Euler Angles

Euler angles are three angles that provide coordinates on $SO(3)$. Denote by $R_1(\theta)$ and $R_3(\theta)$ rotations about the $x-$ and $z-$ axes, respectively, by an angle θ. That is

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad R_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For each $(\varphi, \theta, \psi) \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)$, the matrix $R_3(\varphi)R_1(\theta)R_3(\psi)$ is certainly in $SO(3)$. The following theorem says that each $R \in SO(3)$ has a representation of the form $R_3(\varphi)R_1(\theta)R_3(\psi)$ with $(\varphi, \theta, \psi) \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)$ and that this representation is unique unless θ is 0 or π (in which case $R_3(\varphi)R_1(\theta)R_3(\psi)$ depends only on $\varphi + \psi$ or $\varphi - \psi$, respectively).

Theorem.

(a) (Existence) For each $R \in SO(3)$, there an $(\varphi, \theta, \psi) \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)$ such that $R = R_3(\varphi)R_1(\theta)R_3(\psi)$.

(b) (Uniqueness) Let $(\varphi, \theta, \psi), (\varphi', \theta', \psi') \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)$ and assume that $R_3(\varphi)R_1(\theta)R_3(\psi) = R_3(\varphi')R_1(\theta')R_3(\psi')$. Then $\theta = \theta'$. If $\theta \neq 0, \pi$, then $\varphi = \varphi'$ and $\psi = \psi'$.

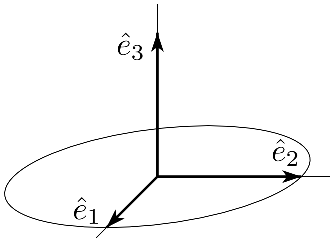
(c) If $R = \begin{bmatrix} \hat\imath_1 & \hat\imath_2 & \hat\imath_3 \end{bmatrix}$, then

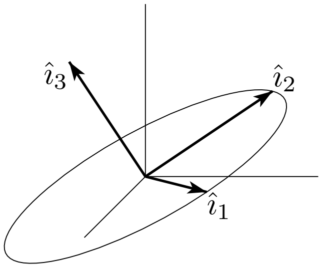
- *θ is the angle between $\hat\imath_3$ and the z–axis and*
- *if $\theta \notin \{0, \pi\}$, then φ is the angle between the projection of $\hat\imath_3$ on the xy–plane and the x–axis, plus $\frac{\pi}{2}$*

Proof: (a) Write $R = \begin{bmatrix} \hat\imath_1 & \hat\imath_2 & \hat\imath_3 \end{bmatrix}$ where $\hat\imath_1$, $\hat\imath_2$ and $\hat\imath_3$ are three mutually perpendicular unit vectors that form a right handed triple and use

$$\hat e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

to denote the standard basis for $\mathbb{R}^3$.



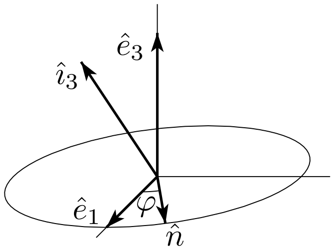


Set

$$\hat{n} = \begin{cases} \frac{\hat{e}_3 \times \hat{\imath}_3}{|\hat{e}_3 \times \hat{\imath}_3|} & \text{if } \hat{\imath}_3 \neq \pm\hat{e}_3 \\ \hat{\imath}_1 & \text{if } \hat{\imath}_3 = \hat{e}_3 \\ -\hat{\imath}_1 & \text{if } \hat{\imath}_3 = -\hat{e}_3 \end{cases}$$

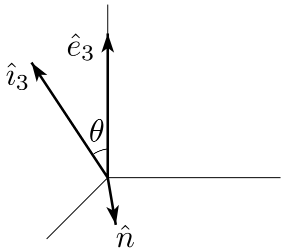
Since $\hat{e}_1$ and $\hat{n}$ are both unit vectors that are perpendicular to $\hat{e}_3$, there is an angle $\varphi \in [0, 2\pi)$ such that rotation about $\hat{e}_3$ by φ maps $\hat{e}_1$ to $\hat{n}$. This rotation leaves $\hat{e}_3$ invariant. Thus

$$\hat{n} = R_3(\varphi)\hat{e}_1 \qquad \hat{e}_3 = R_3(\varphi)\hat{e}_3$$



Now $\hat{e}_3$ and $\hat{\imath}_3$ are both unit vectors that are perpendicular to $\hat{n}$. So there is an angle $\theta \in [0, 2\pi)$ such that rotation about $\hat{n}$ by θ maps $\hat{e}_3$ to $\hat{\imath}_3$. In fact $\theta \in [0, \pi]$ because(1) $\hat{n}$ has direction $\hat{e}_3 \times \hat{\imath}_3$. The rotation about $\hat{n}$ by θ is implemented by $R_3(\varphi)R_1(\theta)R_3(-\varphi)$ (since $\frac{d}{d\theta}R_3(\varphi)R_1(\theta)R_3(-\varphi)\vec{x} = R_3(\varphi)\big(\hat{e}_1 \times (R_3(-\varphi)\vec{x})\big) = \big(R_3(\varphi)\hat{e}_1\big) \times \vec{x} = \hat{n} \times \vec{x}$). So

$$\begin{aligned} \hat{\imath}_3 &= R_3(\varphi)R_1(\theta)R_3(-\varphi)\hat{e}_3 = R_3(\varphi)R_1(\theta)\hat{e}_3 \\ \hat{n} &= R_3(\varphi)R_1(\theta)R_3(-\varphi)\hat{n} = R_3(\varphi)R_1(\theta)\hat{e}_1 \end{aligned}$$



Finally, $\hat{\imath}_1$ and $\hat{n}$ are both unit vectors that are perpendicular to $\hat{\imath}_3$. So there is an angle $\psi \in [0, 2\pi)$ such that rotation about $\hat{\imath}_3$ by ψ maps $\hat{n}$ to $\hat{\imath}_1$. The rotation about $\hat{\imath}_3$ by ψ is implemented by $R_3(\varphi)R_1(\theta)R_3(\psi)R_1(-\theta)R_3(-\varphi)$. (Note that $R_3(\varphi)R_1(\theta)R_3(\psi)R_1(-\theta)R_3(-\varphi)\hat{\imath}_3 = \hat{\imath}_3$.) So

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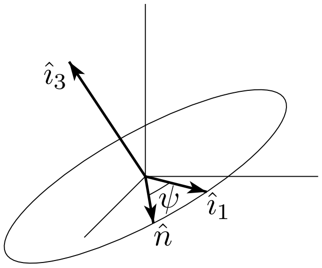
(1) Use a coordinate system with $\hat{e}_3$ on the positive z–axis and $\hat{n}$ on the positive y–axis. If $\hat{\imath}_3 = a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3$, then $\hat{e}_3 \times \hat{\imath}_3 = a\hat{e}_2 - b\hat{e}_1$. For this to be a positive multiple of $\hat{e}_2$, we need $b = 0$ and $a > 0$. So $\hat{\imath}_3$ must be in the xz–plane with positive x–component.

$$\hat{\imath}_1 = R_3(\varphi)R_1(\theta)R_3(\psi)R_1(-\theta)R_3(-\varphi)\hat{n} = R_3(\varphi)R_1(\theta)R_3(\psi)\hat{e}_1$$
$$\hat{\imath}_3 = R_3(\varphi)R_1(\theta)R_3(\psi)R_1(-\theta)R_3(-\varphi)\hat{\imath}_3 = R_3(\varphi)R_1(\theta)R_3(\psi)\hat{e}_3$$



In other words, the two matrices R and $R_3(\varphi)R_1(\theta)R_3(\psi)$ have the same first column (namely $\hat{\imath}_1$) and the same third column (namely $\hat{\imath}_3$). The columns of any matrix in $SO(3)$ form a right handed triple of mutually perpendicular unit vectors. As both R and $R_3(\varphi)R_1(\theta)R_3(\psi)$ are in $SO(3)$, they both must have the same middle column too (namely $\hat{\imath}_2$).

Uniqueness: Let $(\varphi, \theta, \psi), (\varphi', \theta', \psi') \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)$ and assume that

$$R_3(\varphi)R_1(\theta)R_3(\psi) = R_3(\varphi')R_1(\theta')R_3(\psi')$$

Multiplying on the left by $R_3(-\varphi')$ and on the right by $R_3(-\psi')$ gives

$$R_3(\varphi - \varphi')R_1(\theta)R_3(\psi - \psi') = R_1(\theta')$$

In particular $R_3(\varphi - \varphi')R_1(\theta)R_3(\psi - \psi')\hat{e}_3 = R_1(\theta')\hat{e}_3$. As

$$R_1(\theta')\hat{e}_3 = \begin{bmatrix} 0 \\ -\sin\theta' \\ \cos\theta' \end{bmatrix}$$

and

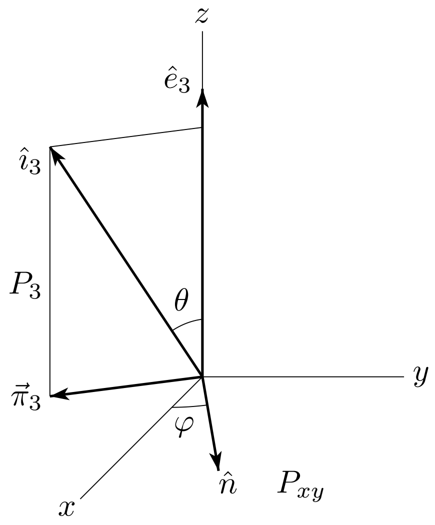
$$\begin{aligned} R_3(\varphi - \varphi')R_1(\theta)R_3(\psi - \psi')\hat{e}_3 &= R_3(\varphi - \varphi')R_1(\theta)\hat{e}_3 \\ &= \begin{bmatrix} \cos(\varphi - \varphi') & -\sin(\varphi - \varphi') & 0 \\ \sin(\varphi - \varphi') & \cos(\varphi - \varphi') & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \sin(\varphi - \varphi')\sin\theta \\ -\cos(\varphi - \varphi')\sin\theta \\ \cos\theta \end{bmatrix} \end{aligned}$$

we have

$$\begin{bmatrix} 0 \\ -\sin\theta' \\ \cos\theta' \end{bmatrix} = \begin{bmatrix} \sin(\varphi - \varphi')\sin\theta \\ -\cos(\varphi - \varphi')\sin\theta \\ \cos\theta \end{bmatrix}$$

As $0 \le \theta, \theta' \in [0, \pi]$ and cos is 1–1 on $[0\pi]$, we now have that $\theta = \theta'$. As long as $\theta \neq 0, \pi$, we also have $\sin(\varphi - \varphi') = 0$ and $\cos(\varphi - \varphi') = 1$, which forces $\varphi - \varphi'$ to be an integer multiple of 2π. Since $0 \le \varphi, \varphi' < 2\pi$, this in turn forces $\varphi = \varphi'$. Finally, $R_3(\varphi)R_1(\theta)R_3(\psi) = R_3(\varphi')R_1(\theta')R_3(\psi')$ has now reduced to $R_3(\varphi)R_1(\theta)R_3(\psi) = R_3(\varphi)R_1(\theta)R_3(\psi')$, which is equivalent to $R_3(\psi) = R_3(\psi')$. Since $0 \le \psi, \psi' < 2\pi$, this forces $\psi = \psi'$.

(c) Assume that $\theta \neq 0$. Then $\hat{n} = \frac{\hat{e}_3 \times \hat{\imath}_3}{|\hat{e}_3 \times \hat{\imath}_3|}$ is perpendicular to both $\hat{\imath}_3$ and $\hat{e}_3$. We defined θ to be the angle of rotation about $\hat{n}$ that maps $\hat{e}_3$ to $\hat{\imath}_3$. By definition, this is the angle between $\hat{e}_3$ and $\hat{\imath}_3$. Let P_{xy} denote the xy–plane and P_3 denote the plane containing $\hat{e}_3$ and $\hat{\imath}_3$. Then $\hat{n}$ lies in P_{xy} (because it is perpendicular to $\hat{e}_3$) and is perpendicular to P_3. The projection of $\hat{\imath}_3$ onto the xy–plane (denoted $\vec{\pi}_3$ in the figure below) lies in both P_3 and P_{xy} and so is an angle $\frac{\pi}{2}$ from $\hat{n}$. We defined φ to be the angle between $\hat{e}_1$ and $\hat{n}$.



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