Tempered Distributions

The theory of tempered distributions allows us to give a rigorous meaning to the Dirac delta function. It is "defined", on a hand waving level, by the properties that

(i) $\delta(x) = 0$ except when $x = 0$

(ii) $\delta(0)$ is "so infinite" that

(iii) the area under its graph is one.

Still on a handwaving level, if f is any continuous function, then the functions $f(x)\delta(x)$ and $f(0)\delta(x)$ are the same since they are both zero for every $x \neq 0$. Consequently

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = \int_{-\infty}^{\infty} f(0)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty} \delta(x)\,dx = f(0) \tag{1}$$

That $\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)$ is by far the most important property of the Dirac delta function. But we already have seen in a problem set that there is no Riemann integrable function $\delta(x)$ that satisfies (1).

The basic idea which allows us to make make rigorous sense of (1) is to generalize the meaning of "a function on $\mathbb{R}$". We shall call the generalization a "tempered distribution on $\mathbb{R}$". Of course a function on $\mathbb{R}$, in the conventional sense, is a rule which assigns a number to each $x \in \mathbb{R}$. A tempered distribution will be a rule which assigns a number to each nice (to be made precise shortly) function on $\mathbb{R}$. We will associate to the conventional function $f : \mathbb{R} \to \mathbb{C}$ the tempered distribution which assigns to the nice function $\varphi(x)$ the number $\int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$. The tempered distribution which corresponds to the Dirac delta function will assign to $\varphi(x)$ the number $\varphi(0)$.

Our first order of business is to make precise "nice function".

Definition 1 Schwartz space is the vector space

$$\mathcal{S}(\mathbb{R}) = \Big\{\ \varphi : \mathbb{R} \to \mathbb{C}\ \Big|\ \varphi \text{ is } C^\infty,\ \sup_{x\in\mathbb{R}} \big|x^n \varphi^{(m)}(x)\big| < \infty \text{ for all integers } n, m \ge 0\ \Big\}$$

Observe that

(1) $\mathcal{S}(\mathbb{R})$ is indeed a vector space. That is,

$$\varphi, \psi \in \mathcal{S}(\mathbb{R}),\ a, b \in \mathbb{C} \implies a\varphi + b\psi \in \mathcal{S}(\mathbb{R})$$

(2) If $f(x)$ is any continuous function on $\mathbb{R}$ which is bounded by a constant times $1 + |x|^p$ for some $p \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R})$, then $f(x)\varphi(x)$ is a continuous function that is bounded

by some constant times $\frac{1}{1+x^2}$ (take $n = p + 2$ and $m = 0$ in Definition 1) so that the integral $\int_{-\infty}^{\infty} f(x)\varphi(x)\, dx$ converges.

(3) Define, for each $n, m \in \mathbb{Z}$ with $n, m \geq 0$ and each $\varphi \in \mathcal{S}(\mathbb{R})$

$$\|\varphi\|_{n,m} = \sup_{x \in \mathbb{R}} \left| x^n \varphi^{(m)}(x) \right|$$

Then

(a) $\|\varphi\|_{n,m} \geq 0$

(b) $\|a\varphi\|_{n,m} = |a|\, \|\varphi\|_{n,m}$

(c) $\|\varphi + \psi\|_{n,m} \leq \|\varphi\|_{n,m} + \|\psi\|_{n,m}$

for all $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ and $a \in \mathbb{C}$. These are precisely the defining conditions for $\|\cdot\|_{n,m}$ to be a semi–norm.

(4) In order for $\|\cdot\|_{n,m}$ to be a norm it must also obey $\|\varphi\|_{n,m} = 0 \iff \varphi = 0$. This is the case if and only if $m = 0$. If $m \neq 0$ the constant function $\varphi(x) = 1$ has $\|\varphi\|_{n,m} = 0$.

Example 2

(a) For any polynomial $P(x)$, the function $\varphi(x) = P(x)e^{-x^2}$ is in Schwartz space. This is because, firstly, for any $n, m \geq 0$, $x^n \varphi^{(m)}(x)$ is again a polynomial times e^{-x^2} and, secondly,

$$e^{-x^2} = \frac{1}{e^{x^2}} \leq \frac{1}{1 + x^2 + \frac{1}{2!}x^2 + \cdots + \frac{1}{p!}x^{2p}} \tag{2}$$

for every $p \in \mathbb{N}$. (The terms that we have dropped from the Taylor expansion of e^{x^2} are all positive.) Consequently, $x^n \varphi^{(m)}(x)$ is bounded.

(b) If φ is C^∞ and of compact support (which means that there is some $M > 0$ such that $\varphi(x) = 0$ for all $|x| > M$) then $\varphi \in \mathcal{S}(\mathbb{R})$. One such function is

$$\varphi(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ e^{-\frac{1}{(x-1)^2}} e^{-\frac{1}{(x+1)^2}} & \text{if } -1 < x < 1 \end{cases}$$

The heart of the proof that this function really is C^∞ at $x = \pm 1$ is the observation that, for any $p \geq 0$, $\lim\limits_{y \to 0} \frac{1}{|y|^p} e^{-\frac{1}{y^2}} = 0$, which follows immediately from (2) with $x = \frac{1}{y}$.

Next, we introduce a metric on $\mathcal{S}(\mathbb{R})$ which is chosen so that φ and ψ are close together if and only if $\|\varphi - \psi\|_{n,m}$ is small for every n, m. The details are given in the following

Theorem 3 *Define* $d : \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \to \mathbb{R}$ *by*

$$d(\varphi, \psi) = \sum_{n,m=0}^{\infty} 2^{-n-m} \frac{\|\varphi - \psi\|_{n,m}}{1 + \|\varphi - \psi\|_{n,m}}$$

Then

(a) $d(\varphi, \psi)$ *is well–defined for all* $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ *and is a metric.*

(b) With this metric, $\mathcal{S}(\mathbb{R})$ *is a complete metric space.*

(c) In this metric $\varphi = \lim_{k\to\infty} \varphi_k$ *if and only if* $\lim_{k\to\infty} \|\varphi_k - \varphi\|_{n,m} = 0$ *for every* $n, m \geq 0$.

Proof: (a) To prove that $\sum\limits_{n,m=0}^{\infty} 2^{-n-m} \frac{\|\varphi-\psi\|_{n,m}}{1+\|\varphi-\psi\|_{n,m}}$ is well–defined it suffices to observe, firstly, that $\frac{A}{1+A} \leq 1$ for every $A \geq 0$ and, secondly, that $\sum\limits_{n,m=0}^{\infty} 2^{-n-m}$ converges because the geometric series $\sum\limits_{n=0}^{\infty} 2^{-n} = 2$.

- The metric axiom $d(\varphi, \psi) \geq 0$ is obvious.
- The metric axiom that $d(\varphi, \psi) = 0 \implies \varphi = \psi$ is obvious because $d(\varphi, \psi) = 0$ forces the $n = m = 0$ term in its definition, namely $\frac{\|\varphi-\psi\|_{0,0}}{1+\|\varphi-\psi\|_{0,0}}$, to vanish. And that first term is zero if and only if $\|\varphi - \psi\|_{0,0} = \sup_{x\in\mathbb{R}} |\varphi(x) - \psi(x)|$ is zero.
- The metric axiom $d(\varphi, \psi) = d(\psi, \varphi)$ is obvious.
- The triangle inequality follows from

$$\frac{\|\varphi - \psi\|_{n,m}}{1 + \|\varphi - \psi\|_{n,m}} \leq \frac{\|\varphi - \zeta\|_{n,m}}{1 + \|\varphi - \zeta\|_{n,m}} + \frac{\|\zeta - \psi\|_{n,m}}{1 + \|\zeta - \psi\|_{n,m}}$$

which is proven as follows. We supress the subscripts n, m. Because $\frac{x}{1+x} = 1 - \frac{1}{1+x}$ is an increasing function of x

$$\begin{aligned} \tfrac{\|\varphi-\psi\|}{1+\|\varphi-\psi\|} &\leq \tfrac{\|\varphi-\zeta\|+\|\zeta-\psi\|}{1+\|\varphi-\zeta\|+\|\zeta-\psi\|} = \tfrac{\|\varphi-\zeta\|}{1+\|\varphi-\zeta\|+\|\zeta-\psi\|} + \tfrac{\|\zeta-\psi\|}{1+\|\varphi-\zeta\|+\|\zeta-\psi\|} \\ &\leq \tfrac{\|\varphi-\zeta\|}{1+\|\varphi-\zeta\|} + \tfrac{\|\zeta-\psi\|}{1+\|\zeta-\psi\|} \end{aligned}$$

(c) For the "only if" part, assume that $\varphi = \lim_{k\to\infty} \varphi_k$ and let $n, m \geq 0$. Then

$$d(\varphi, \varphi_k) \geq 2^{-n-m} \frac{\|\varphi - \varphi_k\|_{n,m}}{1 + \|\varphi - \varphi_k\|_{n,m}} \implies \lim_{k\to 0} \frac{\|\varphi - \varphi_k\|_{n,m}}{1 + \|\varphi - \varphi_k\|_{n,m}} = 0$$

For any $0 < \varepsilon < \frac{1}{2}$ and $x > 0$,

$$\tfrac{x}{1+x} < \varepsilon \implies x < \varepsilon(1 + x) \implies x - \varepsilon x < \varepsilon \implies x < \tfrac{\varepsilon}{1-\varepsilon} < 2\varepsilon$$

Hence $\lim_{k\to 0}\|\varphi-\varphi_k\|_{n,m}=0$ too.

For the "if" part assume that $\lim\limits_{k\to\infty}\|\varphi_k-\varphi\|_{n,m}=0$ for every $n,m\ge 0$. We must prove that, as a consequence, $\varphi=\lim\limits_{k\to\infty}\varphi_k$. The idea is that, in the definition of $d(\varphi,\psi)$, the sum of all terms with m or n large is small, regardless of what φ and ψ are. Precisely, write $\psi_k=\varphi-\varphi_k$ and note that, for every $M\in\mathbb{N}$

$$\begin{aligned}
d(\varphi_k,\varphi) &= \sum_{n,m=0}^{\infty} 2^{-n-m}\frac{\|\psi_k\|_{n,m}}{1+\|\psi_k\|_{n,m}} \\
&= \sum_{0\le n,m\le M} 2^{-n-m}\frac{\|\psi_k\|_{n,m}}{1+\|\psi_k\|_{n,m}} + \sum_{\substack{n,m=0\\ n\text{ or } m>M}}^{\infty} 2^{-n-m}\frac{\|\psi_k\|_{n,m}}{1+\|\psi_k\|_{n,m}} \\
&\le \sum_{0\le n,m\le M} 2^{-n-m}\frac{\|\psi_k\|_{n,m}}{1+\|\psi_k\|_{n,m}} + \sum_{\substack{n,m=0\\ n\text{ or } m>M}}^{\infty} 2^{-n-m} \\
&\le \sum_{0\le n,m\le M} 2^{-n-m}\frac{\|\psi_k\|_{n,m}}{1+\|\psi_k\|_{n,m}} + 2\Big\{\sum_{m=M+1}^{\infty}2^{-m}\Big\}\Big\{\sum_{n=0}^{\infty}2^{-n}\Big\} \\
&= \sum_{0\le n,m\le M} 2^{-n-m}\frac{\|\psi_k\|_{n,m}}{1+\|\psi_k\|_{n,m}} + 2\Big\{\frac{1}{2^M}\Big\}\{2\}
\end{aligned}$$

Let $\varepsilon>0$ and choose M so that $\frac{1}{2^M}\le\frac{\varepsilon}{8}$. For each $n,m\ge 0$, $\lim\limits_{k\to\infty}\|\psi\|_{n,m}=0$ so that there is a $K_{n,m}$ for which $k\ge K_{n,m}$ implies $\|\psi\|_{n,m}<\frac{\varepsilon}{8}$. Set $K=\max\big\{\,K_{n,m}\,\big|\,0\le n,m\le M\,\big\}$. If $k\ge K$, then

$$d(\varphi_k,\varphi)\le\sum_{0\le n,m\le M}2^{-n-m}\frac{\|\psi_k\|_{n,m}}{1+\|\psi_k\|_{n,m}}+2\Big\{\frac{1}{2^M}\Big\}\{2\}<\tfrac{\varepsilon}{2}+\sum_{n,m=0}^{\infty}2^{-n-m}\tfrac{\varepsilon}{8}=\varepsilon$$

(b) Let $\{\varphi_k\}$ be a Cauchy sequence with respect to the metric d. Then, as in part (c), for each $n,m\ge 0$, $\lim\limits_{k,k'\to\infty}\big\|\varphi_k-\varphi_{k'}\big\|_{n,m}=0$. In particular, $\lim\limits_{k,k'\to\infty}\big\|\varphi_k-\varphi_{k'}\big\|_{0,0}=0$, so that the sequence $\{\varphi_k\}$ is Cauchy in the set, $\mathcal{C}(\mathbb{R})$, of all bounded, continuous functions on $\mathbb{R}$ equipped with the uniform metric. We already know that $\mathcal{C}(\mathbb{R})$ is complete, so there exists a continuous function φ such that $\{\varphi_k\}$ converges uniformly to φ. As well, $\lim\limits_{k,k'\to\infty}\big\|\varphi_k-\varphi_{k'}\big\|_{0,1}=0$ so that the sequence $\{\varphi_k'\}$ of first derivatives is Cauchy in $\mathcal{C}(\mathbb{R})$ and there exists a continuous function φ_1 such that $\{\varphi_k'\}$ converges uniformly to φ_1. From our work on uniform limits and differentiability, we know that this ensures that φ is differentiable with $\varphi'=\varphi_1$. Continuing in this way, we see that φ is C^∞ and that, for each $m\ge 0$, the sequence $\{\varphi_k^{(m)}\}$ of m^{th} derivatives converges uniformly to $\varphi^{(m)}$. Finally, we have that, for each $n, m \ge 0$, there is a $K_{n,m}$ such that $|x|^n \big|\varphi_k^{(m)}(x) - \varphi_{k'}^{(m)}(x)\big| < \varepsilon$ for all $k, k' \ge K_{n,m}$ and all $x \in \mathbb{R}$. Consequently, if $k \ge K_{n,m}$,

$$\begin{aligned}\big\|\varphi_k - \varphi\big\|_{n,m} &= \sup_{x\in\mathbb{R}} |x|^n \big|\varphi_k^{(m)}(x) - \varphi^{(m)}(x)\big| = \sup_{x\in\mathbb{R}} \lim_{k'\to\infty} |x|^n \big|\varphi_k^{(m)}(x) - \varphi_{k'}^{(m)}(x)\big| \\ &\le \sup_{x\in\mathbb{R}} \varepsilon = \varepsilon\end{aligned}$$

So, by part (c), $\{\varphi_k\}$ converges to φ with respect to the metric d. ■

Remark. In practice, it is rarely necessary to directly use the definition of the metric d of Theorem 3. One usually just uses part (c) of Theorem 3 instead.

We are now ready to give

Definition 4 (Tempered Distributions) The space of all tempered distributions on $\mathbb{R}$, denoted $\mathcal{S}'(\mathbb{R})$, is the dual space of $\mathcal{S}(\mathbb{R})$. That is, it is the set of all functions

$$f : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$$

that are linear and continuous. One usually denotes by $\langle f, \varphi\rangle$ the value in $\mathbb{C}$ that the distribution $f \in \mathcal{S}'(\mathbb{R})$ assigns to $\varphi \in \mathcal{S}(\mathbb{R})$. In this notation,

- ▷ that f is linear means that $\langle f, a\varphi + b\psi\rangle = a\langle f, \varphi\rangle + b\langle f, \psi\rangle$ for all $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ and all $a, b \in \mathbb{C}$.
- ▷ that f is continuous means that if $\varphi = \lim_{n\to\infty} \varphi_n$ in $\mathcal{S}(\mathbb{R})$, then $\langle f, \varphi\rangle = \lim_{n\to\infty} \langle f, \varphi_n\rangle$.

Example 5

(a) Here is the motivating example for the whole subject. Let $f : \mathbb{R} \to \mathbb{C}$ be any function that is polynomially bounded (that is, there is a polynomial $P(x)$ such that $|f(x)| \le P(x)$ for all $x \in \mathbb{R}$) and that is Riemann integrable on $[-M, M]$ for each $M > 0$. Then

$$f : \varphi \in \mathcal{S}(\mathbb{R}) \mapsto \langle f, \varphi\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\, dx$$

is a tempered distribution. The integral converges because every $\varphi \in \mathcal{S}(\mathbb{R})$ decays faster at infinity than one over any polynomial. See Problem 1, below. The linearity in φ of $\langle f, \varphi\rangle$ is obvious. The continuity in φ of $\langle f, \varphi\rangle$ follows easily from Problem 1 and Theorem 6, below.

(b) The Dirac delta function, and more generally the Dirac delta function translated to $b \in \mathbb{R}$, are defined as tempered distributions by

$$\langle \delta, \varphi \rangle = \varphi(0) \qquad \langle \delta_b, \varphi \rangle = \varphi(b)$$

Once again, the linearity in φ is obvious and the continuity in φ is easily verified if one applies Theorem 6.

(c) The derivative of the Dirac delta function δ_b is defined by

$$\langle \delta_b', \varphi \rangle = -\varphi'(b)$$

The reason for the name "derivative of the Dirac delta function" will be given in the section on differentiation, later.

(d) The principal value of $\frac{1}{x}$ is defined by

$$\left\langle P\tfrac{1}{x}, \varphi \right\rangle = \lim_{\varepsilon \to 0+} \int_{|x|>\varepsilon} \tfrac{\varphi(x)}{x}\, dx$$

The first thing that we have to do is verify that the limit above actually exists. This is not a trivial statement, because not only is $\frac{\varphi(x)}{x}$ not integrable on $[-1,1]$ if $\varphi(0) \neq 0$ (because then $\frac{\varphi(x)}{x}$ is not bounded), but $\int_0^1 \frac{1}{x} dx$ and $\int_{-1}^0 \frac{1}{x}\, dx$ do not even exist as improper integrals:

$$\begin{aligned} \int_0^1 \tfrac{1}{x}\, dx &= \lim_{\varepsilon \to 0+} \int_\varepsilon^1 \tfrac{1}{x}\, dx &&= \lim_{\varepsilon \to 0+} \ln \tfrac{1}{\varepsilon} = \infty \\ \int_{-1}^0 \tfrac{1}{x}\, dx &= \lim_{\varepsilon \to 0+} \int_{-1}^{-\varepsilon} \tfrac{1}{x}\, dx &&= \lim_{\varepsilon \to 0+} \ln \varepsilon = -\infty \end{aligned}$$

Here is the verification that the limit defining $\left\langle P\frac{1}{x}, \varphi \right\rangle$ exists

$$\begin{aligned} \lim_{\varepsilon \to 0+} \int_{|x|>\varepsilon} \tfrac{\varphi(x)}{x}\, dx &= \lim_{\substack{\varepsilon \to 0+ \\ M, M' \to \infty}} \left\{ \int_\varepsilon^M \tfrac{\varphi(x)}{x} dx + \int_{-M'}^{-\varepsilon} \tfrac{\varphi(x)}{x}\, dx \right\} \\ &= \lim_{\substack{\varepsilon \to 0+ \\ M, M' \to \infty}} \left\{ \int_\varepsilon^1 \tfrac{\varphi(x)}{x} dx + \int_1^M \tfrac{\varphi(x)}{x} dx + \int_{-M'}^{-1} \tfrac{\varphi(x)}{x}\, dx + \int_{-1}^{-\varepsilon} \tfrac{\varphi(x)}{x}\, dx \right\} \\ &= \lim_{\substack{\varepsilon \to 0+ \\ M, M' \to \infty}} \left\{ \int_\varepsilon^1 \tfrac{\varphi(x) - \varphi(-x)}{x}\, dx + \int_1^M \tfrac{\varphi(x)}{x}\, dx + \int_{-M'}^{-1} \tfrac{\varphi(x)}{x}\, dx \right\} \end{aligned}$$

The first integral converges because, by the mean value theorem, we have, for some ξ between x and $-x$,

$$\left| \tfrac{\varphi(x) - \varphi(-x)}{x} \right| = \left| \tfrac{\varphi'(\xi)\, 2x}{x} \right| \leq 2\|\varphi\|_{0,1}$$

The second and third integrals converge because, for $|x| \ge 1$

$$\left|\tfrac{\varphi(x)}{x}\right| \le \tfrac{1}{x^2}|x\varphi(x)| \le \tfrac{1}{x^2}\|\varphi\|_{1,0}$$

These bounds give both that $\left\langle P\frac{1}{x}, \varphi\right\rangle$ is well–defined and

$$\begin{aligned}\left|\left\langle P\tfrac{1}{x}, \varphi\right\rangle\right| &\le 2\|\varphi\|_{0,1}\int_0^1 dx + \|\varphi\|_{1,0}\int_1^\infty \tfrac{1}{x^2}\,dx + \|\varphi\|_{1,0}\int_{-\infty}^{-1} \tfrac{1}{x^2}\,dx \\ &= 2\|\varphi\|_{0,1} + 2\|\varphi\|_{1,0}\end{aligned}$$

Linearity is again obvious. Continuity again follows by Theorem 6, below.

Problem 1 Let $f : \mathbb{R} \to \mathbb{C}$ be Riemann integrable on $[-M, M]$ for all $M > 0$ and obey the bound $|f(x)| \le P(x)$ for all $\mathbf{x} \in \mathbb{R}$, where $P(x)$ is the polynomial $P(x) = \sum_{n=N_-}^{N_+} a_n x^n$ and $N_\pm$ are nonnegative integers.

(a) Prove that there is a constant $C > 0$ such that $|f(x)|(1 + x^2) \le C\big(|x|^{N_-} + |x|^{N_+ + 2}\big)$ for all $x \in \mathbb{R}$.

(b) Prove that

$$\int_{-\infty}^{\infty} |f(x)\varphi(x)|\,dx \le \pi C\big(\|\varphi\|_{N_-,0} + \|\varphi\|_{N_+ + 2,0}\big)$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$.

Theorem 6 (Continuity Test) *A linear map $f : \varphi \in \mathcal{S}(\mathbb{R}) \mapsto \langle f, \varphi\rangle \in \mathbb{C}$ is continuous if and only if there are constants $C > 0$ and $N \in \mathbb{N}$ such that*

$$\big|\langle f, \varphi\rangle\big| \le C \sum_{0 \le n,m \le N} \|\varphi\|_{n,m}$$

Proof: $\Leftarrow$: Assume that $\big|\langle f, \varphi\rangle\big| \le C\sum_{0\le n,m\le N}\|\varphi\|_{n,m}$ and that the sequence $\big\{\varphi_k\big\}_{k\in\mathbb{N}}$ converges to φ in $\mathcal{S}(\mathbb{R})$. Then

$$\big|\langle f, \varphi\rangle - \langle f, \varphi_k\rangle\big| = \big|\langle f, \varphi - \varphi_k\rangle\big| \le C \sum_{0\le n,m\le N} \|\varphi - \varphi_k\|_{n,m}$$

converges to zero as $k \to \infty$. So f is continuous.

$\Rightarrow$: Assume that $f \in \mathcal{S}'(\mathbb{R})$. In particular f is continuous at $\varphi = 0$. Then there is a $\delta > 0$ such that

$$d(\psi, 0) < \delta \implies \big|\langle f, \psi\rangle\big| < 1$$

Choose N so that $\sum\limits_{n \text{ or } m>N} 2^{-n-m} < \frac{\delta}{2}$. Then

$$\begin{aligned}\sum_{0\le n,m\le N} \|\psi\|_{n,m} \le \tfrac{\delta}{2} &\implies d(\psi,0) = \sum_{n,m\ge 0} 2^{-n-m}\tfrac{\|\psi\|_{n,m}}{1+\|\psi\|_{n,m}} \le \sum_{n,m\le N}\|\psi\|_{n,m} + \sum_{n \text{ or } m>N} 2^{-n-m}\\ &\implies d(\psi,0) < \delta\\ &\implies \big|\langle f,\psi\rangle\big| < 1\end{aligned}$$

Consequently, for any $0\ne\varphi\in\mathcal{S}(\mathbb{R})$, setting

$$\psi = \tfrac{\delta}{2}\Big[\sum_{n,m\le N}\|\varphi\|_{n,m}\Big]^{-1}\varphi$$

we have

$$\sum_{0\le n,m\le N}\|\psi\|_{n,m} = \sum_{0\le n,m\le N}\tfrac{\delta}{2}\Big[\sum_{n,m\le N}\|\varphi\|_{n,m}\Big]^{-1}\|\varphi\|_{n,m} = \tfrac{\delta}{2}$$

and hence

$$\big|\langle f,\varphi\rangle\big| = \tfrac{2}{\delta}\Big[\sum_{n,m\le N}\|\varphi\|_{n,m}\Big]\,\big|\langle f,\psi\rangle\big| < \tfrac{2}{\delta}\sum_{n,m\le N}\|\varphi\|_{n,m}$$

as desired. ■

Operations on Tempered Distributions

We now define a number of operations like, for example, addition and differentiation, on tempered distributions. The motivation for all of these definitions come from Example 5.a with $f\in\mathcal{S}(\mathbb{R})$. Then we can view f both as a conventional function and as a tempered distribution. We will define each operation in such a way that when it is applied to $f\in\mathcal{S}(\mathbb{R})$, viewed as a distribution, it yields the same answer as when the operation is applied to f viewed as an ordinary function, with the result viewed as a distribution. As a trivial example, suppose that we wish to define multiplication by 7. If $f\in\mathcal{S}(\mathbb{R})$ is viewed as an ordinary function, applying the operation of multiplication by 7 to it gives the ordinary function $7f$. But $7f$ can again be viewed as the distribution $\langle 7f,\varphi\rangle = \int 7f(x)\,\varphi(x)\,dx = 7\,\langle f,\varphi\rangle$. So we would define the operation of multiplication by 7 applied to any distribution f as the distribution $7f$ defined by $\langle 7f,\varphi\rangle = 7\,\langle f,\varphi\rangle$.

Addition and Scalar Multiplication

Motivation. If $f,g\in\mathcal{S}(\mathbb{R})$ and $a,b\in\mathbb{C}$, then

$$\int_{-\infty}^{\infty}\big[af(x)+bg(x)\big]\,\varphi(x)\,dx = a\int_{-\infty}^{\infty} f(x)\,\varphi(x)\,dx + b\int_{-\infty}^{\infty} g(x)\,\varphi(x)\,dx = a\,\langle f,\varphi\rangle + b\,\langle g,\varphi\rangle$$

Definition. If $f, g \in \mathcal{S}'(\mathbb{R})$ and $a, b \in \mathbb{C}$, then we define $af + bg \in \mathcal{S}'(\mathbb{R})$ by

$$\langle af + bg, \varphi \rangle = a \langle f, \varphi \rangle + b \langle g, \varphi \rangle$$

Theorem. *If $f, g \in \mathcal{S}'(\mathbb{R})$ and $a, b \in \mathbb{C}$, then $af + bg$, defined above, is a well–defined element of $\mathcal{S}'(\mathbb{R})$. The operations of addition and scalar multiplication so defined obey the usual vector space axioms.*

Proof: Trivial. ■

Differentiation

Motivation. If $f \in \mathcal{S}(\mathbb{R})$, then, by integration by parts,

$$\int_{-\infty}^{\infty} f'(x)\,\varphi(x)\ dx = -\int_{-\infty}^{\infty} f(x)\,\varphi'(x)\ dx \qquad \text{(the boundary terms vanish)}$$

Definition. We define the first derivative of $f \in \mathcal{S}'(\mathbb{R})$ by

$$\langle f', \varphi \rangle = -\langle f, \varphi' \rangle$$

More generally, we define the p^{th} derivative of $f \in \mathcal{S}'(\mathbb{R})$ by

$$\big\langle f^{(p)}, \varphi \big\rangle = (-1)^p \big\langle f, \varphi^{(p)} \big\rangle$$

Since $\|\varphi^{(p)}\|_{n,m} = \|\varphi\|_{n,m+p}$ the right hand side gives a well–defined element of $\mathcal{S}'(\mathbb{R})$.

Remark. Note that *every* derivative of *every* distribution *always* exists.

Example. The Heavyside unit function

$$H(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

may also be viewed as the tempered distribution

$$\langle H, \varphi \rangle = \int_0^{\infty} \varphi(x)\,dx$$

via Example 5.a. The derivative of this distribution is

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_0^{\infty} \varphi'(x)\,dx = -\big[\varphi(x)\big]_0^{\infty} = \varphi(0) = \langle \delta, \varphi \rangle$$

Thus H' is the Dirac delta function.

The Fourier Transform

Definition 7 The Fourier transform $\hat{f}(k)$ of a function $f \in \mathcal{S}(\mathrm{I\!R})$ is defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\ dx$$

Since $f(x)$, and hence $e^{-ikx}f(x)$, is a continuous function of x which is bounded by a contant times $\frac{1}{1+x^2}$, the integral exists and $\hat{f}(k)$ is a well–defined complex number for each $k \in \mathrm{I\!R}$. We shall show in Theorem 9, below that the map $f \mapsto \hat{f}$ is a continuous, linear map from $\mathcal{S}(\mathrm{I\!R})$ to $\mathcal{S}(\mathrm{I\!R})$ and furthermore that this map is one–to–one and onto with the inverse map being the inverse Fourier transform given by

$$\check{g}(x) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} g(k)\ dk$$

The computational properties of the Fourier transform are given in

Theorem 8 *Let $f, g \in \mathcal{S}(\mathrm{I\!R})$ and $\alpha, \beta \in \mathbb{C}$. Then*

(a) The Fourier transform of $af(x)+bg(x)$ is $a\hat{f}(k) + b\hat{g}(k)$.

(b) If $n \in \mathrm{I\!N}$, then the Fourier transform of $f^{(n)}(x)$ is $(ik)^n \hat{f}(k)$.

(c) The Fourier transform, $\hat{f}(k)$, of $f(x)$ is infinitely differentiable and, for each $n \in \mathrm{I\!N}$, $\frac{d^n}{dk^n}\hat{f}(k)$ is the Fourier transform of $(-ix)^n f(x)$.

(d) Let $a \in \mathrm{I\!R}$. The Fourier transform of the translated function $(T_a f)(x) = f(x-a)$ is $e^{-iak}\hat{f}(k)$.

(e) The Fourier transform of $f(x) = e^{-x^2/2}$ is $\hat{f}(k) = \sqrt{2\pi}e^{-k^2/2}$.

(f) $\int_{-\infty}^{\infty} f(x)\overline{g(x)}\ dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}\ dk$

Proof: (a) The Fourier transform of $af + bg$ is

$$\int_{-\infty}^{\infty} e^{-ikx}[af(x) + bg(x)]\ dx = a\int_{-\infty}^{\infty} e^{-ikx} f(x)\ dx + b\int_{-\infty}^{\infty} e^{-ikx} g(x)\ dx = a\hat{f}(k) + b\hat{g}(k)$$

(b) By induction, it suffices to prove the case $n = 1$. By integration by parts, the Fourier transform of the first derivative $f'(x)$ is

$$\int_{-\infty}^{\infty} e^{-ikx} f'(x)\ dx = -\int_{-\infty}^{\infty} f(x)\big(\tfrac{d}{dx}e^{-ikx}\big)\ dx = ik\int_{-\infty}^{\infty} e^{-ikx} f(x)\ dx = ik\hat{f}(k)$$

The boundary terms vanished because $\lim\limits_{x\to\infty} e^{-ikx}f(x) = \lim\limits_{x\to-\infty} e^{-ikx}f(x) = 0$.

(c) Again, by induction, it suffices to prove the case $n = 1$.

$$\tfrac{d}{dk}\hat{f}(k) = \tfrac{d}{dk}\int_{-\infty}^{\infty} e^{-ikx}f(x)\ dx = \int_{-\infty}^{\infty} \tfrac{\partial}{\partial k}\big(e^{-ikx}f(x)\big)\ dx = \int_{-\infty}^{\infty} (-ix)e^{-ikx}f(x)\ dx$$

is indeed $-i$ times the Fourier transform of $xf(x)$.

The second equality, which moved the derivative with respect to k past the integral sign is justified by the following minor variant of problem #4b in Problem Set 5.

Lemma. *Let $f : (-\infty,\infty)\times[c,d] \to \mathbb{C}$ be continuous. Assume that $\frac{\partial f}{\partial y}$ exists and is continuous and that there is a constant C such that*

$$|f(x,y)|, \big|\tfrac{\partial f}{\partial y}(x,y)\big| \le \tfrac{C}{1+x^2} \qquad \text{and} \qquad \big|\tfrac{\partial f}{\partial y}(x,y) - \tfrac{\partial f}{\partial y}(x,y')\big| \le C\tfrac{|y-y'|}{1+x^2}$$

for all $-\infty < x < \infty$ and $c \le y, y' \le d$. Then $g(y) = \int_{-\infty}^{\infty} f(x,y)\ dx$ is differentiable with $g'(y) = \int_{-\infty}^{\infty} \frac{\partial f}{\partial y}(x,y)\ dx$.

Proof: The assumptions that, for each $c \le y \le d$, $f(x,y)$, $\frac{\partial f}{\partial y}(x,y)$ are continuous and are bounded in absolute value by $\frac{C}{1+x^2}$ ensure that the integrals $\int_{-\infty}^{\infty} f(x,y)\ dx$ and $\int_{-\infty}^{\infty} \frac{\partial f}{\partial y}(x,y)\ dx$ exist. By the Mean Value Theorem (the usual MVT in one dimension), there is for each $x \in \mathbb{R}$ and each pair $y, y' \in [c,d]$ with $y \ne y'$, a number y'' between y and y' such that

$$\tfrac{f(x,y')-f(x,y)}{y'-y} = \tfrac{\partial f}{\partial y}(x,y'')$$

so that

$$\big|\tfrac{f(x,y')-f(x,y)}{y'-y} - \tfrac{\partial f}{\partial y}(x,y)\big| = \big|\tfrac{\partial f}{\partial y}(x,y'') - \tfrac{\partial f}{\partial y}(x,y)\big| \le C\tfrac{|y-y''|}{1+x^2} \le C\tfrac{|y-y'|}{1+x^2}$$

Consequently, if $y \ne y'$,

$$\begin{aligned}\left|\tfrac{g(y')-g(y)}{y'-y} - \int_{-\infty}^{\infty} \tfrac{\partial f}{\partial y}(x,y)\ dx\right| &= \left|\int_{-\infty}^{\infty} \big\{\tfrac{f(x,y')-f(x,y)}{y'-y} - \tfrac{\partial f}{\partial y}(x,y)\big\}\ dx\right| \\ &\le \int_{-\infty}^{\infty} C\tfrac{|y-y'|}{1+x^2}\ dx = \pi C|y-y'|\end{aligned}$$

This converges to zero as $y' \to y$ and so verifies the definition that $\lim\limits_{y'\to y} \frac{g(y')-g(y)}{y'-y}$ exists and equals $\int_{-\infty}^{\infty} \frac{\partial f}{\partial y}(x,y)\ dx$. ∎

(d) The Fourier transform of $T_a f$ is

$$\int_{-\infty}^{\infty} e^{-ikx} f(x-a)\ dx \overset{x'=x-a}{=} \int_{-\infty}^{\infty} e^{-ik(x'+a)} f(x')\ dx' = e^{-ika}\hat{f}(k)$$

(e) Denote by $\hat{f}(k)$ the Fourier transform of the function $f(x)=e^{-x^2/2}$. By part (c) of this Theorem, $\frac{d}{dk}\hat{f}(k)$ is the Fourier transform of $-ixf(x)=-ixe^{-x^2/2}=i\frac{d}{dx}e^{-x^2/2}=if'(x)$. Thus by parts (a) and (b) of this Theorem, $\frac{d}{dk}\hat{f}(k)=-k\hat{f}(k)$ and

$$\tfrac{d}{dk}\big(\hat{f}(k)e^{k^2/2}\big) = e^{k^2/2}\big(\tfrac{d}{dk}\hat{f}(k)+k\hat{f}(k)\big)=0$$

for all $k\in\mathbb{R}$. Consequently $\hat{f}(k)e^{k^2/2}$ must be some constant, independent of k. Hence to determine $\hat{f}(k)$ we need only to determine the value of that constant, which we may do by computing $\hat{f}(k)e^{k^2/2}\big|_{k=0}=\hat{f}(0)$. Since $\hat{f}(0)=\int_{-\infty}^{\infty} e^{-x^2/2}\ dx>0$, it is determined by

$$\hat{f}(0)^2 = \left[\int_{-\infty}^{\infty} e^{-x^2/2}\ dx\right]^2 = \left[\int_{-\infty}^{\infty} e^{-x^2/2}\ dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2/2}\ dx\right] = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}dxdy$$

Changing to polar coordinates,

$$\hat{f}(0)^2 = \int_0^{\infty} dr\ r\int_0^{2\pi} d\theta\ e^{-r^2/2} = 2\pi\int_0^{\infty} dr\ re^{-r^2/2} = 2\pi\Big[-e^{-r^2/2}\Big]_0^{\infty} = 2\pi$$

Thus $\hat{f}(0)=\sqrt{2\pi}$ which tells us that $\hat{f}(k)e^{k^2/2}=\sqrt{2\pi}$ and hence that $\hat{f}(k)=\sqrt{2\pi}e^{-k^2/2}$ for all k.

(f)

$$\begin{aligned}\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)\overline{\hat{g}(k)}\ dk &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dx\ e^{-ikx}f(x)\overline{\hat{g}(k)} = \tfrac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dk\ e^{-ikx}f(x)\overline{\hat{g}(k)}\\ &= \int_{-\infty}^{\infty}dx\ f(x)\left[\overline{\tfrac{1}{2\pi}\int_{-\infty}^{\infty}dk\ e^{ikx}\hat{g}(k)}\right] = \int_{-\infty}^{\infty}dx\ f(x)\overline{g(x)}\end{aligned}$$

The exchange of order of integration executed in the second inequality is justified using question #5 of Problem Set 5. The last equality uses Theorem 9, below. ■

Theorem 9 *The maps*

$$f(x)\in\mathcal{S}(\mathbb{R})\mapsto\hat{f}(k)=\int_{-\infty}^{\infty}e^{-ikx}f(x)\ dx$$

$$g(k)\in\mathcal{S}(\mathbb{R})\mapsto\check{g}(x)=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}g(k)\ dk$$

are one–to–one, continuous, linear maps from $\mathcal{S}(\mathbb{R})$ *onto* $\mathcal{S}(\mathbb{R})$ *and are inverses of each other.*

Proof: That $\hat{f}$ is linear in f was Theorem 8.a.

We now assume that $f \in \mathcal{S}(\mathrm{I\!R})$ and prove that $\hat{f}(k) \in \mathcal{S}(\mathrm{I\!R})$. Let m, n be nonnegative integers. By parts (b) and (c) of Theorem 8 followed by the product rule, $k^n \frac{d^m}{dk^m}\hat{f}(k)$ is the Fourier transform of

$$\begin{aligned}
(-i)^n \tfrac{d^n}{dx^n}\big((-ix)^m f(x)\big) &= (-i)^{m+n} \sum_{\ell=0}^{\min\{m,n\}} \tbinom{n}{\ell}\big(\tfrac{d^\ell}{dx^\ell}x^m\big)\big(\tfrac{d^{n-\ell}}{dx^{n-\ell}}f(x)\big) \\
&= (-i)^{m+n} \sum_{\ell=0}^{\min\{m,n\}} \tbinom{n}{\ell}\tfrac{m!}{(m-\ell)!}x^{m-\ell}f^{(n-\ell)}(x)
\end{aligned}$$

Hence

$$\begin{aligned}
\|\hat{f}(k)\|_{n,m} &= \sup_{k\in\mathrm{I\!R}} \big|k^n \tfrac{d^m}{dk^m}\hat{f}(k)\big| \\
&= \sup_{k\in\mathrm{I\!R}} \Big|\int_{-\infty}^{\infty} e^{-ikx}\Big[\sum_{\ell=0}^{\min\{m,n\}} \tbinom{n}{\ell}\tfrac{m!}{(m-\ell)!}x^{m-\ell}f^{(n-\ell)}(x)\Big]dx\Big| \\
&\le \sum_{\ell=0}^{\min\{m,n\}} \tbinom{n}{\ell}\tfrac{m!}{(m-\ell)!}\int_{-\infty}^{\infty}\big|x^{m-\ell}f^{(n-\ell)}(x)\big|dx \\
&= \sum_{\ell=0}^{\min\{m,n\}} \tbinom{n}{\ell}\tfrac{m!}{(m-\ell)!}\int_{-\infty}^{\infty}\tfrac{1}{1+x^2}\big\{|x|^{m-\ell}+|x|^{m-\ell+2}\big\}\big|f^{(n-\ell)}(x)\big|dx \\
&\le \sum_{\ell=0}^{\min\{m,n\}} \tbinom{n}{\ell}\tfrac{m!}{(m-\ell)!}\big\{\|f\|_{m-\ell,n-\ell}+\|f\|_{m-\ell+2,n-\ell}\big\}\int_{-\infty}^{\infty}\tfrac{1}{1+x^2}\ dx \\
&= \sum_{\ell=0}^{\min\{m,n\}} \pi\tbinom{n}{\ell}\tfrac{m!}{(m-\ell)!}\big\{\|f\|_{m-\ell,n-\ell}+\|f\|_{m-\ell+2,n-\ell}\big\}
\end{aligned}$$

Since $f \in \mathcal{S}(\mathrm{I\!R})$, the right hand side is finite. This proves that $\|\hat{f}\|_{m,n}$ is finite for all nonnegative integers m, n, so that $\hat{f} \in \mathcal{S}(\mathrm{I\!R})$.

It also proves that the map $f \mapsto \hat{f}$ is continuous, since if the sequence $\{f_j\}_{j\in\mathrm{I\!N}}$ converges to f in $\mathcal{S}(\mathrm{I\!R})$, then

$$\|\hat{f}-\hat{f}_j\|_{m,n} \le \sum_{\ell=0}^{\min\{m,n\}} \pi\tbinom{n}{\ell}\tfrac{m!}{(m-\ell)!}\big\{\|f-f_j\|_{m-\ell,n-\ell}+\|f-f_j\|_{m-\ell+2,n-\ell}\big\}$$

converges to zero as $j \to \infty$, for all nonnegative integers m, n. So $\{\hat{f}_j\}_{j\in\mathrm{I\!N}}$ converges to $\hat{f}$ in $\mathcal{S}(\mathrm{I\!R})$ too.

The proof that the map $g(k) \mapsto \check g(x)$ is a continuous, linear map from $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$ is similar.

We now assume that $f(x) \in \mathcal{S}(\mathbb{R})$ and prove that the inverse Fourier transform of $\hat f(k)$ is $f(x)$. In symbols, that

$$f(x) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat f(k)\ dk \tag{3}$$

We first prove the $(x = 0)$ special case that

$$f(0) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat f(k)\ dk \tag{4}$$

Write

$$f(x) = f(0)e^{-x^2/2} + xh(x) \qquad \text{where } h(x) = \begin{cases} \frac{1}{x}\big(f(x) - f(0)e^{-x^2/2}\big) & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases}$$

By Problem 2, below, the function $h \in \mathcal{S}(\mathbb{R})$. So, by parts (e) and (c) of Theorem 8,

$$\hat f(k) = \sqrt{2\pi} f(0)e^{-k^2/2} + i\tfrac{d}{dk}\hat h(k)$$

and

$$\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\hat f(k)\ dk = \tfrac{f(0)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k^2/2}\ dk + \tfrac{i}{2\pi}\int_{-\infty}^{\infty}\tfrac{d}{dk}\hat h(k)\ dk$$

The first term

$$\tfrac{f(0)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k^2/2}\ dk = f(0)$$

by the computation at the end of the proof of Theorem 8.e. The second term is $\frac{i}{2\pi}$ times

$$\int_{-\infty}^{\infty}\tfrac{d}{dk}\hat h(k)\ dk = \lim_{A,B\to\infty}\int_{-A}^{B}\tfrac{d}{dk}\hat h(k)\ dk = \lim_{A,B\to\infty}\big[\hat h(B) - \hat h(-A)\big] = 0$$

Here we have used the fundamental theorem of calculus and the decay at $\pm\infty$ which follows from the fact that $\hat h \in \mathcal{S}(\mathbb{R})$, which, in turn, follows from $h \in \mathcal{S}(\mathbb{R})$. This completes the proof of (4). Replacing f by $T_{-x}f$ and using $f(x) = \big(T_{-x}f\big)(0)$ and $\widehat{T_{-x}f}(k) = e^{ikx}\hat f(k)$ gives (3).

The proof that

$$g(k) = \int_{-\infty}^{\infty} e^{-ikx}\check g(x)\ dx \tag{5}$$

is similar. The formulae (3) and (5) show that the maps $f(x) \mapsto \hat f(k)$ and $g(k) \mapsto \check g(x)$ are onto $\mathcal{S}(\mathbb{R})$ and are inverses of each other. ■

Problem 2 Let $f \in \mathcal{S}(\mathbb{R})$ and define

$$h(x) = \begin{cases} \frac{1}{x}\big(f(x) - f(0)e^{-x^2/2}\big) & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases}$$

Prove that $h \in \mathcal{S}(\mathbb{R})$.

Definition 10 We define the Fourier transform of the tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ to be the tempered distribution

$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle$$

The motivation for this definition is the computation that, if f and φ are both in $\mathcal{S}(\mathbb{R})$, then, writing $\varphi(k) = \overline{\psi(k)}$

$$\begin{aligned} \left\langle \hat{f}, \varphi \right\rangle &= \int_{-\infty}^{\infty} \hat{f}(k)\varphi(k)\ dk = \int_{-\infty}^{\infty} \hat{f}(k)\overline{\psi(k)}\ dk \\ &= 2\pi \int_{-\infty}^{\infty} f(x)\overline{\check{\psi}(x)}\ dx \qquad \text{(by Theorem 8.e and Theorem 9)} \\ &= \int_{-\infty}^{\infty} f(x)\hat{\varphi}(x)\ dx \end{aligned}$$

since

$$\overline{\check{\psi}(x)} = \tfrac{1}{2\pi}\overline{\int_{-\infty}^{\infty} e^{ikx}\psi(k)\ dk} = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\overline{\psi(k)}\ dk = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\varphi(k)\ dk = \tfrac{1}{2\pi}\hat{\varphi}(x)$$

Example 11 The Fourier transform of the Dirac delta function is given by

$$\left\langle \hat{\delta}, \varphi \right\rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int_{-\infty}^{\infty} \varphi(x)\ dx = \langle 1, \varphi \rangle$$

That is, $\hat{\delta}$ is the constant function 1.

Example 12 The Fourier transform of the constant function 1, viewed as a tempered distribution, is

$$\left\langle \hat{1}, \varphi \right\rangle = \langle 1, \hat{\varphi} \rangle = \int_{-\infty}^{\infty} \hat{\varphi}(k)\ dk = 2\pi\varphi(0)$$

by (4). That is, the Fourier transform of the constant function 1 is $2\pi\delta(k)$.

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