

Dini's Theorem

Theorem (Dini's Theorem) *Let K be a compact metric space. Let $f : K \rightarrow \mathbb{R}$ be a continuous function and $f_n : K \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of continuous functions. If $\{f_n\}_{n \in \mathbb{N}}$ converges pointwise to f and if*

$$f_n(x) \geq f_{n+1}(x) \quad \text{for all } x \in K \text{ and all } n \in \mathbb{N}$$

then $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly to f .

Proof: Set, for each $n \in \mathbb{N}$, $g_n(x) = f_n(x) - f(x)$. Then $\{g_n\}_{n \in \mathbb{N}}$ is a sequence of continuous functions on the compact metric space K that converges pointwise to 0. Furthermore $g_n(x) \geq g_{n+1}(x) \geq 0$ for all $x \in K$ and $n \in \mathbb{N}$. Set $M_n = \sup \{g_n(x) \mid x \in K\}$. We must prove that $\lim_{n \rightarrow \infty} M_n = 0$.

We use the finite open subcover property of K . Let $\varepsilon > 0$ and define $\mathcal{O}_n = g_n^{-1}((-\infty, \varepsilon))$. Since g_n is continuous, this is an open set. Since $g_n(x) \geq g_{n+1}(x)$, $\mathcal{O}_n \subset \mathcal{O}_{n+1}$. For each $x \in K$, $\lim_{n \rightarrow \infty} g_n(x) = 0$ so that there is an $n \in \mathbb{N}$ with $g_n(x) < \varepsilon$ so that $x \in \mathcal{O}_n$. Thus $\bigcup_{n=1}^{\infty} \mathcal{O}_n = K$. Since K is compact, there is a finite collection of \mathcal{O}_n 's that also covers K . Since $\mathcal{O}_n \subset \mathcal{O}_{n+1}$, the \mathcal{O}_n in that finite collection with the largest index covers K . Thus there is an $N \in \mathbb{N}$ with $\mathcal{O}_N = K$. That is $g_N(x) < \varepsilon$ for all $x \in K$. Thus $M_N \leq \varepsilon$. Since M_n decreases with n and every $M_n \geq 0$, this forces $\lim_{n \rightarrow \infty} M_n = 0$. ■

Here are examples that show that the hypotheses

- (a) K is compact
- (b) f is continuous
- (c) $f_n(x)$ decreases as n increases

are each necessary.

Example (a) The set $K = (0, 1)$ is not compact. As $n \rightarrow \infty$, the sequence $f_n(x) = x^n$ of continuous functions decreases pointwise to zero. But the convergence is not uniform because

$$\sup \{ x^n \mid 0 < x < 1 \} = 1$$

for all $n \in \mathbb{N}$.

Example (a') The set $K = \mathbb{R}$ is not compact. As $n \rightarrow \infty$, the sequence

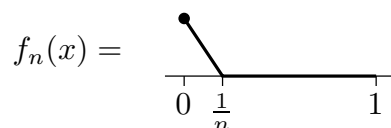


of continuous functions decreases pointwise to zero. But the convergence is not uniform because

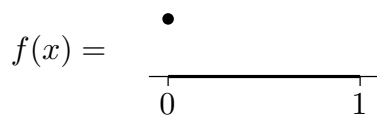
$$\sup \{ f_n(x) \mid x \in \mathbb{R} \} = 1$$

for all $n \in \mathbb{N}$.

Example (b) The set $K = [0, 1]$ is compact. As $n \rightarrow \infty$, the sequence



of continuous functions (the dot is at height 1) decreases pointwise to the discontinuous function

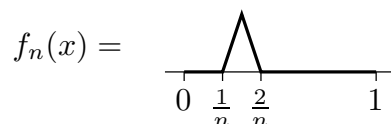


But the convergence is not uniform because

$$\sup \{ f_n(x) - f(x) \mid x \in [0, 1] \} = \sup \left(\begin{array}{c} \text{graph of } f_n(x) - f(x) \\ \text{on } [0, 1] \end{array} \right) = 1$$

for all $n \in \mathbb{N}$.

Example (c) The set $K = [0, 1]$ is compact. As $n \rightarrow \infty$, the sequence



of continuous functions (the triangle has height 1) converges pointwise to zero. But the convergence is not uniform because

$$\sup \{ f_n(x) \mid x \in \mathbb{R} \} = 1$$

for all $n \in \mathbb{N}$.