32}. ■

Problem M.7 Prove that the two charts $(\mathcal{U}_2, \varphi_2)$ and $(\mathcal{U}_2, \psi_2)$ of Example M.10 are not compatible.

Solution. The range of φ_2 is $\big[0, \frac{1}{4}\big) \times (-1,1) \cup \big(-\frac{1}{4}, 0\big) \times (-1,1) = \big(-\frac{1}{4}, \frac{1}{4}\big) \times (-1,1)$. On this range

$$\varphi_2^{-1}(x,y) = \begin{cases} (x,y) & \text{if } 0 \le x < \frac{1}{4} \\ (x+1,-y) & \text{if } -\frac{1}{4} < x < 0 \end{cases}$$

so that

$$\psi_2 \circ \varphi_2^{-1}(x,y) = \left\{ \begin{matrix} \psi_2(x,y) & \text{if } 0 \le x < \frac{1}{4} \\ \psi_2(x+1,-y) & \text{if } -\frac{1}{4} < x < 0 \end{matrix} \right\} = \begin{cases} (x,y) & \text{if } 0 \le x < \frac{1}{4} \\ (x,-y) & \text{if } -\frac{1}{4} < x < 0 \end{cases}$$

This is not continuous across $x = 0$, since

$$\lim_{x\to 0+} \psi_2 \circ \varphi_2^{-1}(x,y) = (0,y) \qquad \text{while} \qquad \lim_{x\to 0-} \psi_2 \circ \varphi_2^{-1}(x,y) = (0,-y)$$

■

Problem M.8 Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds. Prove that $f : \mathcal{M} \to \mathcal{N}$ is C^∞ at $m \in \mathcal{M}$ if and only if $\psi \circ f \circ \phi^{-1}$ is C^∞ at $\phi(m)$ for every chart $(\mathcal{U}, \phi)$ for $\mathcal{M}$ with $m \in \mathcal{U}$ and every chart $(\mathcal{V}, \psi)$ for $\mathcal{N}$ with $f(m) \in \mathcal{V}$.

Solution. The "if" part is trivial. To prove the "only if" part, observe that, by definition, there are charts $\{\tilde{\mathcal{U}}, \tilde{\phi}\}$ and $\{\tilde{\mathcal{V}}, \tilde{\psi}\}$ such that $m \in \tilde{\mathcal{U}}$, $f(m) \in \tilde{\mathcal{V}}$ and $\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}$ being C^∞ at $\tilde{\phi}(m)$. Let $\{\mathcal{U}, \phi\}$ and $\{\mathcal{V}, \psi\}$ be any charts with $m \in \mathcal{U}$ and $f(m) \in \mathcal{V}$. Then $m \in \tilde{\mathcal{U}} \cap \mathcal{U}$ and $f(m) \in \tilde{\mathcal{V}} \cap \mathcal{V}$. By compatibility $\tilde{\phi} \circ \phi^{-1}$ and $\psi \circ \tilde{\psi}^{-1}$ are C^∞ at $\phi(m)$ and $\tilde{\psi}(f(m))$ respectively. Consequently,

$$\psi \circ f \circ \phi^{-1} = \left(\psi \circ \tilde{\psi}^{-1}\right) \circ \left(\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}\right) \circ \left(\tilde{\phi} \circ \phi^{-1}\right)$$

is C^∞ at $\phi(m)$. ■

Problem M.9 Prove that $\mathbb{R}^n$ is diffeomorphic to $\left\{\ \mathbf{x} \in \mathbb{R}^n \ \middle|\ \sum_{i=1}^n x_i^2 < 1\ \right\}$.

Solution. The map

$$\Phi(x) = \frac{x}{\sqrt{1 - |x|^2}}$$

is a diffeomorphism from $\left\{\ \mathbf{x} \in \mathbb{R}^n \ \middle|\ \sum_{i=1}^n x_i^2 < 1\ \right\}$ to $\mathbb{R}^n$. The inverse map is

$$\Phi^{-1}(y) = \frac{y}{\sqrt{1 + |y|^2}}$$

■

Problem M.10 Prove that $\mathbb{R}^n$ is not diffeomorphic to S^n.

Solution. Suppose that $\Phi : S^n \to \mathbb{R}^n$ were a diffeomorphism. Let $g : \mathbb{R}^n \to \mathbb{R}$ be defined by $g(x) = x_1$. Then $g \circ \Phi : S^n \to \mathbb{R}$ is C^∞ and is onto $\mathbb{R}$. But that is impossible because S^n is compact so that every C^∞ function on S^n is bounded. ■

Problem M.11 Outline an argument to prove that the disk $\left\{\ \mathbf{x} \in \mathbb{R}^2 \ \middle|\ x^2 + y^2 < 2\ \right\}$ is not diffeomorphic to the annulus $\left\{\ \mathbf{x} \in \mathbb{R}^2 \ \middle|\ 1 < x^2 + y^2 < 2\ \right\}$.

Solution. The disk $\mathcal{M} = \left\{\ \mathbf{x} \in \mathbb{R}^2 \ \middle|\ x^2 + y^2 < 2\ \right\}$ has the property that every C^∞ closed curve may be continuously deformed to a point. To see this parametrize any curve by a function $f : \mathbb{R} \to \mathcal{M}$ that has period one. Then $f_s(t) = (1-s)f(t)$ implements the deformation, since $f_0(t) = f(t)$ and $f_1(t) = 0$ has range the single point $\{0\}$. If the disk and annulus were diffeomorphic, the annulus would also have this property. It doesn't. For example, the circle $C = \left\{\ (x,y)\ \middle|\ x^2+y^2=1\ \right\}$ cannot be deformed to a point in the annulus. If it could, Green's theorem would yield that $\oint_C \left[-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right] = 0$, which is false. ■

Problem M.12 In this problem $G = SO(3)$.

a) Fix any $a \in G$. Denote by $I = \left\{\ (i,j) \in \mathbb{N}^2\ \middle|\ 1 \le i \le 3,\ 1 \le j \le 3\ \right\}$ the set of indices for the matrix elements of the matrices in G. Prove that there exist $\alpha, \beta, \gamma \in I$ such that every matrix element $g_\delta,\ \delta \in I$ is a C^∞ function of $g_\alpha,\ g_\beta,\ g_\gamma$ for matrices $g \in G$ in a neighbourhood of a.

b) Prove that a curve $q : (c,d) \to G$ is C^∞ if and only if every matrix element $q(t)_{i,j}$ is C^∞.

c) Prove that matrix multiplication $(a,b) \mapsto ab$ is a C^∞ function from $G \times G$ to G.

d) Prove that the inverse function $a \mapsto a^{-1}$ is a C^∞ function from G to G.

Solution. (a) Name the matrix elements of $g \in G$ by

$$\begin{bmatrix} a_1(g) & b_1(g) & c_1(g) \\ a_2(g) & b_2(g) & c_2(g) \\ a_3(g) & b_3(g) & c_3(g) \end{bmatrix}$$

and set

$$\begin{aligned} f_1 &= a_1^2 + a_2^2 + a_3^2 & f_2 &= b_1^2 + b_2^2 + b_3^2 & f_3 &= c_1^2 + c_2^2 + c_3^2 \\ f_4 &= a_1b_1 + a_2b_2 + a_3b_3 & f_5 &= a_1c_1 + a_2c_2 + a_3c_3 & f_6 &= b_1c_1 + b_2c_2 + b_3c_3 \end{aligned}$$

We have to show that, with $\nabla = \left(\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3}, \frac{\partial}{\partial b_1}, \frac{\partial}{\partial b_2}, \frac{\partial}{\partial b_3}, \frac{\partial}{\partial c_1}, \frac{\partial}{\partial c_2}, \frac{\partial}{\partial c_3}, \right)$,

$$\begin{aligned} \nabla f_1 &= \big(2a_1, 2a_2, 2a_3, 0,0,0,0,0,0\big) & \nabla f_2 &= \big(0,0,0,2b_1,2b_2,2b_3,0,0,0\big) \\ \nabla f_3 &= \big(0,0,0,0,0,0,2c_1,2c_2,2c_3\big) & \nabla f_4 &= \big(b_1,b_2,b_3,a_1,a_2,a_3,0,0,0\big) \\ \nabla f_5 &= \big(c_1,c_2,c_3,0,0,0,a_1,a_2,a_3\big) & \nabla f_6 &= \big(0,0,0,c_1,c_2,c_3,b_1,b_2,b_3\big) \end{aligned}$$

are linearly independent vectors at any point obeying $f_1 = f_2 = f_3 = 1$ and $f_4 = f_5 = f_6 = 0$. In other words, we have to show that

$$\alpha_1\nabla f_1 + \alpha_2\nabla f_2 + \alpha_3\nabla f_3 + \alpha_4\nabla f_4 + \alpha_5\nabla f_5 + \alpha_6\nabla f_6 = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$$

at every such point. This follows from

$$\begin{aligned}
\alpha_1 &= \tfrac{1}{2}\big(2|a|^2\,\alpha_1 + a\cdot b\,\alpha_4 + a\cdot c\,\alpha_5\big)\\
&= \tfrac{1}{2}\big(a_1,a_2,a_3,0,0,0,0,0,0\big)\cdot\big(\alpha_1\nabla f_1+\alpha_2\nabla f_2+\alpha_3\nabla f_3+\alpha_4\nabla f_4+\alpha_5\nabla f_5+\alpha_6\nabla f_6\big)\\
\alpha_2 &= \tfrac{1}{2}\big(2|b|^2\,\alpha_2 + a\cdot b\,\alpha_4 + b\cdot c\,\alpha_6\big)\\
&= \tfrac{1}{2}\big(b_1,b_2,b_3,0,0,0\big)\cdot\big(\alpha_1\nabla f_1+\alpha_2\nabla f_2+\alpha_3\nabla f_3+\alpha_4\nabla f_4+\alpha_5\nabla f_5+\alpha_6\nabla f_6\big)\\
\alpha_3 &= \tfrac{1}{2}\big(2|c|^2\,\alpha_3 + a\cdot c\,\alpha_5 + b\cdot c\,\alpha_6\big)\\
&= \tfrac{1}{2}\big(0,0,0,0,0,0,c_1,c_2,c_3\big)\cdot\big(\alpha_1\nabla f_1+\alpha_2\nabla f_2+\alpha_3\nabla f_3+\alpha_4\nabla f_4+\alpha_5\nabla f_5+\alpha_6\nabla f_6\big)\\
\alpha_4 &= 2a\cdot b\,\alpha_1 + |b|^2\,\alpha_4 + b\cdot c\,\alpha_5\\
&= \big(b_1,b_2,b_3,0,0,0,0,0,0\big)\cdot\big(\alpha_1\nabla f_1+\alpha_2\nabla f_2+\alpha_3\nabla f_3+\alpha_4\nabla f_4+\alpha_5\nabla f_5+\alpha_6\nabla f_6\big)\\
\alpha_5 &= 2a\cdot c\,\alpha_1 + b\cdot c\,\alpha_4 + |c|^2\,\alpha_5\\
&= \big(c_1,c_2,c_3,0,0,0,0,0,0\big)\cdot\big(\alpha_1\nabla f_1+\alpha_2\nabla f_2+\alpha_3\nabla f_3+\alpha_4\nabla f_4+\alpha_5\nabla f_5+\alpha_6\nabla f_6\big)\\
\alpha_6 &= 2b\cdot c\,\alpha_2 + a\cdot c\,\alpha_4 + |c|^2\,\alpha_6\\
&= \big(0,0,0,c_1,c_2,c_3,0,0,0\big)\cdot\big(\alpha_1\nabla f_1+\alpha_2\nabla f_2+\alpha_3\nabla f_3+\alpha_4\nabla f_4+\alpha_5\nabla f_5+\alpha_6\nabla f_6\big)
\end{aligned}$$

(b) Pick any $c < t_0 < d$. Pick three matrix elements as coordinates in some neighbourhood of $q_0(t)$. By Problem M.8, $q(t)$ is C^∞ at t_0 if and only if those three matrix elements are C^∞ at t_0. If those three matrix elements are C^∞ at t_0, the remaining matrix elements are also C^∞ by part (a).

(c) Every matrix element of ab is a polynomial in the matrix elements of a and the matrix elements of b and hence is C^∞.

(d) Every matrix element of a^{-1} is a polynomial in the matrix elements of a (since $\det a = 1$) and hence is C^∞.

■

Problem M.13 Let $\mathcal{M}$ be a manifold, ω be a 1–form on $\mathcal{M}$ and $c(t) : [0,1] \to \mathcal{M}$ be a path in $\mathcal{M}$. Prove that the definition of $\int_c \omega$ given in part (c) of Definition M.13 is independent of the decomposition of c into finitely many pieces and of the choice of coordinate patches.

Solution. "Decomposing c into finitely many pieces" means picking $0 < t_1 < t_2 < \cdots < t_m$ with $m \in \mathbb{N}$ and $\big\{ c(t) \bigm| t_{\ell-1} \le t \le t_\ell \big\}$ contained in a single coordinate patch for each $\ell = 1,\ 2,\ \cdots,\ m$ and then applying the single patch algorithm of Definition M.13 to the part of $c(t)$ with $t_{\ell-1} \le t \le t_\ell$ for each $\ell = 1,\ 2,\ \cdots,\ m$.

Make any two such decompositions and choices of coordinate patches for each piece. Since

$$\int_{t_{\ell-1}}^{t_\ell} h(t)\ dt = \int_{t_{\ell-1}}^{s} h(t)\ dt + \int_{s}^{t_\ell} h(t)\ dt$$

for all $t_{\ell-1} < s < t_\ell$ we are free to add any finite number of decomposition points. So we may assume that the two sets of decomposition times are identical. Thus it suffices to prove that if $0 \le t_\ell < t_{\ell+1} \le 1$ and

- ○ if $\{U, \zeta\}$ and $\{\tilde U, \tilde\zeta\}$ are two patches with $c(t) \in U \cap \tilde U$ for all $t_\ell \le t \le t_{\ell+1}$ and
- ○ if ω assigns $\{U, \zeta\}$ the pair of functions (f, g) and assigns $\{\tilde U, \tilde\zeta\}$ the pair of functions $(\tilde f, \tilde g)$

then

$$\int_0^1 \Big[f\big(\zeta(c(t))\big)\tfrac{dx(c(t))}{dt} + g\big(\zeta(c(t))\big)\tfrac{dy(c(t))}{dt}\Big]\ dt = \int_0^1 \Big[\tilde f\big(\tilde\zeta(c(t))\big)\tfrac{d\tilde x(c(t))}{dt} + \tilde g\big(\tilde\zeta(c(t))\big)\tfrac{d\tilde y(c(t))}{dt}\Big]\ dt$$

But this is the computation of part (c) of Remark M.14. ■

Problem M.14 Let $\mathcal{M}$ be a manifold of dimension $n \in \mathbb{N}$ (not necessarily 2) and suppose that we have defined a wedge product for $\mathcal{M}$ that is bilinear, graded anticommutative and associative (i.e. is satisfies properties (a), (b) and (c) above). Let, for each $\le j \le n$, ω_j be a 1–form on $\mathcal{M}$ and, for each $1 \le i, j \le n$, f_{ij} be a function on $\mathcal{M}$. Prove that

$$\Big(\sum_{j=1}^n f_{1j}\omega_j\Big) \bigwedge \Big(\sum_{j=1}^n f_{2j}\omega_j\Big) \bigwedge \cdots \bigwedge \Big(\sum_{j=1}^n f_{nj}\omega_j\Big) = \det\big[f_{ij}\big]_{1\le i,j\le n}\ \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$$

Solution. By linearity

$$\Big(\sum_{j=1}^n f_{1j}\omega_j\Big) \bigwedge \Big(\sum_{j=1}^n f_{2j}\omega_j\Big) \bigwedge \cdots \bigwedge \Big(\sum_{j=1}^n f_{nj}\omega_j\Big) = \sum_{j_1,\cdots,j_n=1}^{n} f_{1j_1}\cdots f_{nj_n}\ \omega_{j_1} \wedge \cdots \wedge \omega_{j_n}$$

By anticummutativity

$$\omega_{j_\ell} \wedge \omega_{j_{\ell+1}} = \begin{cases} -\omega_{j_{\ell+1}} \wedge \omega_{j_\ell} & \text{if } j_\ell \ne j_{\ell+1} \\ 0 & \text{if } j_\ell = j_{\ell+1} \end{cases}$$

so that

$$\omega_{j_1} \wedge \cdots \wedge \omega_{j_\ell} \wedge \omega_{j_{\ell+1}} \wedge \cdots \wedge \omega_{j_n} = \begin{cases} -\omega_{j_1} \wedge \cdots \wedge \omega_{j_{\ell+1}} \wedge \omega_{j_\ell} \wedge \cdots \wedge \omega_{j_n} & \text{if } j_\ell \ne j_{\ell+1} \\ 0 & \text{if } j_\ell = j_{\ell+1} \end{cases} \qquad (1)$$

Repeatedly applying this, we have that $\omega_{j_1} \wedge \cdots \wedge \omega_{j_n}$ is zero unless $j_1, \cdots, j_n$ are all different, that is unless $j_1 = \pi(1)$, $\cdots$, $j_n = \pi(n)$ for some permutation of $(1, \cdots, n)$. Repeatedly applying (1) also gives that, for any permutation$^{(1)}$ π,

$$\omega_{\pi(1)} \wedge \cdots \wedge \omega_{\pi(n)} = \operatorname{sgn}\pi \ \omega_1 \wedge \cdots \wedge \omega_n$$

where $\operatorname{sgn}\pi$ is the sign$^{(1)}$ of the permutation. Denoting by S_n the set of all permutations of $(1, \cdots, n)$, we have

$$\begin{aligned}
\Big(\sum_{j=1}^n f_{1j}\omega_j\Big) \bigwedge \Big(\sum_{j=1}^n f_{2j}\omega_j\Big) \bigwedge \cdots \bigwedge \Big(\sum_{j=1}^n f_{nj}\omega_j\Big) &= \sum_{j_1,\cdots,j_n=1}^n f_{1j_1} \cdots f_{nj_n} \ \omega_{j_1} \wedge \cdots \wedge \omega_{j_n} \\
&= \sum_{\pi \in S_n} f_{1\pi(1)} \cdots f_{n\pi(n)} \ \omega_{\pi(1)} \wedge \cdots \wedge \omega_{\pi(n)} \\
&= \sum_{\pi \in S_n} \operatorname{sgn}\pi \ f_{1\pi(1)} \cdots f_{n\pi(n)} \ \omega_1 \wedge \cdots \wedge \omega_n \\
&= \det \big[f_{ij}\big]_{1 \le i,j \le n} \ \omega_1 \wedge \cdots \wedge \omega_n
\end{aligned}$$

Problem M.15 Prove that Definition M.18 is independent of the choice of coordinate patch.

Solution. Assume that

- $\{U, \zeta\}$ and $\{\tilde U, \tilde\zeta\}$ are two charts with $U \cap \tilde U \ne \emptyset$ and
- the transition function $\tilde\zeta \circ \zeta^{-1}$ $\big($from $\zeta(U \cap \tilde U) \subset \mathbb{R}^2$ to $\tilde\zeta(U \cap \tilde U) \subset \mathbb{R}^2\big)$ is $\big(\tilde x(x,y), \tilde y(x,y)\big)$

(a) Let $F : \mathcal{M} \to \mathbb{C}$ be a C^1 0–form and set $\varphi = F \circ \zeta^{-1}$ and $\tilde\varphi = F \circ \tilde\zeta^{-1}$. Then $\varphi(x,y) = \tilde\varphi\big(\tilde x(x,y), \tilde y(x,y)\big)$ and

$$\begin{aligned}
dF\big|_{\{\tilde U, \tilde\zeta\}} &= \tilde f(\tilde x, \tilde y)\, d\tilde x + \tilde g(\tilde x, \tilde y)\, d\tilde y \text{ with } \tilde f(\tilde x, \tilde y) = \tfrac{\partial\tilde\varphi}{\partial\tilde x}(\tilde x, \tilde y),\ \tilde g(\tilde x, \tilde y) = \tfrac{\partial\tilde\varphi}{\partial\tilde y}(\tilde x, \tilde y) \\
dF\big|_{\{U, \zeta\}} &= \tfrac{\partial\varphi}{\partial x}(x,y)\, dx + \tfrac{\partial\varphi}{\partial y}(x,y)\, dy \\
&= \Big\{\tfrac{\partial\tilde\varphi}{\partial\tilde x}\big(\tilde x(x,y), \tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial x}(x,y) + \tfrac{\partial\tilde\varphi}{\partial\tilde y}\big(\tilde x(x,y), \tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial x}(x,y)\Big\}\, dx \\
&\quad + \Big\{\tfrac{\partial\tilde\varphi}{\partial\tilde x}\big(\tilde x(x,y), \tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial y}(x,y) + \tfrac{\partial\tilde\varphi}{\partial\tilde y}\big(\tilde x(x,y), \tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial y}(x,y)\Big\}\, dy \\
&= f(x,y)\, dx + g(x,y)\, dy
\end{aligned}$$

---

(1) A permutation of $(1, \cdots, n)$ is a 1–1 map from $\{1, \cdots, n\}$ onto $\{1, \cdots, n\}$. A transposition is a permutation that just exchanges two neighbouring elements. Any permutation may be expressed as a product (i.e. composition) of transpositions. The sign of a permutation is $+1$ if it is the product of an even number of transpositions and -1 if it is the product of an odd number of transpositions.

with

$$\begin{aligned} f(x,y) &= \tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial x}(x,y) + \tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial x}(x,y) \\ g(x,y) &= \tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial y}(x,y) + \tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial y}(x,y) \end{aligned}$$

This agrees with the coordinate transformation rule of Definition M.13.

(b) If ω is a C^1 1–form with

$$\begin{aligned} \omega\big|_{\{U,\zeta\}} &= f(x,y)\,dx + g(x,y)\,dy \\ \omega\big|_{\{\tilde U,\tilde\zeta\}} &= \tilde f(\tilde x,\tilde y)\,d\tilde x + \tilde g(\tilde x,\tilde y)\,d\tilde y \end{aligned}$$

then

$$\begin{aligned} f(x,y) &= \tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial x}(x,y) + \tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial x}(x,y) \\ g(x,y) &= \tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial y}(x,y) + \tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial y}(x,y) \end{aligned}$$

Since

$$\begin{aligned} \tfrac{\partial g}{\partial x}(x,y) &= \tfrac{\partial\tilde f}{\partial\tilde x}\tfrac{\partial\tilde x}{\partial x}\tfrac{\partial\tilde x}{\partial y} + \tfrac{\partial\tilde f}{\partial\tilde y}\tfrac{\partial\tilde y}{\partial x}\tfrac{\partial\tilde x}{\partial y} + \tilde f\tfrac{\partial^2\tilde x}{\partial x\partial y} + \tfrac{\partial\tilde g}{\partial\tilde x}\tfrac{\partial\tilde x}{\partial x}\tfrac{\partial\tilde y}{\partial y} + \tfrac{\partial\tilde g}{\partial\tilde y}\tfrac{\partial\tilde y}{\partial x}\tfrac{\partial\tilde y}{\partial y} + \tilde g\tfrac{\partial^2\tilde y}{\partial x\partial y} \\ \tfrac{\partial f}{\partial y}(x,y) &= \tfrac{\partial\tilde f}{\partial\tilde x}\tfrac{\partial\tilde x}{\partial y}\tfrac{\partial\tilde x}{\partial x} + \tfrac{\partial\tilde f}{\partial\tilde y}\tfrac{\partial\tilde y}{\partial y}\tfrac{\partial\tilde x}{\partial x} + \tilde f\tfrac{\partial^2\tilde x}{\partial x\partial y} + \tfrac{\partial\tilde g}{\partial\tilde x}\tfrac{\partial\tilde x}{\partial y}\tfrac{\partial\tilde y}{\partial x} + \tfrac{\partial\tilde g}{\partial\tilde y}\tfrac{\partial\tilde y}{\partial y}\tfrac{\partial\tilde y}{\partial x} + \tilde g\tfrac{\partial^2\tilde y}{\partial x\partial y} \end{aligned}$$

we have that

$$\begin{aligned} d\omega\big|_{\{U,\zeta\}} &= \big[\tfrac{\partial g}{\partial x}(x,y) - \tfrac{\partial f}{\partial y}(x,y)\big]\,dx\wedge dy \\ &= \big[\tfrac{\partial\tilde g}{\partial\tilde x} - \tfrac{\partial\tilde f}{\partial y}\big]\ \big[\tfrac{\partial\tilde x}{\partial x}\tfrac{\partial\tilde y}{\partial y} - \tfrac{\partial\tilde x}{\partial y}\tfrac{\partial\tilde y}{\partial x}\big]\,dx\wedge dy \end{aligned}$$

which agrees with the coordinate transformation rule of Definition M.15.

(c) The 2–form case is trivial. ■

Problem M.16 Prove the graded product rule that if ω is a k–form and ω' is a k'–form, then

$$d(\omega\wedge\omega') = (d\omega)\wedge\omega' + (-1)^k\omega\wedge(d\omega')$$

Solution. It suffices to consider $k+k'\le 1$, since all of $d(\omega\wedge\omega')$, $(d\omega)\wedge\omega'$ and $\omega\wedge(d\omega')$ vanish if $k+k'\ge 2$ because the manifold $\mathcal{M}$ is assumed to be two dimensional. It suffices to verify the specified formula in any chart $\{U,\zeta\}$,

(a) If $k=k'=0$, $\omega\circ\zeta^{-1}(x,y)=f(x,y)$ and $\omega'\circ\zeta^{-1}(x,y)=g(x,y)$, then

$$\begin{aligned} d\omega &= \tfrac{\partial f}{\partial x}(x,y)\,dx + \tfrac{\partial f}{\partial y}(x,y)\,dy \\ d\omega' &= \tfrac{\partial g}{\partial x}(x,y)\,dx + \tfrac{\partial g}{\partial y}(x,y)\,dy \\ (d\omega)\wedge\omega' + (-1)^k\omega\wedge(d\omega') &= \tfrac{\partial f}{\partial x}(x,y)g(x,y)\,dx + \tfrac{\partial f}{\partial y}(x,y)g(x,y)\,dy \\ &\qquad + f(x,y)\tfrac{\partial g}{\partial x}(x,y)\,dx + f(x,y)\tfrac{\partial g}{\partial y}(x,y)\,dy \end{aligned}$$

and

$$d(\omega\wedge\omega') = \tfrac{\partial}{\partial x}(fg)\,dx + \tfrac{\partial}{\partial y}(fg)\,dy$$

agree by the product rule.

(b) If $k=0$, $k'=1$, $\omega\circ\zeta^{-1}(x,y)=f(x,y)$ and $\omega'\big|_{\{\mathcal{U},\zeta\}} = g(x,y)\,dx + h(x,y)\,dy$, then

$$\begin{aligned}
d\omega &= \tfrac{\partial f}{\partial x}(x,y)\,dx + \tfrac{\partial f}{\partial y}(x,y)\,dy \\
d\omega' &= \big[\tfrac{\partial h}{\partial x}(x,y) - \tfrac{\partial g}{\partial y}(x,y)\big]\,dx\wedge dy \\
(d\omega)\wedge\omega' + (-1)^k\omega\wedge(d\omega') &= \big[\tfrac{\partial f}{\partial x}(x,y)\,dx + \tfrac{\partial f}{\partial y}(x,y)\,dy\big]\wedge\big[g(x,y)\,dx + h(x,y)\,dy\big] \\
&\qquad + f(x,y)\big[\tfrac{\partial h}{\partial x}(x,y) - \tfrac{\partial g}{\partial y}(x,y)\big]\,dx\wedge dy \\
&= \big[h\tfrac{\partial f}{\partial x} - g\tfrac{\partial f}{\partial y} + f\tfrac{\partial h}{\partial x} - f\tfrac{\partial g}{\partial y}\big]\,dx\wedge dy
\end{aligned}$$

and

$$\begin{aligned}
d(\omega\wedge\omega') &= d\big(f(x,y)g(x,y)\,dx + f(x,y)h(x,y)\,dy\big) \\
&= \big[-\tfrac{\partial}{\partial y}\big(f(x,y)g(x,y)\big) + \tfrac{\partial}{\partial x}\big(f(x,y)h(x,y)\big)\big]\,dx\wedge dy
\end{aligned}$$

agree by the product rule.

(c) If $k=1$ and $k'=0$, then, by the previous case

$$d(\omega\wedge\omega') = d(\omega'\wedge\omega) = (d\omega')\wedge\omega + \omega'\wedge(d\omega) = (d\omega)\wedge\omega' - \omega\wedge(d\omega')$$

■

Problem M.17 (Vector analysis in $\mathbb{R}^3$) Let $\mathcal{M}$ be $\mathbb{R}^3$ with atlas $\big(U=\mathbb{R}^3,\ \zeta(\vec x)=\vec x=(x,y,z)\big)$. Let $f:\mathbb{R}^3\to\mathbb{R}$ be any C^∞ function on $\mathbb{R}^3$ and $\vec a(\vec x) = \big(a^1(\vec x)\ ,\ a^2(\vec x)\ ,\ a^3(\vec x)\big)$ and $\vec b(\vec x) = \big(b^1(\vec x)\ ,\ b^2(\vec x)\ ,\ b^3(\vec x)\big)$ be any two vector fields (i.e. vector valued functions) on $\mathbb{R}^3$. We can associate to $\vec a(\vec x)$ a 1–form $\omega^1_{\vec a}$ and a 2–form $\omega^2_{\vec a}$ by

$$\begin{aligned}
\omega^1_{\vec a} &= a^1(\vec x)\,dx + a^2(\vec x)\,dy + a^3(\vec x)\,dz \\
\omega^2_{\vec a} &= a^1(\vec x)\,dy\wedge dz + a^2(\vec x)\,dz\wedge dx + a^3(\vec x)\,dx\wedge dy
\end{aligned}$$

Prove that

(a) $\omega^1_{\vec a}\wedge\omega^1_{\vec b} = \omega^2_{\vec a\times\vec b}$

(b) $\omega^1_{\vec a}\wedge\omega^2_{\vec b} = \vec a(\vec x)\cdot\vec b(\vec x)\ dx\wedge dy\wedge dz$

(c) $df = \omega^1_{\vec\nabla f}$

(d) $d\omega^1_{\vec a} = \omega^2_{\vec\nabla\times\vec a}$

(e) $d\omega^2_{\vec a} = \vec\nabla\cdot\vec a(\vec x)\ dx\wedge dy\wedge dz$

Observe that

- $d^2 f = 0$ says that $0 = d\omega^1_{\vec{\nabla} f} = \omega^2_{\vec{\nabla}\times\vec{\nabla} f}$ i.e. that $\vec{\nabla}\times\vec{\nabla} f = 0$.
- $d^2\omega^1_{\vec{a}} = 0$ says that $0 = d\omega^2_{\vec{\nabla}\times\vec{a}} = \vec{\nabla}\cdot\vec{\nabla}\times\vec{a}(\vec{x})\ dx\wedge dy\wedge dz$ i.e. that $\vec{\nabla}\cdot\vec{\nabla}\times\vec{a} = 0$.

Solution. (a)

$$\begin{aligned}\omega^1_{\vec{a}}\wedge\omega^1_{\vec{b}} &= \big(a^1(\vec{x})\,dx + a^2(\vec{x})\,dy + a^3(\vec{x})\,dz\big)\wedge\big(b^1(\vec{x})\,dx + b^2(\vec{x})\,dy + b^3(\vec{x})\,dz\big)\\ &= a^1(\vec{x})b^2(\vec{x})\,dx\wedge dy + a^1(\vec{x})b^3(\vec{x})\,dx\wedge dz + a^2(\vec{x})b^1(\vec{x})\,dy\wedge dx\\ &\qquad + a^2(\vec{x})b^3(\vec{x})\,dy\wedge dz + a^3(\vec{x})b^1(\vec{x})\,dz\wedge dx + a^3(\vec{x})b^2(\vec{x})\,dz\wedge dy\\ &= \big(a^1(\vec{x})b^2(\vec{x}) - a^2(\vec{x})b^1(\vec{x})\big)\,dx\wedge dy + \big(a^3(\vec{x})b^1(\vec{x}) - a^1(\vec{x})b^3(\vec{x})\big)\,dz\wedge dx\\ &\qquad + \big(a^2(\vec{x})b^3(\vec{x}) - a^3(\vec{x})b^2(\vec{x})\big)\,dy\wedge dz\\ &= \omega^2_{\vec{a}\times\vec{b}}\end{aligned}$$

(b)

$$\begin{aligned}\omega^1_{\vec{a}}\wedge\omega^2_{\vec{b}} &= \big(a^1(\vec{x})\,dx + a^2(\vec{x})\,dy + a^3(\vec{x})\,dz\big)\wedge\big(b^1(\vec{x})\,dy\wedge dz + b^2(\vec{x})\,dz\wedge dx + b^3(\vec{x})\,dx\wedge dy\big)\\ &= a^1(\vec{x})b^1(\vec{x})\,dx\wedge dy\wedge dz + a^2(\vec{x})b^2(\vec{x})\,dy\wedge dz\wedge dx + a^3(\vec{x})b^3(\vec{x})\,dz\wedge dx\wedge dy\\ &= \vec{a}(\vec{x})\cdot\vec{b}(\vec{x})\ dx\wedge dy\wedge dz\end{aligned}$$

(c)

$$df = \tfrac{\partial f}{\partial x}\,dx + \tfrac{\partial f}{\partial y}\,dy + \tfrac{\partial f}{\partial z}\,dz = \omega^1_{\vec{\nabla} f}$$

(d)

$$\begin{aligned}d\omega^1_{\vec{a}} &= d\big(a^1(\vec{x})\,dx + a^2(\vec{x})\,dy + a^3(\vec{x})\,dz\big)\\ &= \tfrac{\partial a^1}{\partial y}\,dy\wedge dx + \tfrac{\partial a^1}{\partial z}\,dz\wedge dx + \tfrac{\partial a^2}{\partial x}\,dx\wedge dy + \tfrac{\partial a^2}{\partial z}\,dz\wedge dy + \tfrac{\partial a^3}{\partial x}\,dx\wedge dz + \tfrac{\partial a^3}{\partial y}\,dy\wedge dz\\ &= \big(\tfrac{\partial a^3}{\partial y} - \tfrac{\partial a^2}{\partial z}\big)\,dy\wedge dz + \big(\tfrac{\partial a^1}{\partial z} - \tfrac{\partial a^3}{\partial x}\big)\,dz\wedge dx + \big(\tfrac{\partial a^2}{\partial x} - \tfrac{\partial a^1}{\partial y}\big)\,dx\wedge dy\\ &= \omega^2_{\vec{\nabla}\times\vec{a}}\end{aligned}$$

(e)

$$\begin{aligned}d\omega^2_{\vec{a}} &= d\big(a^1(\vec{x})\,dy\wedge dz + a^2(\vec{x})\,dz\wedge dx + a^3(\vec{x})\,dx\wedge dy\big)\\ &= \tfrac{\partial a^1}{\partial z}\,dz\wedge dx\wedge dy + \tfrac{\partial a^2}{\partial y}\,dy\wedge dz\wedge dx + \tfrac{\partial a^3}{\partial z}\,dz\wedge dx\wedge dy\\ &= \vec{\nabla}\cdot\vec{a}(\vec{x})\ dx\wedge dy\wedge dz\end{aligned}$$

Problem M.18 Let Ω be an open connected, simply connected subset of $\mathbb{R}^2$. Think of Ω as a two dimensional manifold as in Example M.2. Let $F_1(x,y),\ F_2(x,y) \in C^\infty(\Omega)$ obey the compatibility condition that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. The goal of this problem is to prove that there exists a function $\varphi(x,y) \in C^\infty(\Omega)$ such that

$$F_1(x,y) = \tfrac{\partial\varphi}{\partial x}(x,y) \qquad \text{and} \qquad F_2(x,y) = \tfrac{\partial\varphi}{\partial y}(x,y)$$

This is the analog in two dimensions of the statement that, if Ω is a simply connected region in $\mathbb{R}^3$ and $\vec{F}(\vec{x})$ is a vector field in Ω that obeys $\vec{\nabla} \times \vec{F}(\vec{x}) = \vec{0}$, then there is a "potential" $\varphi(\vec{x})$ such that $\vec{F}(\vec{x}) = \vec{\nabla}\varphi(\vec{x})$.

(a) Define the 1–form $\omega = F_1(x,y)\,dx + F_2(x,y)\,dy$. Prove that ω is closed.

(b) Let $C_1(t), C_2(t) : [0,1] \to \Omega$ be any two paths in Ω with $C_1(0) = C_2(0)$ and $C_1(1) = C_2(1)$. That is, the two paths have the same initial and final points. Prove that $\int_{C_1} \omega = \int_{C_2} \omega$.

(c) Fix any point $(x_0, y_0) \in \Omega$. For each point $(x,y) \in \Omega$, select a path $C_{x,y}(t) : [0,1] \to \Omega$ such that $C_{x,y}(0) = (x_0, y_0)$ and $C_{x,y}(1) = (x,y)$. Define $\varphi(x,y) = \int_{C_{x,y}} \omega$. Prove that

$$\tfrac{\partial\varphi}{\partial x}(x,y) = F_1(x,y) \qquad \text{and} \qquad \tfrac{\partial\varphi}{\partial y}(x,y) = F_2(x,y)$$

(d) Let $\phi(x,y)$ and $\psi(x,y)$ be any two functions on Ω that obey

$$\tfrac{\partial\phi}{\partial x}(x,y) = \tfrac{\partial\psi}{\partial x}(x,y) = F_1(x,y) \qquad \text{and} \qquad \tfrac{\partial\phi}{\partial y}(x,y) = \tfrac{\partial\psi}{\partial y}(x,y) = F_2(x,y)$$

Prove that $\phi(x,y) - \psi(x,y)$ is a constant independent of x and y.

Solution. (a) By the hypothesised compatibility,

$$d\omega = \tfrac{\partial F_1}{\partial y}\, dy \wedge dx + \tfrac{\partial F_2}{\partial x}\, dx \wedge dy = \left(\tfrac{\partial F_2}{\partial x} - \tfrac{\partial F_1}{\partial y}\right) dx \wedge dy = 0$$

(b) Since Ω is simply connected, it is possible to deform C_1 continouously to C_2. That is, there is a smooth function $D : [0,1] \times [0,1] \to \Omega$ with

$$\begin{aligned} D(t,0) &= C_1(t) && \text{for all } 0 \le t \le 1 \\ D(t,1) &= C_2(t) && \text{for all } 0 \le t \le 1 \\ D(0,s) &= C_1(0) = C_2(0) && \text{for all } 0 \le s \le 1 \\ D(1,s) &= C_1(1) = C_2(1) && \text{for all } 0 \le s \le 1 \end{aligned}$$

The boundary $\partial D = C_1 + E_1 - C_2 - E_2$ of D consists of $C_1 - C_2$ and the two degenerate curves

$$E_1(s) = C_1(1) = C_2(1) \qquad E_2(s) = C_1(0) = C_2(0) \qquad \text{for all } 0 \le s \le 1$$

So $\int_{E_1} \omega = \int_{E_2} \omega = 0$ and, by Stoke's theorem,

$$\int_{C_1} \omega - \int_{C_2} \omega = \int_{\partial D} \omega = \int_D d\omega = 0$$

(c) Fix any point (x, y) in the open set Ω. We prove that $\frac{\partial \varphi}{\partial x}(x, y) = F_1(x, y)$. The proof of other case is similar. Fix another point (x_1, y) in Ω with the property that $(x'', y) \in \Omega$ for all x'' between x_1 and x. Choose, for each (x', y) with x' sufficiently close to x, the curve $C_{x',y}$ to obey

- ∘ $C_{x',y}(0) = (x_0, y_0)$
- ∘ $C_{x',y}(\frac{1}{2}) = (x_1, y)$
- ∘ $C_{x',y}(t)$ is independent of x' for $0 \le t \le \frac{1}{2}$
- ∘ $C_{x',y}(t) = \big(x_1 + 2(x' - x_1)(t - \frac{1}{2}), y\big)$ for $\frac{1}{2} \le t \le 1$.

That is, $C_{x'y}$ starts at (x_0, y_0) at $t = 0$, moves to (x_1, y), along a path independent of x', at $t = \frac{1}{2}$ and then follows a horizontal straight line to (x', y) at $t = 1$. The contribution to $\varphi(x', y) = \int_{C_{x',y}} \omega$ arising from $0 \le t \le \frac{1}{2}$ is independent of x' and will give contribution zero to $\frac{\partial \varphi}{\partial x}(x, y)$. The contribution to $\varphi(x', y) = \int_{C_{x',y}} \omega$ arising from $\frac{1}{2} \le t \le 1$ is precisely

$$\int_{\frac{1}{2}}^{1} F_1\big(x_1 + 2(x' - x_1)(t - \tfrac{1}{2}), y\big)\, 2(x' - x_1)\, dt = \int_{x_1}^{x'} F_1(\xi, y)\, d\xi \qquad \text{where } \xi = x_1 + 2(x' - x_1)(t - \tfrac{1}{2})$$

The derivative of this with respect to x' is $F_1(x', y)$. Hence $\frac{\partial \varphi}{\partial x}(x, y) = F_1(x, y)$ as desired.

(d) The gradient of $\theta = \phi - \psi$ vanishes identically. So $\theta = \phi - \psi$ is a constant. Here is another proof in the language of forms. If (x_0, y_0) and (x, y) are any two points in Ω and C is a curve from (x_0, y_0) to (x, y) in Ω, then

$$\theta(x, y) - \theta(x_0, y_0) = \int_{\partial C} \theta = \int_C d\theta = 0$$

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