Integration on Manifolds

Manifolds

A manifold is a generalization of a surface. We shall give the precise definition shortly. These notes are intended to provide a lightning fast introduction to integration on manifolds. For a more thorough, but still elementary discussion, see

- ▷ M.P. do Carmo, *Differential forms and applications*
- ▷ Barrett O'Neill, *Elementary Differential Geometry*, Chapter 4
- ▷ Walter Rudin, *Principles of Mathematical Analysis*, Chapter 10
- ▷ Michael Spivak, *Calculus on Manifolds; A Modern Approach to the Classical Theorems of Advanced Calculus*

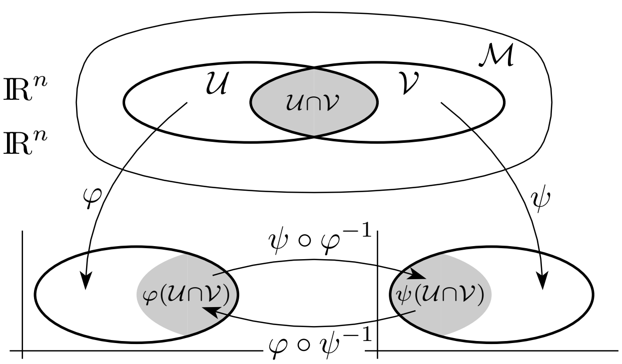
Roughly speaking, an n–dimensional manifold is a set that looks locally like $\mathbb{R}^n$. It is a union of subsets each of which may be equipped with a coordinate system with coordinates running over an open subset of $\mathbb{R}^n$. Here is a precise definition.

Definition M.1 Let $\mathcal{M}$ be a metric space. We now define what is meant by the statement that $\mathcal{M}$ is an n–dimensional C^∞ manifold.

(a) A *chart* on $\mathcal{M}$ is a pair $\{\mathcal{U}, \varphi\}$ with $\mathcal{U}$ an open subset of $\mathcal{M}$ and φ a homeomorphism (a 1–1, onto, continuous function with continuous inverse) from $\mathcal{U}$ to an open subset of $\mathbb{R}^n$. Think of φ as assigning coordinates to each point of $\mathcal{U}$.

(b) Two charts $\{\mathcal{U}, \varphi\}$ and $\{\mathcal{V}, \psi\}$ are said to be *compatible* if the transition functions

$$\psi\circ\varphi^{-1}:\ \varphi(\mathcal{U}\cap\mathcal{V})\subset\mathbb{R}^n\to\psi(\mathcal{U}\cap\mathcal{V})\subset\mathbb{R}^n$$
$$\varphi\circ\psi^{-1}:\ \psi(\mathcal{U}\cap\mathcal{V})\subset\mathbb{R}^n\to\varphi(\mathcal{U}\cap\mathcal{V})\subset\mathbb{R}^n$$



are C^∞. That is, all partial derivatives of all orders of $\psi\circ\varphi^{-1}$ and $\varphi\circ\psi^{-1}$ exist and are continuous.

(c) An *atlas* for $\mathcal{M}$ is a family $\mathcal{A} = \big\{\ \{\mathcal{U}_i, \varphi_i\}\ \big|\ i\in\mathcal{I}\ \big\}$ of charts on $\mathcal{M}$ such that $\big\{\mathcal{U}_i\big\}_{i\in\mathcal{I}}$ is an open cover of $\mathcal{M}$ and such that every pair of charts in $\mathcal{A}$ are compatible. The index set $\mathcal{I}$ is completely arbitrary. It could consist of just a single index. It could consist of uncountably many indices. An atlas $\mathcal{A}$ is called *maximal* if every chart $\{\mathcal{U}, \varphi\}$ on $\mathcal{M}$ that is compatible with every chart of $\mathcal{A}$ is itself in $\mathcal{A}$.

(d) An n-dimensional manifold consists of a metric space $\mathcal{M}$ together with a maximal atlas $\mathcal{A}$.

Problem M.1 Let $\mathcal{A}$ be an atlas for the metric space $\mathcal{M}$. Prove that there is a unique maximal atlas for $\mathcal{M}$ that contains $\mathcal{A}$.

Problem M.2 Let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of a metric space $\mathcal{M}$. Let φ be a homeomorphism from $\mathcal{U}$ to an open subset of $\mathbb{R}^n$ and ψ be a homeomorphism from $\mathcal{V}$ to an open subset of $\mathbb{R}^m$. Prove that if $\mathcal{U} \cap \mathcal{V}$ is nonempty and

$$\psi \circ \varphi^{-1}\colon\ \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n \to \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m$$
$$\varphi \circ \psi^{-1}\colon\ \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m \to \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n$$

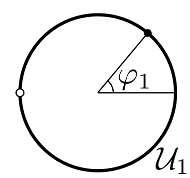
are C^∞, then $m = n$.

Thanks to Problem M.1, it suffices to supply any, not necessarily maximal, atlas for a metric space to turn it into a manifold. We do exactly that in each of the following examples.

Example M.2 (Open Subset of $\mathbb{R}^n$) Let $\mathbb{1}_n$ be the identity map on $\mathbb{R}^n$. Then $\big\{\{\mathbb{R}^n, \mathbb{1}_n\}\big\}$ is an atlas for $\mathbb{R}^n$. Indeed, if $\mathcal{U}$ is any nonempty, open subset of $\mathbb{R}^n$, then $\big\{\{\mathcal{U}, \mathbb{1}_n\}\big\}$ is an atlas for $\mathcal{U}$. So every open subset of $\mathbb{R}^n$ is naturally a C^∞ manifold.

Example M.3 (The Circle) The circle $S^1 = \big\{\ (x, y) \in \mathbb{R}^2 \ \big|\ x^2 + y^2 = 1\ \big\}$ is a manifold of dimension one when equipped with, for example, the atlas $\mathcal{A} = \{(\mathcal{U}_1, \varphi_1),\ (\mathcal{U}_2, \varphi_2)\}$ where

$$\mathcal{U}_1 = S^1 \setminus \{(-1, 0)\} \quad \varphi_1(x, y) = \arctan\tfrac{y}{x} \text{ with } -\pi < \varphi_1(x, y) < \pi$$
$$\mathcal{U}_2 = S^1 \setminus \{(1, 0)\} \quad \varphi_2(x, y) = \arctan\tfrac{y}{x} \text{ with } 0 < \varphi_2(x, y) < 2\pi$$



My use of $\arctan\frac{y}{x}$ here is pretty sloppy. To define, φ_1 carefully, we can say that $\varphi_1(x, y)$ is the unique $-\pi < \theta < \pi$ such that $(x, y) = (\cos\theta, \sin\theta)$. To verify that these two charts are compatible, we first determine the domain intersection $\mathcal{U}_1 \cap \mathcal{U}_2 = S^1 \setminus \{(-1, 0), (1, 0)\}$ and then the ranges $\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) = (-\pi, 0) \cup (0, \pi)$ and $\varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2) = (0, \pi) \cup (\pi, 2\pi)$ and finally, we check that

$$\varphi_2 \circ \varphi_1^{-1}(\theta) = \begin{Bmatrix} \theta & \text{if } 0 < \theta < \pi \\ \theta + 2\pi & \text{if } -\pi < \theta < 0 \end{Bmatrix} \qquad \varphi_1 \circ \varphi_2^{-1}(\theta) = \begin{Bmatrix} \theta & \text{if } 0 < \theta < \pi \\ \theta - 2\pi & \text{if } \pi < \theta < 2\pi \end{Bmatrix}$$

are indeed C^∞.

Example M.4 (The n–Sphere) The n–sphere

$$S^n = \left\{ \mathbf{x} = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; x_1^2 + \cdots + x_{n+1}^2 = 1 \right\}$$

is a manifold of dimension n when equipped with the atlas

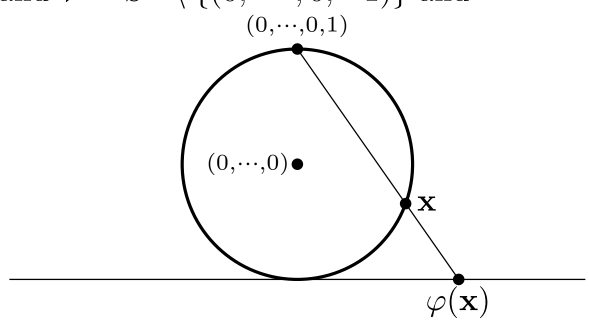
$$\mathcal{A}_1 = \left\{ (\mathcal{U}_i, \varphi_i),\ (\mathcal{V}_i, \psi_i) \;\middle|\; 1 \le i \le n+1 \right\}$$

where, for each $1 \le i \le n+1$,

$$\mathcal{U}_i = \left\{ (x_1, \cdots, x_{n+1}) \in S^n \;\middle|\; x_i > 0 \right\} \quad \varphi_i(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$$
$$\mathcal{V}_i = \left\{ (x_1, \cdots, x_{n+1}) \in S^n \;\middle|\; x_i < 0 \right\} \quad \psi_i(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$$

So both φ_i and ψ_i just discard the coordinate x_i. They project onto $\mathbb{R}^n$, viewed as the hyperplane $x_i = 0$. Another possible atlas, compatible with $\mathcal{A}_1$, is $\mathcal{A}_2 = \left\{ (\mathcal{U}, \varphi)\ ,\ (\mathcal{V}, \psi) \right\}$ where the domains $\mathcal{U} = S^n \setminus \{(0, \cdots, 0, 1)\}$ and $\mathcal{V} = S^n \setminus \{(0, \cdots, 0, -1)\}$ and

$$\varphi(x_1, \cdots, x_{n+1}) = \left(\tfrac{2x_1}{1-x_{n+1}}, \cdots, \tfrac{2x_n}{1-x_{n+1}} \right)$$
$$\psi(x_1, \cdots, x_{n+1}) = \left(\tfrac{2x_1}{1+x_{n+1}}, \cdots, \tfrac{2x_n}{1+x_{n+1}} \right)$$



are the stereographic projections from the north and south poles, respectively. Both φ and ψ have range $\mathbb{R}^n$. So we can think of S^n as $\mathbb{R}^n$ plus an additional single "point at infinity".

Problem M.3 In this problem we use the notation of Example M.4.

(a) Prove that $\mathcal{A}_1$ is an atlas for S^n.

(b) Prove that $\mathcal{A}_2$ is an atlas for S^n.

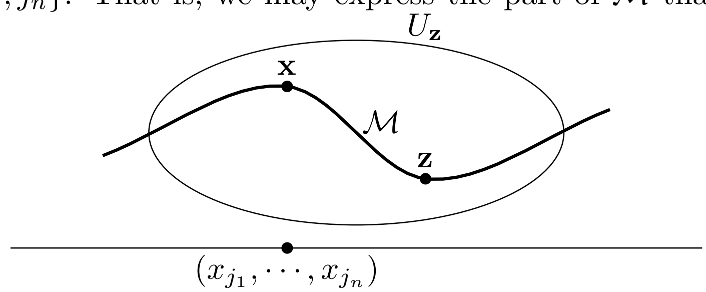
Example M.5 (Surfaces) Any smooth n–dimensional surface in $\mathbb{R}^{n+m}$ is an n–dimensional manifold. Roughly speaking, a subset of $\mathbb{R}^{n+m}$ is an n–dimensional surface if, locally, m of the $m+n$ coordinates of points on the surface are determined by the other n coordinates in a C^∞ way. For example, the unit circle S^1 is a one dimensional surface in $\mathbb{R}^2$. Near $(0, 1)$ a point $(x, y) \in \mathbb{R}^2$ is on S^1 if and only if $y = \sqrt{1 - x^2}$, and near $(-1, 0)$, (x, y) is on S^1 if and only if $x = -\sqrt{1 - y^2}$.

The precise definition is that $\mathcal{M}$ is an n–dimensional surface in $\mathbb{R}^{n+m}$ if $\mathcal{M}$ is a subset of $\mathbb{R}^{n+m}$ with the property that for each $\mathbf{z} = (z_1, \cdots, z_{n+m}) \in \mathcal{M}$, there are

- a neighbourhood $U_{\mathbf{z}}$ of $\mathbf{z}$ in $\mathbb{R}^{n+m}$
- n integers $1 \le j_1 < j_2 < \cdots < j_n \le n+m$
- and m C^∞ functions $f_k(x_{j_1}, \cdots, x_{j_n})$, $k \in \{1, \cdots, n+m\} \setminus \{j_1, \cdots, j_n\}$

such that the point $\mathbf{x} = (x_1, \cdots, x_{n+m}) \in U_{\mathbf{z}}$ is in $\mathcal{M}$ if and only if $x_k = f_k(x_{j_1}, \cdots, x_{j_n})$ for all $k \in \{1, \cdots, n+m\} \setminus \{j_1, \cdots, j_n\}$. That is, we may express the part of $\mathcal{M}$ that is near $\mathbf{z}$ as

$$
\begin{aligned}
x_{i_1} &= f_{i_1}\big(x_{j_1},\ x_{j_2},\ \cdots,\ x_{j_n}\big) \\
x_{i_2} &= f_{i_2}\big(x_{j_1},\ x_{j_2},\ \cdots,\ x_{j_n}\big) \\
&\ \vdots \\
x_{i_m} &= f_{i_m}\big(x_{j_1},\ x_{j_2},\ \cdots,\ x_{j_n}\big)
\end{aligned}
$$



where $\big\{i_1, \cdots, i_m\big\} = \{1, \cdots, n+m\} \setminus \{j_1, \cdots, j_n\}$

for some C^∞ functions $f_1,\ \cdots,\ f_m$. We may use $x_{j_1},\ x_{j_2},\ \cdots,\ x_{j_n}$ as coordinates for $\mathcal{M}$ in $\mathcal{M} \cap U_{\mathbf{z}}$. Of course, an atlas is $\mathcal{A} = \big\{\ (U_{\mathbf{z}} \cap \mathcal{M}, \varphi_{\mathbf{z}})\ \big|\ \mathbf{z} \in M\ \big\}$, with $\varphi_{\mathbf{z}}(\mathbf{x}) = (x_{j_1}, \cdots, x_{j_n})$.

Equivalently, $\mathcal{M}$ is an n–dimensional surface in $\mathbb{R}^{n+m}$, if, for each $\mathbf{z} \in \mathcal{M}$, there are

- a neighbourhood $U_{\mathbf{z}}$ of $\mathbf{z}$ in $\mathbb{R}^{n+m}$
- and m C^∞ functions $g_k : U_{\mathbf{z}} \to \mathbb{R}$, with the vectors $\big\{\ \nabla g_k(\mathbf{z})\ \big|\ 1 \le k \le m\ \big\}$ linearly independent

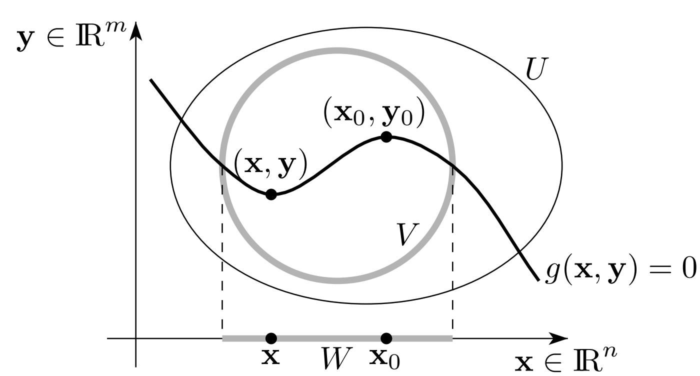
such that the point $\mathbf{x} \in U_{\mathbf{z}}$ is in $\mathcal{M}$ if and only if $g_k(\mathbf{x}) = 0$ for all $1 \le k \le m$. To get from the implicit equations for $\mathcal{M}$ given by the g_k's to the explicit equations for $\mathcal{M}$ given by the f_k's one need only invoke (possible after renumbering the components of $\mathbf{x}$) the

Implicit Function Theorem

Let $m, n \in \mathbb{N}$ and let $U \subset \mathbb{R}^{n+m}$ be an open set. Let $\mathbf{g} : U \to \mathbb{R}^m$ be C^∞ with $\mathbf{g}(\mathbf{x}_0, \mathbf{y}_0) = 0$ for some $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{y}_0 \in \mathbb{R}^m$ with $(\mathbf{x}_0, \mathbf{y}_0) \in U$. Assume that $\det \big[\frac{\partial g_i}{\partial y_j}(\mathbf{x}_0, \mathbf{y}_0)\big]_{1 \le i,j \le m} \ne 0$. Then there exist open sets $V \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^n$ with $\mathbf{x}_0 \in W$ and $(\mathbf{x}_0, \mathbf{y}_0) \in V$ such that

$$\text{for each } \mathbf{x} \in W, \text{ there is a unique } (\mathbf{x}, \mathbf{y}) \in V \text{ with } \mathbf{g}(\mathbf{x}, \mathbf{y}) = 0.$$

If the $\mathbf{y}$ above is denoted $\mathbf{f}(\mathbf{x})$, then $\mathbf{f} : W \to \mathbb{R}^m$ is C^∞, $\mathbf{f}(\mathbf{x}_0) = \mathbf{y}_0$ and $\mathbf{g}\big(\mathbf{x}, \mathbf{f}(\mathbf{x})\big) = 0$ for all $\mathbf{x} \in W$.



The n–sphere S^n is the n–dimensional surface in $\mathbb{R}^{n+1}$ given implicitly by the equation $g(x_1, \cdots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2 - 1 = 0$. In a neighbourhood of the north pole (for example, the northern hemisphere), S^n is given explicitly by the equation $x_{n+1} = \sqrt{x_1^2 + \cdots + x_n^2}$.
If you think of the set of all 3×3 real matrices as $\mathbb{R}^9$ (because a 3×3 matrix has 9 matrix elements) then

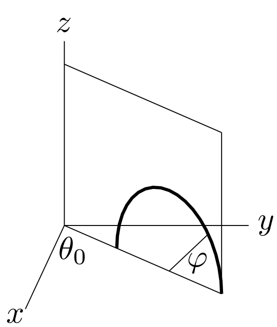
$$SO(3) = \left\{ \; 3 \times 3 \text{ real matrices } R \;\middle|\; R^t R = \mathbb{1},\ \det R = 1 \; \right\}$$

is a 3–dimensional surface in $\mathbb{R}^9$. We shall look at it more closely in Example M.7, below. $SO(3)$ is the group of all rotations about the origin in $\mathbb{R}^3$ and is also the set of all orientations of a rigid body with one point held fixed.

Example M.6 (A Torus) The torus T^2 is the two dimensional surface

$$T^2 = \left\{ \; (x, y, z) \in \mathbb{R}^3 \;\middle|\; \left(\sqrt{x^2 + y^2} - 1\right)^2 + z^2 = \tfrac{1}{4} \; \right\}$$

in $\mathbb{R}^3$. In cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the equation of the torus is $(r-1)^2 + z^2 = \frac{1}{4}$. Fix any θ, say θ_0. Recall that the set of all points in



$\mathbb{R}^3$ that have $\theta = \theta_0$ is like one page in an open book. It is a half–plane that starts at the z axis. The intersection of the torus with that half plane is a circle of radius $\frac{1}{2}$ centred on $r = 1$, $z = 0$. As φ runs from 0 to 2π, the point $r = 1 + \frac{1}{2}\cos\varphi$, $z = \frac{1}{2}\sin\varphi$, $\theta = \theta_0$ runs over that circle. If we now run θ from 0 to 2π, the circle on the page sweeps out the whole torus. So, as φ runs from 0 to 2π and θ runs from 0 to 2π, the point $(x, y, z) = \left((1 + \frac{1}{2}\cos\varphi)\cos\theta, (1 + \frac{1}{2}\cos\varphi)\sin\theta, \frac{1}{2}\sin\varphi\right)$ runs over the whole torus. So we may build coordinate patches for T^2 using θ and φ (with ranges $(0, 2\pi)$ or $(-\pi, \pi)$) as coordinates.

Example M.7 ($O(3)$, $SO(3)$) As a special case of Example M.5 we have the groups

$$SO(3) = \left\{ \; 3 \times 3 \text{ real matrices } R \;\middle|\; R^t R = \mathbb{1}_3,\ \det R = 1 \; \right\}$$
$$O(3) = \left\{ \; 3 \times 3 \text{ real matrices } R \;\middle|\; R^t R = \mathbb{1}_3 \; \right\}$$

of rotations and rotations/reflections in $\mathbb{R}^3$. (Rotations and reflections are the angle and length preserving linear maps. In classical mechanics, $SO(3)$ is the set of all possible configurations of rigid body with one point held fixed.) We can identify the set of all 3×3 real matrices with $\mathbb{R}^9$, because a 3×3 matrix has 9 matrix elements. The restriction that

$$R = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \in O(3)$$

is given implicitly by the following six equations.

$$\begin{aligned}
\big(R^tR\big)_{1,1} &= a_1^2 + a_2^2 + a_3^2 = 1 && \text{i.e. } |\mathbf{a}| = 1 \\
\big(R^tR\big)_{2,2} &= b_1^2 + b_2^2 + b_3^2 = 1 && \text{i.e. } |\mathbf{b}| = 1 \\
\big(R^tR\big)_{3,3} &= c_1^2 + c_2^2 + c_3^2 = 1 && \text{i.e. } |\mathbf{c}| = 1 \\
\big(R^tR\big)_{1,2} = \big(R^tR\big)_{2,1} &= a_1b_1 + a_2b_2 + a_3b_3 = 0 && \text{i.e. } \mathbf{a}\perp\mathbf{b} \\
\big(R^tR\big)_{1,3} = \big(R^tR\big)_{3,1} &= a_1c_1 + a_2c_2 + a_3c_3 = 0 && \text{i.e. } \mathbf{a}\perp\mathbf{c} \\
\big(R^tR\big)_{2,3} = \big(R^tR\big)_{3,2} &= b_1c_1 + b_2c_2 + b_3c_3 = 0 && \text{i.e. } \mathbf{b}\perp\mathbf{c}
\end{aligned} \tag{M.1}$$

We can verify the independence conditions of Example M.5 (that the gradients of the left hand sides are independent) directly. See Problems M.5 and M.6, below. Or we can argue geometrically. In a neighbourhood of any fixed element, $\tilde R$, of $SO(3)$, we may use two of the three $\mathbf{a}$–components as coordinates. (In fact we may use any two $\mathbf{a}$–coordinates whose magnitude at $\tilde R$ is not one.) Once two components of $\mathbf{a}$ have been chosen, the third $\mathbf{a}$–component is determined up to a sign by the requirement that $|\mathbf{a}| = 1$. The sign is chosen so as to remain in the neighbourhood. Once $\mathbf{a}$ has been chosen, the set $\{\ \mathbf{b}\in\mathbb{R}^3 \ \big|\ \mathbf{b}\perp\mathbf{a}\ \}$ is a plane through the origin so that $\{\ \mathbf{b}\in\mathbb{R}^3 \ \big|\ \mathbf{b}\perp\mathbf{a},\ |\mathbf{b}| = 1\ \}$ is the intersection of that plane with the unit sphere. So $\mathbf{b}$ lies on a great circle of the unit sphere. Thus $\mathbf{b}$ is determined up to a single rotation angle by the requirements that $\mathbf{b}\perp\mathbf{a}$ and $|\mathbf{b}| = 1$. That rotation angle is the third coordinate. Once $\mathbf{a}$ and $\mathbf{b}$ have been chosen, the set $\{\ \mathbf{c}\in\mathbb{R}^3 \ \big|\ \mathbf{c}\perp\mathbf{a},\ \mathbf{c}\perp\mathbf{b}\ \}$ is a line through the origin. So $\mathbf{c}$ is determined up to a sign by the requirements that $\mathbf{c}\perp\mathbf{a},\mathbf{b}$ and $|\mathbf{c}| = 1$. Again, the sign is chosen so as to remain in the neighbourhood. So $O(3)$ is a manifold of dimension 3. Any element of $O(3)$ automatically obeys

$$\big(\det R\big)^2 = \det R^tR = \det \mathbb{1}_3 = 1 \implies \det R = \pm 1$$

So $SO(3)$ is just one of the two connected components of $O(3)$. It is an important example of a Lie group, which is, by definition, a C^∞ manifold that is also a group with the operations of multiplication and taking inverses continuous.

Problem M.4 Let $R \in O(3)$.

(a) Prove that if λ is an eigenvalue of R, then $|\lambda| = 1$ and $\bar{\lambda}$ is an eigenvalue of R.
(b) Prove that at least one eigenvalue of R is either $+1$ or -1.
(c) Prove that the columns of R are mutually perpendicular and are each of unit length.
(d) Prove that R is either a rotation, a reflection or a composition of a rotation and a reflection.

Problem M.5 Denote by $g_1, \cdots, g_6$ the left hand sides of (M.1). Prove that the gradients of $g_1, \cdots, g_6$, evaluated at any $R \in O(3)$, are linearly independent.

Problem M.6 Use the implicit function theorem to prove that for each $1 \le i, j \le 3$, the (i, j) matrix element, a_{ij}, of matrices $R = \big[a_{ij}\big]_{1 \le i,j \le 3}$ in a neighbourhood of $\mathbb{1}$ in $SO(3)$, is a C^∞ function of the matrix elements a_{21}, a_{31} and a_{32}.

Example M.8 **(More Tori)** Define an equivalence relation on $\mathbb{R}^n$ by

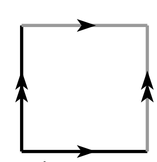
$$x \sim y \iff x - y \in \mathbb{Z}^n$$

In this example, when $x \sim y$ we want to think of x and y as two different names for the same object. The set of all possible names for the object whose name is also x is $[x] = \{\ y \in \mathbb{R}^n \ \big|\ y \sim x\ \}$ and is called the equivalence class of $x \in \mathbb{R}^n$. The set of equivalence classes is denoted $\mathbb{R}^n/\mathbb{Z}^n = \{\ [x] \ \big|\ x \in \mathbb{R}^n\ \}$. Each equivalence class $[x]$ contains exactly one representative $\tilde{x} \in [x]$ obeying $0 \le \tilde{x}_j < 1$ for each $1 \le j \le n$. So we can also think of $\mathbb{R}^n/\mathbb{Z}^n$ as being

$$\{\ x \in \mathbb{R}^n \ \big|\ 0 \le x_j < 1 \text{ for all } 1 \le j \le n\ \}$$

But then we should also identify, for each $1 \le j \le n$, the edges

$$\{\ x \in \mathbb{R}^n \ \big|\ x_j = 1,\ 0 \le x_i \le 1\ \forall\ i \ne j\ \}$$
$$\text{and } \{\ x \in \mathbb{R}^n \ \big|\ x_j = 0,\ 0 \le x_i \le 1\ \forall\ i \ne j\ \}$$



We can turn the set $\mathbb{R}^n/\mathbb{Z}^n$, which is also called a torus, into a metric space by imposing the metric

$$\rho\big([x],[y]\big) = \min\{\ |\tilde{x} - \tilde{y}| \ \big|\ \tilde{x} \in [x],\ \tilde{y} \in [y]\ \}$$

So we only need an atlas to turn the torus into a manifold. If $\mathcal{U}$ is any open subset of $\mathbb{R}^n$ with the property that no two points of $\mathcal{U}$ are equivalent (any open ball of radius at most $\frac{1}{2}$ has this property), then $[\mathcal{U}] = \{\ [x] \ \big|\ x \in \mathcal{U}\ \}$ is an open subset of $\mathbb{R}^n/\mathbb{Z}^n$ and each element of $[\mathcal{U}]$ contains a unique representative $\tilde{x} \in [x]$ that is in $\mathcal{U}$. Define

$$\begin{aligned} \Phi_{\mathcal{U}} : [\mathcal{U}] &\to \mathbb{R}^n \\ [x] &\mapsto \tilde{x} \text{ with } \tilde{x} \in [x],\ \tilde{x} \in \mathcal{U} \end{aligned}$$

Then $\{[\mathcal{U}], \Phi_{\mathcal{U}}\}$ is a chart and the set of all such charts is an atlas.

Example M.9 (The Cartesian Product) If $\mathcal{M}$ is a manifold of dimension m with atlas $\mathcal{A}$ and $\mathcal{N}$ is a manifold of dimension n with atlas $\mathcal{B}$ then

$$\mathcal{M} \times \mathcal{N} = \left\{ \ (x,y) \ \middle| \ x \in \mathcal{M},\ y \in \mathcal{N} \ \right\}$$

is an $(m+n)$–dimensional manifold with atlas

$$\left\{ \ \big(U \times V, \varphi \oplus \psi\big) \ \middle| \ (U,\varphi) \in \mathcal{A},\ (V,\psi) \in \mathcal{B} \ \right\} \qquad \text{where } \varphi \oplus \psi\big((x,y)\big) = \big(\varphi(x), \psi(y)\big)$$

For example, $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$, $S^1 \times \mathbb{R}$ is a cylinder, $S^1 \times S^1$ is a torus and the configuration space of a rigid body is $\mathbb{R}^3 \times SO(3)$ (with the $\mathbb{R}^3$ components giving the location of the centre of mass of the body and the $SO(3)$ components giving the orientation).

Example M.10 (The Möbius Strip) We are now going to turn the set

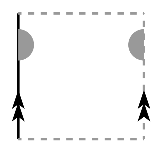
$$\mathcal{M} = [0,1) \times (-1,1)$$

into two very different manifolds by assigning two different, incompatible, atlases. Both atlases will contain two charts with

$$\mathcal{U}_1 = \big(\tfrac{1}{8}, \tfrac{7}{8}\big) \times (-1,1) \qquad \mathcal{U}_2 = \big[0, \tfrac{1}{4}\big) \times (-1,1) \cup \big(\tfrac{3}{4}, 1\big) \times (-1,1)$$

The first atlas attaches each point $(0,t)$ on the left hand edge to the point $(1,t)$ on the right hand edge by using the coordinate functions

$$\psi_1(x,y) = (x,y)$$
$$\psi_2(x,y) = \begin{cases} (x,y) & \text{if } 0 \le x < \frac{1}{4} \\ (x-1,y) & \text{if } \frac{3}{4} < x < 1 \end{cases}$$



The range of ψ_2 is

$$\psi_2\big(\big[0,\tfrac{1}{4}\big) \times (-1,1)\big) \ \cup \ \psi_2\big(\big(\tfrac{3}{4},1\big) \times (-1,1)\big) = \big[0,\tfrac{1}{4}\big) \times (-1,1) \ \cup \ \big(-\tfrac{1}{4},0\big) \times (-1,1)$$
$$= \big(-\tfrac{1}{4},\tfrac{1}{4}\big) \times (-1,1)$$

The inverse map for ψ_2 is

$$\psi_2^{-1}(x,y) = \begin{cases} (x,y) & \text{if } 0 \le x < \frac{1}{4} \\ (x+1,y) & \text{if } -\frac{1}{4} < x < 0 \end{cases}$$

The inverse image under ψ_2 of the disk $x^2 + \big(y - \frac{1}{2}\big)^2 < \frac{1}{16}$ (denote it $B_{\frac{1}{4}}(0,\frac{1}{2})$) is

$$\psi_2^{-1}\big(B_{\frac{1}{4}}(0,\tfrac{1}{2}) \cap \{x \ge 0\}\big) \ \cup \ \psi_2^{-1}\big(B_{\frac{1}{4}}(0,\tfrac{1}{2}) \cap \{x < 0\}\big)$$
$$= B_{\frac{1}{4}}(0,\tfrac{1}{2}) \cap \{x \ge 0\} \ \cup \ \left\{ \ (x+1,y) \ \middle| \ (x,y) \in B_{\frac{1}{4}}(0,\tfrac{1}{2}),\ x < 0 \ \right\}$$

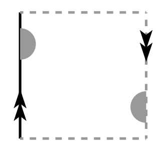
That is the union of the two shaded half disks displayed in the figure above. The union is connected in the manifold with atlas $\{\{\mathcal{U}_1, \psi_1\}, \{\mathcal{U}_1, \psi_2\}\}$. This manifold may be constructed from a strip of paper by gluing the left and right hand edges together. To complete the definition of this manifold, we must provide it with a metric and then verify that $\{(\mathcal{U}_1, \psi_1), (\mathcal{U}_1, \psi_2)\}$ really is an atlas and, in particular, that ψ_2 and its inverse are continuous. The metric (similar to the metric of Example M.8)

$$\rho_\psi\big((x,y), (x',y')\big) = \min\big\{|(x-x', y-y')|, |(x-x'+1, y-y')|, |(x-x'-1, y-y')|\big\}$$

works.

The second atlas attaches each point $(0,t)$ on the left hand edge to the point $(1,-t)$ on the right hand edge by using the coordinate functions

$$\varphi_1(x,y) = (x,y)$$
$$\varphi_2(x,y) = \begin{cases} (x,y) & \text{if } 0 \le x < \frac{1}{4} \\ (x-1,-y) & \text{if } \frac{3}{4} < x < 1 \end{cases}$$



The range of φ_2 is $\big(-\frac{1}{4}, \frac{1}{4}\big) \times (-1,1)$, the same as the range of ψ_2. The inverse map for φ_2 is

$$\varphi_2^{-1}(x,y) = \begin{cases} (x,y) & \text{if } 0 \le x < \frac{1}{4} \\ (x+1,-y) & \text{if } -\frac{1}{4} < x < 0 \end{cases}$$

The union of the two shaded half disks in the figure above is the inverse image under φ_2 of the disk $x^2 + \big(y - \frac{1}{2}\big)^2 < \frac{1}{16}$. That union is connected in the manifold with atlas $\{\{\mathcal{U}_1, \varphi_1\}, \{\mathcal{U}_1, \varphi_2\}\}$. This manifold may be constructed from a strip of paper by gluing the left and right hand edges together, after putting a half twist in the strip. It is called a Möbius strip. It has metric

$$\rho_\psi\big((x,y), (x',y')\big) = \min\big\{|(x-x', y-y')|, |(x-x'+1, y+y')|, |(x-x'-1, y+y')|\big\}$$

Problem M.7 Prove that the two charts $(\mathcal{U}_2, \varphi_2)$ and $(\mathcal{U}_2, \psi_2)$ of Example M.10 are not compatible.

Definition M.11

(a) A function f from a manifold $\mathcal{M}$ to a manifold $\mathcal{N}$ (it is traditional to omit the atlas from the notation) is said to be C^∞ at $m \in \mathcal{M}$ if there exists a chart $\{\mathcal{U}, \varphi\}$ for $\mathcal{M}$ and a chart $\{\mathcal{V}, \psi\}$ for $\mathcal{N}$ such that $m \in \mathcal{U}$, $f(m) \in \mathcal{V}$ and $\psi \circ f \circ \varphi^{-1}$ is C^∞ at $\varphi(m)$.

(b) Two manifolds $\mathcal{M}$ and $\mathcal{N}$ are *diffeomorphic* if there exists a function $f : \mathcal{M} \to \mathcal{N}$ that is 1–1 and onto with f and f^{-1} C^∞ everywhere. Then you should think of $\mathcal{M}$ and $\mathcal{N}$ as the same manifold with m and $f(m)$ being two different names for the same point, for each $m \in \mathcal{M}$.

Problem M.8 Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds. Prove that $f : \mathcal{M} \to \mathcal{N}$ is C^∞ at $m \in \mathcal{M}$ if and only if $\psi \circ f \circ \phi^{-1}$ is C^∞ at $\phi(m)$ for every chart $(\mathcal{U}, \phi)$ for $\mathcal{M}$ with $m \in \mathcal{U}$ and every chart $(\mathcal{V}, \psi)$ for $\mathcal{N}$ with $f(m) \in \mathcal{V}$.

Problem M.9 Prove that $\mathbb{R}^n$ is diffeomorphic to $\left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n x_i^2 < 1 \right\}$.

Problem M.10 Prove that $\mathbb{R}^n$ is not diffeomorphic to S^n.

Problem M.11 Outline an argument to prove that the disk $\left\{ \mathbf{x} \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 < 2 \right\}$ is not diffeomorphic to the annulus $\left\{ \mathbf{x} \in \mathbb{R}^2 \,\middle|\, 1 < x^2 + y^2 < 2 \right\}$.

Problem M.12 In this problem $G = SO(3)$.

a) Fix any $a \in G$. Denote by $I = \left\{ (i,j) \in \mathbb{N}^2 \,\middle|\, 1 \le i \le 3,\ 1 \le j \le 3 \right\}$ the set of indices for the matrix elements of the matrices in G. Prove that there exist $\alpha, \beta, \gamma \in I$ such that every matrix element $g_\delta,\ \delta \in I$ is a C^∞ function of $g_\alpha,\ g_\beta,\ g_\gamma$ for matrices $g \in G$ in a neighbourhood of a.

b) Prove that a curve $q : (c,d) \to G$ is C^∞ if and only if every matrix element $q(t)_{i,j}$ is C^∞.

c) Prove that matrix multiplication $(a,b) \mapsto ab$ is a C^∞ function from $G \times G$ to G.

d) Prove that the inverse function $a \mapsto a^{-1}$ is a C^∞ function from G to G.

Integration

We now move onto integration. I shall explicitly define integrals over 0–, 1– and 2–dimensional regions of a two dimensional manifold and prove a generalization of Stokes' theorem. I am restricting to low dimensions purely for pedagogical reasons. The same ideas also work for higher dimensions. Before getting into the details, here is a little motivational discussion.

A curve, i.e. a region that can be parametrized by a function of one real variable, is a 1–dimensional region. We shall allow, as a domain of integration for a 1–dimensional integral any finite union of, possibly disconnected, curves. We shall call this a 1–chain. In your multivariable calculus course you probably considered two types of integrals over curves.

The first was used to compute, for example, the lengths of curves. You took a curve, cut it up into a union of "infinitesimal pieces", computed the length of each infinitesimal piece by viewing it as a line segment, and added up the lengths of all of the different pieces using an integral. There is a class of manifolds, called Riemannian manifolds, on which this construction can be implemented. But we shall not consider that type of integral here.

The second type of line integral is a "work–type" integral. Here is one way such an integral can arise. Suppose we have a particle moving in a force field (for example, a gravitational field) in $\mathbb{R}^3$. A force field is just a function $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ with $\vec{F}(\vec{r})$ being the force felt by the particle when it is at $\vec{r}$. If the particle has mass m and is at $\vec{r}(t)$ at time t, then according to Newton's law of motion, the acceleration $\vec{a}(t) = \frac{d^2\vec{r}}{dt^2}(t)$ of the particle at time t is determined by

$$m\tfrac{d^2\vec{r}}{dt^2}(t) = \vec{F}\big(\vec{r}(t)\big)$$

Dot both sides of this equation with the velocity vector $\vec{v}(t) = \frac{d\vec{r}}{dt}(t)$ and observe that the resulting left hand side $m\frac{d^2\vec{r}}{dt^2}(t) \cdot \frac{d\vec{r}}{dt}(t) = \frac{d}{dt}\big(\frac{1}{2}m\vec{v}(t)^2\big)$ is a perfect derivative. If we integrate both sides of

$$\tfrac{d}{dt}\big(\tfrac{1}{2}m\vec{v}(t)^2\big) = \vec{F}\big(\vec{r}(t)\big) \cdot \tfrac{d\vec{r}}{dt}(t)$$

with respect to t from an initial time t_1 to a final time t_2 and apply the fundamental theorem of calculus, we get

$$\tfrac{1}{2}m\vec{v}(t_2)^2 - \tfrac{1}{2}m\vec{v}(t_1)^2 = \int_{t_1}^{t_2} \vec{F}\big(\vec{r}(t)\big) \cdot \tfrac{d\vec{r}}{dt}(t)\ dt$$

The quantity $\frac{1}{2}m\vec{v}(t)^2$ is called the kinetic energy of the particle at time t. So the change in kinetic energy between time t_1 and time t_2 is given by the, so called, work integral $\int_{t_1}^{t_2} \vec{F}\big(\vec{r}(t)\big) \cdot \frac{d\vec{r}}{dt}(t)\ dt$. If we call C the path followed by the particle, this integral is often denoted $\int_C \vec{F} \cdot d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz$, where $\vec{F} = \big(F_1, F_2, F_3\big)$. This is the type of integral that we shall define on general manifolds. The object being integrated, $F_1 dx + F_2 dy + F_3 dz$ will be called a 1–form.

Similarly a 0–dimensional domain of integration will consist of a finite union of points and will be called a 0–chain. A 2–dimensional domain of integration will consist of a finite union of surfaces, i.e. regions that we can parametrize by functions of two real variables, and will be called a 2–chain. The object integrated in an n–dimensional integral will be called an n–form. The definitions will be chosen so that (a) we can use local coordinate systems to express our integrals in terms of ordinary first and second year calculus integrals for evaluation, but at the same time (b) the answer to the integral so obtained does not depend on which coordinate systems are used.

We now move on to formulating the definitions associated with integration on manifolds. The formulations I have chosen are far from the most elegant ones available. But they are what you actually use when you compute integrals and they get us to integration and to Stokes' theorem relatively quickly. For the rest of these notes, assume that $\mathcal{M}$ is a two dimensional C^∞ manifold with maximal atlas $\mathcal{A}$. Except where explicitly stated otherwise, all functions are assumed to be C^∞.

0–dimensional Integrals

Definition M.12

(a) A 0–form is a function $F : \mathcal{M} \to \mathbb{C}$.

(b) A 0–chain is an expression of the form $n_1 P_1 + \cdots + n_k P_k$ with $P_1, \cdots, P_k$ distinct points of $\mathcal{M}$ and $n_1, \cdots n_k \in \mathbb{Z}$.

(c) If F is a 0–form and $n_1 P_1 + \cdots + n_k P_k$ is a 0–chain, then we define the integral

$$\int_{n_1 P_1 + \cdots + n_k P_k} F = n_1 F(P_1) + \cdots + n_k F(P_k)$$

The definition of a chain given in part (b) is somewhat casual. Under a more formal definition, a 0–chain is a function $\sigma : \mathcal{M} \to \mathbb{Z}$ for which $\sigma(P)$ is zero for all but finitely many $P \in \mathcal{M}$. The function $\sigma : \mathcal{M} \to \mathbb{Z}$ which corresponds to $n_1 P_1 + \cdots + n_k P_k$ has $\sigma(P) = n_i$ when $P = P_i$ for some $1 \le i \le k$ and $\sigma(P) = 0$ if $P \notin \{P_1, \cdots, P_k\}$. Addition of 0–chains and multiplication of a 0–chain by an integer are defined by

$$(\sigma + \sigma')(P) = \sigma(P) + \sigma'(P) \qquad\qquad (n\sigma)(P) = n\sigma(P)$$

1–dimensional Integrals

Definition M.13

(a) A 1–form ω is a rule which assigns to each coordinate chart $\{U\ ,\ \zeta=(x,y)\}$ a pair (f,g) of (complex valued) functions on $\zeta(U)$ in a coordinate invariant manner (to be defined in one sentence). We write

$$\omega\big|_{\{U,\zeta\}} = f\,dx + g\,dy$$

to indicate that ω assigns the pair (f,g) to the chart $\{U,\zeta\}$. That ω is coordinate invariant means that

- if $\{U,\zeta\}$ and $\{\tilde U,\tilde\zeta\}$ are two charts with $U\cap\tilde U\neq\emptyset$ and
- if ω assigns to $\{U,\zeta\}$ the pair of functions (f,g) and assigns to $\{\tilde U,\tilde\zeta\}$ the pair of functions $(\tilde f,\tilde g)$ and
- if the transition function $\tilde\zeta\circ\zeta^{-1}$ $\big($from $\zeta(U\cap\tilde U)\subset\mathbb{R}^2$ to $\tilde\zeta(U\cap\tilde U)\subset\mathbb{R}^2\big)$ is $\big(\tilde x(x,y),\tilde y(x,y)\big)$,

then

$$\begin{aligned} f(x,y) &= \tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial x}(x,y) + \tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial x}(x,y)\\ g(x,y) &= \tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial y}(x,y) + \tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial y}(x,y)\end{aligned}$$

Motivation, and a memory aid, for the above coordinate transformation rule is provided in Remark M.14, below.

(b) A path is a map $C:[0,1]\to\mathcal{M}$.
A 1–chain is an expression of the form $n_1C_1+\cdots+n_kC_k$ with $C_1,\cdots,C_k$ distinct paths and $n_1,\cdots n_k\in\mathbb{Z}$.

(c) Let $\big\{U,\zeta=(x,y)\big\}$ be a coordinate chart for $\mathcal{M}$ and let $\omega\big|_{\{U,\zeta\}}=f\,dx+g\,dy$. If $c(t):[0,1]\to U\subset\mathcal{M}$ is a path with range in U, then we define the integral

$$\int_c\omega = \int_0^1 \Big[\ \underbrace{f\big(\ \overbrace{\zeta(\underbrace{c(t)}_{\in\mathcal{M}})}^{\in\mathbb{R}^2}\ \big)}_{\in\mathbb{C}}\ \tfrac{d}{dt}x(c(t)) + g\big(\zeta(c(t))\big)\tfrac{d}{dt}y(c(t))\Big]\ dt$$

If c does not have range in a single chart, split it up into a finite number of pieces, each with range in a single chart. This can always be done, since the range of c is always compact. The answer is independent of choice of chart(s), because ω is invariant under coordinate transformations. See part (b) of Remark M.14 and Problem M.13.

(d) If ω is a 1–form and $n_1C_1+\cdots n_kC_k$ is a 1–chain, then we define the integral

$$\int_{n_1C_1+\cdots n_kC_k}\omega = n_1\int_{C_1}\omega+\cdots+n_k\int_{C_k}\omega$$

(e) Addition of 1–forms and multiplication of a 1–form by a function on $\mathcal{M}$ are defined as follows. Let $\alpha : \mathcal{M} \to \mathbb{C}$ and let $\big\{U, \zeta = (x, y)\big\}$ be a coordinate chart for $\mathcal{M}$. If $\omega_1\big|_{\{U,\zeta\}} = f_1\, dx + g_1\, dy$ and $\omega_2\big|_{\{U,\zeta\}} = f_2\, dx + g_2\, dy$, then

$$\begin{aligned}\big(\omega_1 + \omega_2\big)\big|_{\{U,\zeta\}} &= \big(f_1 + f_2\big)\, dx + \big(g_1 + g_2\big)\, dy \\ \big(\alpha\omega_1\big)\big|_{\{U,\zeta\}} &= \big(\alpha \circ \zeta^{-1} f_1\big)\, dx + \big(\alpha \circ \zeta^{-1} g_1\big)\, dy\end{aligned}$$

.

Remark M.14

(a) For now think of $f\, dx + g\, dy$ just as a piece of notation which specifies the two functions (f, g) that ω assigns to the chart $\{U, \zeta = (x, y)\}$. We will later define an operator d that maps n–forms to $(n + 1)$–forms. In particular, it will map the coordinate function x, which is a zero form (but which is only defined on part of the manifold) to the 1–form $1dx + 0dy$.

(b) The integral of part (c) is a generalization of the calculus definition of a "work–type" integral along a parametrized line.

(c) The motivation for the definition of a 1–form is the ordinary change of variables rule for an integral along a curve. Suppose that $\big(\tilde{x}(x, y), \tilde{y}(x, y)\big)$ expresses $(\tilde{x}, \tilde{y})$–coordinates as a function of (x, y)–coordinates. If $\big(X(t), Y(t)\big)$ is a parametrized curve in (x, y)–coordinates, then $\tilde{X}(t) = \tilde{x}\big(X(t), Y(t)\big)$, $\tilde{Y}(t) = \tilde{y}\big(X(t), Y(t)\big)$ provides a parametrization of the same curve in $(\tilde{x}, \tilde{y})$ coordinates. Substituting

$$\tilde{X}(t) = \tilde{x}\big(X(t), Y(t)\big) \qquad \tilde{Y}(t) = \tilde{y}\big(X(t), Y(t)\big)$$

and

$$\begin{aligned}\tfrac{d\tilde{X}}{dt}(t) &= \tfrac{\partial \tilde{x}}{\partial x}\big(X(t), Y(t)\big)\tfrac{dX}{dt}(t) + \tfrac{\partial \tilde{x}}{\partial y}\big(X(t), Y(t)\big)\tfrac{dY}{dt}(t) \\ \tfrac{d\tilde{Y}}{dt}(t) &= \tfrac{\partial \tilde{y}}{\partial x}\big(X(t), Y(t)\big)\tfrac{dX}{dt}(t) + \tfrac{\partial \tilde{y}}{\partial y}\big(X(t), Y(t)\big)\tfrac{dY}{dt}(t)\end{aligned}$$

into the definition

$$\int \tilde{f}(\tilde{x}, \tilde{y})\, d\tilde{x} + \tilde{g}(\tilde{x}, \tilde{y})\, d\tilde{y} = \int \big\{\tilde{f}\big(\tilde{X}(t), \tilde{Y}(t)\big)\tfrac{d\tilde{X}}{dt}(t) + \tilde{g}\big(\tilde{X}(t), \tilde{Y}(t)\big)\tfrac{d\tilde{Y}}{dt}(t)\big\}\, dt$$

of the line integral gives

$$\begin{aligned}&\int \tilde{f}(\tilde{x}, \tilde{y})\, d\tilde{x} + \tilde{g}(\tilde{x}, \tilde{y})\, d\tilde{y} \\ &= \int \Big\{\big[\tilde{f}\big(\tilde{x}(x, y), \tilde{y}(x, y)\big)\tfrac{\partial \tilde{x}}{\partial x}\big(x, y\big) + \tilde{g}\big(\tilde{x}(x, y), \tilde{y}(x, y)\big)\tfrac{\partial \tilde{y}}{\partial x}\big(x, y\big)\big]_{x=X(t), y=Y(t)}\ \tfrac{dX}{dt}(t) \\ &\qquad + \big[\tilde{f}\big(\tilde{x}(x, y), \tilde{y}(x, y)\big)\tfrac{\partial \tilde{x}}{\partial y}\big(x, y\big) + \tilde{g}\big(\tilde{x}(x, y), \tilde{y}(x, y)\big)\tfrac{\partial \tilde{y}}{\partial y}\big(x, y\big)\big]_{x=X(t), y=Y(t)}\ \tfrac{dY}{dt}(t)\Big\}\, dt \\ &= \int f(x, y)\, dx + g(x, y)\, dy\end{aligned}$$

with

$$f(x,y) = \tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial x}(x,y) + \tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial x}(x,y)$$
$$g(x,y) = \tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde x}{\partial y}(x,y) + \tilde g\big(\tilde x(x,y),\tilde y(x,y)\big)\tfrac{\partial\tilde y}{\partial y}(x,y)$$

This is exactly the coordinate transformation rule of part (a) of Definition M.13.

(d) To remember the coordinate transformation rule, just remember

$$d\tilde x = \tfrac{\partial\tilde x}{\partial x}(x,y)\,dx + \tfrac{\partial\tilde x}{\partial y}(x,y)\,dy$$
$$d\tilde y = \tfrac{\partial\tilde y}{\partial x}(x,y)\,dx + \tfrac{\partial\tilde y}{\partial y}(x,y)\,dy$$

Substituting this into

$$\begin{aligned}\tilde f d\tilde x + \tilde g d\tilde y &= \tilde f\tfrac{\partial\tilde x}{\partial x}\,dx + \tilde f\tfrac{\partial\tilde x}{\partial y}\,dy + \tilde g\tfrac{\partial\tilde y}{\partial x}\,dx + \tilde g\tfrac{\partial\tilde y}{\partial y}\,dy\\ &= \big\{\tilde f\tfrac{\partial\tilde x}{\partial x} + \tilde g\tfrac{\partial\tilde y}{\partial x}\big\}\,dx + \big\{\tilde f\tfrac{\partial\tilde x}{\partial y} + \tilde g\tfrac{\partial\tilde y}{\partial y}\big\}\,dy\end{aligned}$$

and matching the result with $f\,dx + g\,dy$ gives

$$f = \tilde f\tfrac{\partial\tilde x}{\partial x} + \tilde g\tfrac{\partial\tilde y}{\partial x} \qquad g = \tilde f\tfrac{\partial\tilde x}{\partial y} + \tilde g\tfrac{\partial\tilde y}{\partial y}$$

Putting in the only arguments that make sense, gives the detailed coordinate transformation rule.

Problem M.13 Let $\mathcal M$ be a manifold, ω be a 1–form on $\mathcal M$ and $c(t) : [0,1] \to \mathcal M$ be a path in $\mathcal M$. Prove that the definition of $\int_c \omega$ given in part (c) of Definition M.13 is independent of the decomposition of c into finitely many pieces and of the choice of coordinate charts.

2–dimensional Integrals

Definition M.15

(a) A 2–form Ω is a rule which assigns to each chart $\{U,\zeta\}$ a function f on $\zeta(U)$ such that $\Omega\big|_{\{U,\zeta\}} = f\,dx\wedge dy$ is invariant under coordinate transformations. This means that

- if $\{U,\zeta\}$ and $\{\tilde U,\tilde\zeta\}$ are two charts with $U\cap\tilde U \neq \emptyset$ and
- if Ω assigns $\{U,\zeta\}$ the function f and assigns $\{\tilde U,\tilde\zeta\}$ the function $\tilde f$ and
- if the transition function $\tilde\zeta\circ\zeta^{-1}$ $\big($from $\zeta(U\cap\tilde U)\subset \mathbb{R}^2$ to $\tilde\zeta(U\cap\tilde U)\subset\mathbb{R}^2\big)$ is $\big(\tilde x(x,y),\tilde y(x,y)\big)$,

then

$$f(x,y) = \tilde f\big(\tilde x(x,y),\tilde y(x,y)\big)\Big[\tfrac{\partial\tilde x}{\partial x}(x,y)\tfrac{\partial\tilde y}{\partial y}(x,y) - \tfrac{\partial\tilde x}{\partial y}(x,y)\tfrac{\partial\tilde y}{\partial x}(x,y)\Big]$$

(b) The standard 2–simplex is

$$Q^2 = \left\{ \ (x,y) \in \mathbb{R}^2 \ \middle| \ x,y \ge 0, x+y \le 1 \ \right\}$$

A surface is a map $D : Q^2 \to \mathcal{M}$.

A 2–chain is an expression of the form $n_1 D_1 + \cdots n_k D_k$ with $D_1, \cdots, D_k$ surfaces and $n_1, \cdots n_k \in \mathbb{Z}$.

(c) Let $\{U, \zeta = (x,y)\}$ be a chart and let $\Omega\big|_{U,\zeta} = f(x,y)\,dx \wedge dy$. If $D : Q^2 \to U \subset \mathcal{M}$ is a surface with range in U, then we define the integral

$$\begin{aligned}\int_D \Omega = \iint_{Q^2} f\big(\zeta(D(s,t))\big)\big[\tfrac{\partial}{\partial s} x\big(D(s,t)\big)\tfrac{\partial}{\partial t} y\big(D(s,t)\big) \\ - \tfrac{\partial}{\partial t} x\big(D(s,t)\big)\tfrac{\partial}{\partial s} y\big(D(s,t)\big)\big]\ dsdt\end{aligned}$$

If D does not have range in a single chart, split it up into a finite number of pieces, each with range in a single chart. This can always be done, since the range of D is always compact. The answer is independent of choice of chart(s).

(d) If Ω is a 2–form and $n_1 D_1 + \cdots n_k D_k$ is a 2–chain, then we define the integral

$$\int_{n_1 D_1 + \cdots n_k D_k} \Omega = n_1 \int_{D_1} \Omega + \cdots + n_k \int_{D_k} \Omega$$

Remark M.16

(a) Once again think, for now, of $f\,dx \wedge dy$ as just a piece of notation which specifies the function f that Ω assigns to the chart $\{U, \zeta = (x,y)\}$. We will later define a wedge product $\wedge$. Then $dx \wedge dy$ will really be the wedge product of the 1–forms dx and dy and $f\,dx \wedge dy$ will be the wedge product of the 0–form f and the 2–form $dx \wedge dy$. Under the normal notation convention, the wedge product of a 0–form f and any form ω is written $f\omega$, rather than $f \wedge \omega$.

(b) The motivation for the coordinate transformation rule of a 2–form given in Definition M.15.a is the ordinary change of variables rule

$$\begin{aligned}\int \tilde f(\tilde x, \tilde y)\ d\tilde x d\tilde y &= \int \tilde f\big(\tilde x(x,y), \tilde y(x,y)\big) \left| \det \begin{bmatrix} \frac{\partial \tilde x}{\partial x}(x,y) & \frac{\partial \tilde y}{\partial x}(x,y) \\ \frac{\partial \tilde x}{\partial y}(x,y) & \frac{\partial \tilde y}{\partial y}(x,y) \end{bmatrix} \right| \ dxdy \\ &= \int \tilde f \ \left| \tfrac{\partial \tilde x}{\partial x}\tfrac{\partial \tilde y}{\partial y} - \tfrac{\partial \tilde x}{\partial y}\tfrac{\partial \tilde y}{\partial x} \right| \ dxdy\end{aligned}$$

for an integral on a region in $\mathbb{R}^2$, except for the absolute value signs. So we are dealing with oriented (i.e. signed) areas.

(c) The integral over Q^2 in part (c) of Definition M.15 is the standard multivariable calculus expression for an integral over a parametrized region in $\mathbb{R}^2$.

The Wedge Product

We now define a multiplication rule on the space of forms. If ω is a k–form and ω' is a k'–form then the product will be a $(k+k')$–form (zero if $k+k'$ is strictly larger than the dimension of the manifold, which in our case is 2) and will be denoted $\omega \wedge \omega'$ (read "omega wedge omega prime"). It will have the following properties.

(a) $\omega \wedge \omega'$ is linear in ω and in ω'. That is, if $\omega = \alpha_1\omega_1 + \alpha_2\omega_2$ (where α_1,α_2 are complex valued functions on $\mathcal{M}$ and ω_1, ω_2 are forms, then

$$\big(\alpha_1\omega_1 + \alpha_2\omega_2\big) \wedge \omega' = \alpha_1(\omega_1 \wedge \omega') + \alpha_2(\omega_1 \wedge \omega')$$

Similarly, $\omega \wedge \big(\alpha_1'\omega_1' + \alpha_2'\omega_2'\big) = \alpha_1'(\omega \wedge \omega_1') + \alpha_2'(\omega \wedge \omega_1')$.

(b) The product is graded anticommutative. This means that if ω is a k–form and ω' is a k'–form then $\omega \wedge \omega' = (-1)^{kk'}\omega' \wedge \omega$ (so that $\omega \wedge \omega' = \omega' \wedge \omega$ if at least one of k and k' is even and $\omega \wedge \omega' = -\omega' \wedge \omega$ if both k and k' are odd). In particular $\omega \wedge \omega = 0$.

(c) The wedge product is associative. That is $(\omega \wedge \omega') \wedge \omega'' = \omega \wedge \big(\omega' \wedge \omega''\big)$.

Linearity almost determines the product completely. Pick a patch $\big\{\mathcal{U}, \zeta = (x, y)\big\}$. Then every 0-form is a function f multiplying the 0–form 1 (whose value at every point of the manifold is 1). On $\mathcal{U}$, every 1–form, $\omega\big|_{\{\mathcal{U},\zeta\}} = f dx + g dy$, is a linear combination of the two 1–forms dx and dy and every 2–form is a function times the 2–form $dx \wedge dy$. So, on the coordinate patch, the wedge product is completely determined by the wedge products of 1, dx, dy or $dx \wedge dy$ with 1, dx, dy or $dx \wedge dy$.

Since our manifold has dimension 2, the wedge product (in either order) of dx, dy or $dx \wedge dy$ with $dx \wedge dy$ is zero. And of course we define $1 \wedge \omega = \omega \wedge 1 = \omega$ for all forms ω, so that for all 0–forms f we have $f \wedge \omega = \omega \wedge f = f\omega$. By property (b), $dx \wedge dx = dy \wedge dy = 0$. So that leaves only the wedge product of dx with dy, which (surprise!) we define to be $dx \wedge dy$ and the wedge product of dy with dx, which must be $-dx \wedge dy$, by property (b). By way of summary, we have

Definition M.17 If ω is a k–form and ω' is a k'–form then $\omega \wedge \omega'$ is the $(k+k')$–form that is determined by $\omega \wedge \omega' = (-1)^{kk'}\omega' \wedge \omega$ and

(a) if $k = k' = 0$ and $(\omega \wedge \omega')(P) = \omega(P)\omega'(P)$.

(b) if $k = 0$ and $\omega'\big|_{U,\zeta} = f\,dx + g\,dy$ then

$$\omega \wedge \omega'\big|_{U,\zeta} = (\omega \circ \zeta^{-1}) f\,dx + (\omega \circ \zeta^{-1}) g\,dy$$

(c) if $k = 0$ and $\omega'\big|_{U,\zeta} = f\,dx \wedge dy$ then

$$\omega \wedge \omega'\big|_{U,\zeta} = (\omega \circ \zeta^{-1}) f\,dx \wedge dy$$

(d) if $k = k' = 1$ and $\omega\big|_{U,\zeta} = f\,dx + g\,dy$ and $\omega'\big|_{U,\zeta} = f'\,dx + g\,dy'$ then

$$\omega \wedge \omega'\big|_{U,\zeta} = [fg' - gf']\,dx \wedge dy$$

In particular $dx \wedge dx = dy \wedge dy = 0$ and $dx \wedge dy = -dy \wedge dx$.

(e) If $k + k' > 2$, $\omega \wedge \omega' = 0$.

Problem M.14 Let $\mathcal{M}$ be a manifold of dimension $n \in \mathbb{N}$ (not necessarily 2) and suppose that we have defined a wedge product for $\mathcal{M}$ that is bilinear, graded anticommutative and associative (i.e. is satisfies properties (a), (b) and (c) above). Let, for each $\le j \le n$, ω_j be a 1–form on $\mathcal{M}$ and, for each $1 \le i, j \le n$, f_{ij} be a function on $\mathcal{M}$. Prove that

$$\Big(\sum_{j=1}^n f_{1j}\omega_j\Big) \bigwedge \Big(\sum_{j=1}^n f_{2j}\omega_j\Big) \bigwedge \cdots \bigwedge \Big(\sum_{j=1}^n f_{nj}\omega_j\Big) = \det\big[f_{ij}\big]_{1\le i,j\le n}\ \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$$

The Differential Operator d

We now define a differential operator which unifies and generalizes gradient, curl and divergence (see Problem M.17, below). If ω is a k–form, then $d\omega$ will be a $k+1$–form (and will be zero if k is greater than or equal to the dimension of the manifold). It is the unique such operator that obeys

(a) d is linear. That is, if ω_1, ω_2 are k–forms and $\alpha_1, \alpha_2 \in \mathbb{C}$, then $d\big(\alpha_1\omega_1 + \alpha_2\omega_2\big) = \alpha_1 d\omega_1 + \alpha_2\omega_2$.

(b) d obeys a graded product rule. That is, if ω^k is a k–form and ω^ℓ is an ℓ–form, then $d\big(\omega^k \wedge \omega^\ell\big) = \big(d\omega^k\big) \wedge \omega^\ell + (-1)^k \omega^k \wedge \big(d\omega^\ell\big)$.

(c) If F is a 0–form and $\{U, \zeta = (x^1, \cdots, x^n)\}$ (those are superscripts, not powers) is a coordinate chart on $\mathcal{M}$, then

$$dF\big|_{\{U,\zeta\}} = \tfrac{\partial}{\partial x^1}\big(F \circ \zeta^{-1}\big)(\vec{x})\ dx^1 + \cdots + \tfrac{\partial}{\partial x^n}\big(F \circ \zeta^{-1}\big)(\vec{x})\ dx^n$$

(d) For any differential form ω, $d\big(d\omega\big) = 0$.

For two dimensions, this forces

Definition M.18 Let $\mathcal{M}$ be a two dimensional manifold. If $\{U, \zeta\}$ is a coordinate chart on $\mathcal{M}$ and

(a) if $F : \mathcal{M} \to \mathbb{C}$ is a 0–form, then

$$dF\big|_{\{U,\zeta\}} = \tfrac{\partial}{\partial x}\big(F \circ \zeta^{-1}\big)(x,y)\,dx + \tfrac{\partial}{\partial y}\big(F \circ \zeta^{-1}\big)(x,y)\,dy$$

(b) if ω is a 1–form with $\omega\big|_{\{U,\zeta\}} = f(x,y)\,dx + g(x,y)\,dy$, then

$$d\omega\big|_{\{U,\zeta\}} = \big[\tfrac{\partial g}{\partial x}(x,y) - \tfrac{\partial f}{\partial y}(x,y)\big]\,dx \wedge dy$$

(c) if Ω is a 2–form, then $d\Omega = 0$

Lemma M.19 *The differential operator d maps k–forms to $k+1$ forms and obeys*

$$d^2 = 0$$

Proof: In the case $k = 0$, (writing $f = F \circ \zeta^{-1}$)

$$d^2F = d\big(\tfrac{\partial f}{\partial x}\,dx + \tfrac{\partial f}{\partial y}\,dy\big) = \tfrac{\partial}{\partial y}\tfrac{\partial f}{\partial x} dy \wedge dx + \tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial y} dx \wedge dy = \big[-\tfrac{\partial^2 f}{\partial y \partial x} + \tfrac{\partial^2 f}{\partial x \partial y}\big] dx \wedge dy = 0$$

The cases $k = 1, 2$ are trivial, since d applied to any 2–form is zero. ■

Problem M.15 Prove that Definition M.18 is independent of the choice of coordinate chart.

Problem M.16 Prove the graded product rule that if ω is a k–form and ω' is a k'–form, then

$$d(\omega \wedge \omega') = (d\omega) \wedge \omega' + (-1)^k \omega \wedge (d\omega')$$

Problem M.17 (Vector analysis in $\mathbb{R}^3$) Let $\mathcal{M}$ be $\mathbb{R}^3$ with atlas $\big(U = \mathbb{R}^3,\ \zeta(\vec{x}) = \vec{x} = (x, y, z)\big)$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be any C^∞ function on $\mathbb{R}^3$ and $\vec{a}(\vec{x}) = \big(a^1(\vec{x})\ ,\ a^2(\vec{x})\ ,\ a^3(\vec{x})\big)$ and $\vec{b}(\vec{x}) = \big(b^1(\vec{x})\ ,\ b^2(\vec{x})\ ,\ b^3(\vec{x})\big)$ be any two vector fields (i.e. vector valued functions) on $\mathbb{R}^3$. We can associate to $\vec{a}(\vec{x})$ a 1–form $\omega^1_{\vec{a}}$ and a 2–form $\omega^2_{\vec{a}}$ by

$$\omega^1_{\vec{a}} = a^1(\vec{x})\,dx + a^2(\vec{x})\,dy + a^3(\vec{x})\,dz$$
$$\omega^2_{\vec{a}} = a^1(\vec{x})\,dy \wedge dz + a^2(\vec{x})\,dz \wedge dx + a^3(\vec{x})\,dx \wedge dy$$

Prove that

(a) $\omega^1_{\vec{a}} \wedge \omega^1_{\vec{b}} = \omega^2_{\vec{a} \times \vec{b}}$

(b) $\omega^1_{\vec{a}} \wedge \omega^2_{\vec{b}} = \vec{a}(\vec{x}) \cdot \vec{b}(\vec{x})\ dx \wedge dy \wedge dz$

(c) $df = \omega^1_{\vec{\nabla} f}$

(d) $d\omega^1_{\vec{a}} = \omega^2_{\vec{\nabla} \times \vec{a}}$

(e) $d\omega^2_{\vec{a}} = \vec{\nabla} \cdot \vec{a}(\vec{x})\ dx \wedge dy \wedge dz$

Observe that

- $d^2 f = 0$ says that $0 = d\omega^1_{\vec{\nabla} f} = \omega^2_{\vec{\nabla} \times \vec{\nabla} f}$ i.e. that $\vec{\nabla} \times \vec{\nabla} f = 0$.
- $d^2 \omega^1_{\vec{a}} = 0$ says that $0 = d\omega^2_{\vec{\nabla} \times \vec{a}} = \vec{\nabla} \cdot \vec{\nabla} \times \vec{a}(\vec{x})\ dx \wedge dy \wedge dz$ i.e. that $\vec{\nabla} \cdot \vec{\nabla} \times \vec{a} = 0$.

The Boundary Operator δ

In preparation for Stokes' theorem, we now define an operator δ that maps n–chains to $(n-1)$–chains. You should think of δC as the oriented boundary of the chain C. The definition of δ (still assuming that the manifold $\mathcal{M}$ is of dimension two) is

Definition M.20

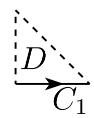
(a) For any 0–chain $\delta\big(n_1 P_1 + \cdots n_k P_k\big) = 0$.

(b) For a path $C : [0,1] \to \mathcal{M}$, δC is the 0–chain $C(1) - C(0)$.

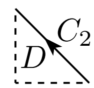
For a 1–chain $\delta\big(n_1 C_1 + \cdots n_k C_k\big) = n_1 \delta(C_1) + \cdots n_k \delta(C_k)$.

(c) For a surface $D : Q^2 \to \mathcal{M}$, δC is the 1–chain $C_1 + C_2 + C_3$ where, for $0 \le t \le 1$,

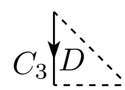
$$C_1(t) = D(t,0)$$



$$C_2(t) = D(1-t,t)$$



$$C_3(t) = D(0,1-t)$$



For a 2–chain $\delta\big(n_1 D_1 + \cdots n_k D_k\big) = n_1 \delta(D_1) + \cdots n_k \delta(D_k)$.

Lemma M.21 *The boundary operator obeys*

$$\delta^2 = 0$$

Proof: For a surface D,

$$\begin{aligned}\delta^2 D &= \delta\big(C_1 + C_2 + C_3\big) \\ &= [C_1(1) - C_1(0)] + [C_2(1) - C_2(0)] + [C_3(1) - C_3(0)] \\ &= [D(1,0) - D(0,0)] + [D(0,1) - D(1,0)] + [D(0,0) - D(0,1)] = 0\end{aligned}$$

The case $n = 2$ follows from this. The cases $n = 0, 1$ are trivial, because δ applied to any 0–chain is zero. ■

Stokes' Theorem

Theorem M.22 (Stokes' Theorem) *If ω is a k–form and D is a $(k+1)$–chain, then*

$$\int_{\delta D} \omega = \int_D d\omega$$

Proof: We give the proof for a manifold of dimension two. For $k = 0$ and D being the path C, this is the fundamental theorem of calculus.

$$F(C(1)) - F(C(0)) = \int_C dF$$

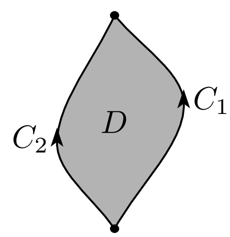
For $k = 1$ and D a surface, this is Green's Theorem.

$$\int_{\delta D} f\,dx + g\,dy = \iint_D \left[\tfrac{\partial g}{\partial x} - \tfrac{\partial f}{\partial y}\right] dx \wedge dy$$

For $k \geq 2$, both sides are zero. ■

Here are two consequences of Stokes' Theorem. Firstly, if ω is a compactly supported 1–form, $\iint_{\mathcal{M}} d\omega = 0$. To see this, let S be a surface in $\mathcal{M}$ that contains the support of ω. Then $\iint_{\mathcal{M}} d\omega = \iint_S d\omega = \int_{\partial S} \omega = 0$, since ω vanishes on ∂S. Secondly, if ω is a closed 1–form (meaning that $d\omega = 0$) and if C_1 and C_2 are two 1–chains with $C_1 - C_2 = \delta D$ for some 2–chain D, then $\int_{C_1} \omega = \int_{C_2} \omega$, since

$$\int_{C_1} \omega - \int_{C_2} \omega = \int_{C_1 - C_2} \omega = \int_D d\omega = 0$$



Problem M.18 Let Ω be an open connected, simply connected subset of $\mathbb{R}^2$. Think of Ω as a two dimensional manifold as in Example M.2. Let $F_1(x,y),\ F_2(x,y) \in C^\infty(\Omega)$ obey the compatibility condition that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. The goal of this problem is to prove that there exists a function $\varphi(x,y) \in C^\infty(\Omega)$ such that

$$F_1(x,y) = \tfrac{\partial \varphi}{\partial x}(x,y) \qquad \text{and} \qquad F_2(x,y) = \tfrac{\partial \varphi}{\partial y}(x,y)$$

This is the analog in two dimensions of the statement that, if Ω is a simply connected region in $\mathbb{R}^3$ and $\vec{F}(\vec{x})$ is a vector field in Ω that obeys $\vec{\nabla} \times \vec{F}(\vec{x}) = \vec{0}$, then there is a "potential" $\varphi(\vec{x})$ such that $\vec{F}(\vec{x}) = \vec{\nabla}\varphi(\vec{x})$.

(a) Define the 1–form $\omega = F_1(x,y)\,dx + F_2(x,y)\,dy$. Prove that ω is closed.

(b) Let $C_1(t), C_2(t) : [0,1] \to \Omega$ be any two paths in Ω with $C_1(0) = C_2(0)$ and $C_1(1) = C_2(1)$. That is, the two paths have the same initial and final points. Prove that $\int_{C_1} \omega = \int_{C_2} \omega$.

(c) Fix any point $(x_0, y_0) \in \Omega$. For each point $(x, y) \in \Omega$, select a path $C_{x,y}(t) : [0, 1] \to \Omega$ such that $C_{x,y}(0) = (x_0, y_0)$ and $C_{x,y}(1) = (x, y)$. Define $\varphi(x, y) = \int_{C_{x,y}} \omega$. Prove that

$$\frac{\partial \varphi}{\partial x}(x, y) = F_1(x, y) \qquad \text{and} \qquad \frac{\partial \varphi}{\partial y}(x, y) = F_2(x, y)$$

(d) Let $\phi(x, y)$ and $\psi(x, y)$ be any two functions on Ω that obey

$$\frac{\partial \phi}{\partial x}(x, y) = \frac{\partial \psi}{\partial x}(x, y) = F_1(x, y) \qquad \text{and} \qquad \frac{\partial \phi}{\partial y}(x, y) = \frac{\partial \psi}{\partial y}(x, y) = F_2(x, y)$$

Prove that $\phi(x, y) - \psi(x, y)$ is a constant independent of x and y.