The Dirichlet Test

**Theorem (The Dirichlet Test)** Let $X$ be a metric space. If the functions $f_n : X \to \mathbb{C}$, $g_n : X \to \mathbb{R}$, $n \in \mathbb{N}$ obey

- $F_n(x) = \sum_{m=1}^{n} f_m(x)$ is bounded uniformly in $n$ and $x$
- $g_{n+1}(x) \leq g_n(x)$ for all $x \in X$ and $n \in \mathbb{N}$
- $\{g_n(x)\}_{n \in \mathbb{N}}$ converges uniformly to zero on $X$

then $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ converges uniformly on $X$.

**Proof:** The trick to this proof is the summation by parts formula, which we now derive.

$$s_n(x) = \sum_{k=1}^{n} f_k(x)g_k(x)$$

$$= F_1(x)g_1(x) + \sum_{k=2}^{n} [F_k(x) - F_{k-1}(x)]g_k(x)$$

$$= F_1(x)g_1(x) + \sum_{k=2}^{n} F_k(x)g_k(x) - \sum_{k=2}^{n} F_{k-1}(x)g_k(x)$$

$$= \sum_{k=1}^{n} F_k(x)g_k(x) - \sum_{k=1}^{n-1} F_k(x)g_{k+1}(x)$$

$$= \sum_{k=1}^{n} F_k(x)[g_k(x) - g_{k+1}(x)] + g_{n+1}(x)F_n(x)$$

So if $m > n$, the difference between the $m^{th}$ and $n^{th}$ partial sums is

$$s_m(x) - s_n(x) = \sum_{k=n+1}^{m} F_k(x)[g_k(x) - g_{k+1}(x)] + g_{m+1}(x)F_m(x) - g_{n+1}(x)F_n(x)$$

If $M = \sup \{ |F_n(x)| \mid x \in X, \ n \in \mathbb{N} \}$,

$$|s_m(x) - s_n(x)| \leq M \sum_{k=n+1}^{m} [g_k(x) - g_{k+1}(x)] + Mg_{m+1}(x) + Mg_{n+1}(x)$$

$$= M[g_{n+1}(x) - g_{m+1}(x)] + Mg_{m+1}(x) + Mg_{n+1}(x)$$

$$= 2Mg_{n+1}(x) \quad (1)$$

since $g_{m+1}(x) \geq 0$, $g_{n+1}(x) \geq 0$ and every $g_k(x) - g_{k+1}(x) \geq 0$. For each fixed $x$, $\lim_{n \to \infty} g_{n+1}(x) = 0$. So (1) guarantees that $\{s_n(x)\}$ is a Cauchy sequence and hence converges. Call the limit $s(x)$. Taking the limit of (1) as $m \to \infty$ gives

$$|s(x) - s_n(x)| \leq 2Mg_{n+1}(x)$$
Since $g_{n+1}(x)$ converges uniformly to zero as $n \to \infty$, we have that $s_n(x)$ converges uniformly to $s(x)$ as $n \to \infty$.

**Example.** We shall consider three different power series: $\sum_{n=0}^{\infty} \left( \frac{z}{R} \right)^n$, $\sum_{n=0}^{\infty} \frac{1}{n} \left( \frac{z}{R} \right)^n$ and $\sum_{n=0}^{\infty} \frac{1}{n^2} \left( \frac{z}{R} \right)^n$, for some fixed $R > 0$. For all three series, the radius of convergence is exactly $R$ since, for $\ell \in \{0, 1, 2\}$,

$$\limsup_{n \to \infty} \frac{1}{n^\ell} = \frac{1}{R} \limsup_{n \to \infty} \left( \frac{1}{n} \right)^\ell = \frac{1}{R}$$

So all three series converge for all complex numbers $z$ with $|z| < R$ and diverge for all complex numbers with $|z| > R$. What if $|z| = R$?

We’ll start with the series $\sum_{n=0}^{\infty} \left( \frac{z}{R} \right)^n$. Then we can compute exactly the partial sum

$$F_n(z) = \sum_{m=0}^{n} \left( \frac{z}{R} \right)^m = \begin{cases} \frac{1 - \left( \frac{z}{R} \right)^{n+1}}{1 - \frac{z}{R}} & \text{if } z \neq R \\ n + 1 & \text{if } z = R \end{cases} \tag{2}$$

As expected, if $|z| < R$ this converges to $\frac{1}{1 - \frac{z}{R}}$ as $n \to \infty$. Also as expected, this diverges for $|z| > R$, because $\left| \left( \frac{z}{R} \right)^{n+1} \right| = \left| \frac{z}{R} \right|^{n+1} \to \infty$. I claim that this also diverges whenever $|z| = R$. For $z = R$, it is obvious because $n + 1 \to \infty$. For $|z| = R$ with $z \neq R$, $\left( \frac{z}{R} \right)^{n+1}$ does not blow up as $n \to \infty$, but it cannot converge either, because

$$\left| \left( \frac{z}{R} \right)^{n+2} - \left( \frac{z}{R} \right)^{n+1} \right| = \left| \frac{z}{R} \right|^{n+1} \left| \frac{z}{R} - 1 \right| = \left| \frac{z}{R} - 1 \right|$$

is independent of $n$. So the geometric series $\sum_{n=0}^{\infty} \left( \frac{z}{R} \right)^n$, which has radius of convergence $R$, converges if and only if $|z| < R$.

The third series, $\sum_{n=0}^{\infty} \frac{1}{n^2} \left( \frac{z}{R} \right)^n$, converges for all $|z| \leq R$, by comparison with $\sum_{n=0}^{\infty} \frac{1}{n^2}$. As the series has radius of convergence $R$, it converges if and only if $|z| \leq R$.

The middle series $\sum_{n=0}^{\infty} \frac{1}{n} \left( \frac{z}{R} \right)^n$ has a more interesting domain of convergence. Of course the radius of convergence is exactly $R$, so the series converges for all complex numbers $z$ with $|z| < R$ and diverges for all complex numbers with $|z| > R$. What if $|z| = R$? Well, if $z = R$, then the series is $\sum_{n=0}^{\infty} \frac{1}{n} \left( \frac{z}{R} \right)^n = \sum_{n=0}^{\infty} \frac{1}{n}$ which diverges. So that leaves $|z| = R$ but with $z \neq R$. This is where the Dirichlet test comes in handy. Fix any $\varepsilon > 0$ and set

$$X = \{ z \in \mathbb{C} \mid |z| = R, |z - R| \geq \varepsilon \}$$

$$f_n(z) = \left( \frac{z}{R} \right)^n$$

$$F_n(z) = \sum_{m=0}^{n} \left( \frac{z}{R} \right)^m \text{ as in (2)}$$

$$g_n = \frac{1}{n}$$

\[\text{Diagram:} \quad X \quad \varepsilon \quad |z - R| = \varepsilon \quad |z| = R \]
For $z \in X$

$$|F_n(z)| = \left| \frac{1 - \left(\frac{z}{R}\right)^{n+1}}{1 - \frac{z}{R}} \right| \leq \frac{1 + \left| \frac{z}{R} \right|^{n+1}}{\left| R - z \right|} \leq \frac{2R}{\varepsilon}$$

so that the hypotheses of the Dirichlet test are satisfied and the series converges uniformly on $X$. We conclude that $\sum_{n=0}^{\infty} \frac{1}{n} \left(\frac{z}{R}\right)^n$ converges for $|z| < R$ and for $|z| = R$, $z \neq R$ and diverges for $|z| > R$ and for $z = R$. 