The Change of Variables $x = g(y)$

Theorem. *Let $a < b$ and $c < d$. Let $g : [c, d] \to [a, b]$ be continuous, strictly monotonic and obey $g(c) = a$ and $g(d) = b$. Let $f, \alpha : [a, b] \to \mathbb{R}$. Set*

$$h(y) = f\big(g(y)\big) \qquad \beta(y) = \alpha\big(g(y)\big)$$

If $f \in \mathcal{R}(\alpha)$ on $[a, b]$, then $h \in \mathcal{R}(\beta)$ on $[c, d]$ and

$$\int_a^b f(x)\, d\alpha(x) = \int_c^d h(y)\, d\beta(y)$$

Proof: For any partition $P = \{c = y_0, \cdots, y_n = d\}$ of $[c, d]$ and any choice $T = \{s_1, \cdots, s_n\}$ for P

$$\begin{aligned}
\Big|S(P,T,h,\beta) - \int_a^b f\, d\alpha\Big| &= \Big|\sum_{i=1}^n h(s_i)\big[\beta(y_i) - \beta(y_{i-1})\big] - \int_a^b f\, d\alpha\Big| \\
&= \Big|\sum_{i=1}^n f\big(g(s_i)\big)\big[\alpha\big(g(y_i)\big) - \alpha\big(g(y_{i-1})\big)\big] - \int_a^b f\, d\alpha\Big| \\
&= \Big|S(g(P), g(T), f, \alpha) - \int_a^b f\, d\alpha\Big|
\end{aligned}$$

where

$$\begin{aligned}
g(P) &= \big\{\ g(y)\ \big|\ y \in P\ \big\} \\
&= \big\{g(y_0) = g(c) = \overbrace{a}^{x_0},\ \overbrace{g(y_1)}^{x_1}, \cdots,\ g(y_n) = g(d) = \overbrace{b}^{x_n}\big\}
\end{aligned}$$

is a partition of $[a, b]$ because g is assumed to be strictly monotone (so that $y_{i-1} < y_i \implies x_{i-1} < x_i$) and is assumed to obey $x_0 = g(y_0) = a$ and $x_n = g(y_n) = b$ and

$$g(T) = \big\{\overbrace{g(s_1)}^{t_1}, \cdots, \overbrace{g(s_n)}^{t_n}\big\}$$

is a choice for $g(P)$ because g is assumed to be strictly monotone (so $y_{i-1} \le s_i \le y_i \implies x_{i-1} = g(y_{i-1}) \le g(s_i) = t_i \le g(y_i) = x_i$).

Now let $\varepsilon > 0$. We have assumed that $f \in \mathcal{R}(\alpha)$ on $[a, b]$, so there is a partition P'_ε of $[a.b]$ such that $P' \supset P'_\varepsilon \implies \big|S(P', T', f, \alpha) - \int_a^b f\, d\alpha\big| < \varepsilon$ for all choices T' for P'. The assumptions that we have made on g guarantee that the inverse function $g^{-1} : [a, b] \to [c, d]$ exists and that $g^{-1}\big(P'_\varepsilon\big)$ is a partition of $[c, d]$. We choose $P_\varepsilon = g^{-1}\big(P'_\varepsilon\big)$. Then

$$\begin{aligned}
P \supset P_\varepsilon &\implies P' = g(P) \supset g(P_\varepsilon) = P'_\varepsilon \\
&\implies \Big|S(P,T,h,\beta) - \int_a^b f\, d\alpha\Big| = \Big|S(\overbrace{g(P)}^{P'}, \overbrace{g(T)}^{T'}, f, \alpha) - \int_a^b f\, d\alpha\Big| < \varepsilon
\end{aligned}$$

as desired. ■