The Dirichlet Test

Theorem (The Dirichlet Test) *Let X be a metric space. If the functions $f_n : X \to \mathbb{C}$, $g_n : X \to \mathbb{R}$, $n \in \mathbb{N}$ obey*

- $F_n(x) = \sum_{m=1}^{n} f_m(x)$ *is bounded uniformly in n and x*
- $g_{n+1}(x) \le g_n(x)$ *for all $x \in X$ and $n \in \mathbb{N}$*
- $\big\{g_n(x)\big\}_{n\in\mathbb{N}}$ *converges uniformly to zero on X*

then $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ *converges uniformly on X.*

Proof: The trick to this proof is the summation by parts formula, which we now derive.

$$\begin{aligned}
s_n(x) &= \sum_{k=1}^{n} f_k(x)g_k(x) \\
&= F_1(x)g_1(x) + \sum_{k=2}^{n} [F_k(x) - F_{k-1}(x)]g_k(x) \\
&= F_1(x)g_1(x) + \sum_{k=2}^{n} F_k(x)g_k(x) - \sum_{k=2}^{n} F_{k-1}(x)g_k(x) \\
&= \sum_{k=1}^{n} F_k(x)g_k(x) - \sum_{k=1}^{n-1} F_k(x)g_{k+1}(x) \\
&= \sum_{k=1}^{n} F_k(x)\big[g_k(x) - g_{k+1}(x)\big] + g_{n+1}(x)F_n(x)
\end{aligned}$$

So if $m > n$, the difference between the m^{th} and n^{th} partial sums is

$$s_m(x) - s_n(x) = \sum_{k=n+1}^{m} F_k(x)\big[g_k(x) - g_{k+1}(x)\big] + g_{m+1}(x)F_m(x) - g_{n+1}(x)F_n(x)$$

If $M = \sup\big\{\ |F_n(x)|\ \big|\ x \in X,\ n \in \mathbb{N}\ \big\}$,

$$\begin{aligned}
\big|s_m(x) - s_n(x)\big| &\le M \sum_{k=n+1}^{m} \big[g_k(x) - g_{k+1}(x)\big] + Mg_{m+1}(x) + Mg_{n+1}(x) \\
&= M\big[g_{n+1}(x) - g_{m+1}(x)\big] + Mg_{m+1}(x) + Mg_{n+1}(x) \\
&= 2Mg_{n+1}(x)
\end{aligned} \tag{1}$$

since $g_{m+1}(x) \ge 0$, $g_{n+1}(x) \ge 0$ and every $g_k(x) - g_{k+1}(x) \ge 0$. For each fixed x, $\lim_{n\to\infty} g_{n+1}(x) = 0$. So (1) guarantees that $\{s_n(x)\}$ is a Cauchy sequence and hence converges. Call the limit $s(x)$. Taking the limit of (1) as $m \to \infty$ gives

$$\big|s(x) - s_n(x)\big| \le 2Mg_{n+1}(x)$$

Since $g_{n+1}(x)$ converges uniformly to zero as $n \to \infty$, we have that $s_n(x)$ converges uniformly to $s(x)$ as $n \to \infty$.

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Example. We shall consider three different power series: $\sum_{n=0}^{\infty} \left(\frac{z}{R}\right)^n$, $\sum_{n=0}^{\infty} \frac{1}{n}\left(\frac{z}{R}\right)^n$ and $\sum_{n=0}^{\infty} \frac{1}{n^2}\left(\frac{z}{R}\right)^n$, for some fixed $R > 0$. For all three series, the radius of convergence is exactly R since, for $\ell \in \{0, 1, 2\}$,

$$\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n^\ell}\frac{1}{R^n}} = \frac{1}{R} \limsup_{n\to\infty} \left(\sqrt[n]{\frac{1}{n}}\right)^\ell = \frac{1}{R}$$

So all three series converge for all complex numbers z with $|z| < R$ and diverge for all complex numbers with $|z| > R$. What if $|z| = R$?

We'll start with the series $\sum_{n=0}^{\infty} \left(\frac{z}{R}\right)^n$. Then we can compute exactly the partial sum

$$F_n(z) = \sum_{m=0}^{n} \left(\frac{z}{R}\right)^m = \begin{cases} \dfrac{1 - \left(\frac{z}{R}\right)^{n+1}}{1 - \frac{z}{R}} & \text{if } z \neq R \\ n+1 & \text{if } z = R \end{cases} \tag{2}$$

As expected, if $|z| < R$ this converges to $\frac{1}{1-\frac{z}{R}}$ as $n \to \infty$. Also as expected, this diverges for $|z| > R$, because $\left|\left(\frac{z}{R}\right)^{n+1}\right| = \left|\frac{z}{R}\right|^{n+1} \to \infty$. I claim that this also diverges whenever $|z| = R$. For $z = R$, it is obvious because $n + 1 \to \infty$. For $|z| = R$ with $z \neq R$, $\left(\frac{z}{R}\right)^{n+1}$ does not blow up as $n \to \infty$, but it cannot converge either, because

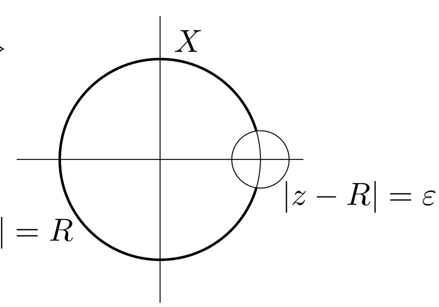
$$\left|\left(\frac{z}{R}\right)^{n+2} - \left(\frac{z}{R}\right)^{n+1}\right| = \left|\frac{z}{R}\right|^{n+1} \left|\frac{z}{R} - 1\right| = \left|\frac{z}{R} - 1\right|$$

is independent of n. So the geometric series $\sum_{n=0}^{\infty} \left(\frac{z}{R}\right)^n$, which has radius of convergence R, converges if and only if $|z| < R$.

The third series, $\sum_{n=0}^{\infty} \frac{1}{n^2}\left(\frac{z}{R}\right)^n$, converges for all $|z| \le R$, by comparison with $\sum_{n=0}^{\infty} \frac{1}{n^2}$. As the series has radius of convergence R, it converges if and only if $|z| \le R$.

The middle series $\sum_{n=0}^{\infty} \frac{1}{n}\left(\frac{z}{R}\right)^n$ has a more interesting domain of convergence. Of course the radius of convergence is exactly R, so the series converges for all complex numbers z with $|z| < R$ and diverges for all complex numbers with $|z| > R$. What if $|z| = R$? Well, if $z = R$, then the series is $\sum_{n=0}^{\infty} \frac{1}{n}\left(\frac{z}{R}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n}$ which diverges. So that leaves $|z| = R$ but with $z \neq R$. This is where the Dirichlet test comes in handy. Fix any $\varepsilon > 0$ and set

$$X = \left\{ z \in \mathbb{C} \;\middle|\; |z| = R,\ |z - R| \ge \varepsilon \right\}$$
$$f_n(z) = \left(\frac{z}{R}\right)^n$$
$$F_n(z) = \sum_{m=0}^{n} \left(\frac{z}{R}\right)^m \text{ as in (2)}$$
$$g_n = \frac{1}{n}$$



For $z \in X$

$$\left|F_n(z)\right| = \left|\frac{1-\left(\frac{z}{R}\right)^{n+1}}{1-\frac{z}{R}}\right| \le \frac{1+\left|\frac{z}{R}\right|^{n+1}}{\frac{1}{R}|R-z|} \le \frac{2R}{\varepsilon}$$

so that the hypotheses of the Dirichlet test are satisfied and the series converges uniformly on X. We conclude that $\sum_{n=0}^{\infty} \frac{1}{n}\left(\frac{z}{R}\right)^n$ converges for $|z| < R$ and for $|z| = R,\ z \ne R$ and diverges for $|z| > R$ and for $z = R$.