The Contraction Mapping Theorem and the Implicit Function Theorem

Theorem (The Contraction Mapping Theorem) *Let* $B_a = \{\ \vec{x} \in \mathbb{R}^d \ \big|\ \|\vec{x}\| < a\ \}$ *denote the open ball of radius* a *centred on the origin in* $\mathbb{R}^d$. *If the function*

$$\vec{g} : B_a \to \mathbb{R}^d$$

obeys

(H1) *there is a constant* $G < 1$ *such that* $\quad \|\vec{g}(\vec{x}) - \vec{g}(\vec{y})\| \le G\,\|\vec{x} - \vec{y}\| \quad$ *for all* $\vec{x}, \vec{y} \in B_a$

(H2) $\|\vec{g}(\vec{0})\| < (1 - G)a$

then the equation

$$\vec{x} = \vec{g}(\vec{x})$$

has exactly one solution.

Discussion of hypothesis (H1): Hypothesis (H1) is responsible for the word "Contraction" in the name of the theorem. Because $G < 1$ (and it is crucial that $G < 1$) the distance between the images $\vec{g}(\vec{x})$ and $\vec{g}(\vec{y})$ of $\vec{x}$ and $\vec{y}$ is smaller than the original distance between $\vec{x}$ and $\vec{y}$. Thus the function g contracts distances. Note that, when the dimension $d = 1$,

$$|g(x) - g(y)| = \Big|\int_x^y g'(t)\,dt\Big| \le \Big|\int_x^y |g'(t)|\,dt\Big| \le \Big|\int_x^y \sup_{t' \in B_a} |g'(t')|\,dt\Big| = |x - y| \sup_{t' \in B_a} |g'(t')|$$

For a once continuously differentiable function, the smallest G that one can pick and still have $|g(x) - g(y)| \le G|x - y|$ for all x, y is $G = \sup_{t' \in B_a} |g'(t')|$. In this case (H1) comes down to the requirement that there exist a constant $G < 1$ such that $|g'(t)| \le G < 1$ for all $t' \in B_a$. For dimensions $d > 1$, one has a whole matrix $\mathcal{G}(\vec{x}) = \big[\frac{\partial g_i}{\partial x_j}(\vec{x})\big]_{1 \le i,j \le d}$ of first partial derivatives. There is a measure of the size of this matrix, called the norm of the matrix and denoted $\big\|\mathcal{G}(\vec{x})\big\|$ such that

$$\|\vec{g}(\vec{x}) - \vec{g}(\vec{y})\| \le \|\vec{x} - \vec{y}\| \sup_{\vec{t} \in B_a} \big\|\mathcal{G}(\vec{t})\big\|$$

Once again (H1) comes down to $\big\|\mathcal{G}(\vec{t})\big\| \le G < 1$ for all $\vec{t} \in B_a$. Roughly speaking, (H1) forces the derivative of $\vec{g}$ to be sufficiently small, which forces the derivative of $\vec{x} - \vec{g}(\vec{x})$ to be bounded away from zero.

If we were to relax (H1) to $G \le 1$, the theorem would fail. For example, $g(x) = x$ obeys $|g(x) - g(y)| = |x - y|$ for all x and y. So G would be one in this case. But every x obeys $g(x) = x$, so the solution is certainly not unique.

Discussion of hypothesis (H2): If $\vec{g}$ only takes values that are outside of B_a, then $\vec{x} = \vec{g}(\vec{x})$ cannot possibly have any solutions. So there has to be a requirement that $\vec{g}(\vec{x})$ lies in B_a for at least some values of $\vec{x} \in B_a$. Our hypotheses are actually somewhat stronger than this:

$$\|\vec{g}(\vec{x})\| = \|\vec{g}(\vec{x}) - \vec{g}(\vec{0}) + \vec{g}(\vec{0})\| \le \|\vec{g}(\vec{x}) - \vec{g}(\vec{0})\| + \|\vec{g}(\vec{0})\| \le G\|\vec{x} - \vec{0}\| + (1 - G)a$$

by (H1) and (H2). So, for all $\vec{x}$ in B_a, that is, all $\vec{x}$ with $\|\vec{x}\| < a$, $\|\vec{g}(\vec{x})\| < Ga + (1 - G)a = a$. With our hypotheses $\vec{g} : B_a \to B_a$. Roughly speaking, (H2) requires that $\vec{g}(\vec{x})$ be sufficiently small for at least one $\vec{x}$.

If we were to relax (H2) to $\|\vec{g}(\vec{0})\| \le (1 - G)a$, the theorem would fail. For example, let $d = 1$, pick any $a > 0$, $0 < G < 1$ and define $g : B_a \to \mathbb{R}$ by $g(x) = a(1 - G) + Gx$. Then $g'(x) = G$ for all x and $g(0) = a(1 - G)$. For this g,

$$g(x) = x \quad \iff \quad a(1 - G) + Gx = x \quad \iff \quad a(1 - G) = (1 - G)x \quad \iff \quad x = a$$

As $x = a$ is not in the domain of definition of g, there is no solution.

Proof that there is at most one solution: Suppose that $\vec{x}^*$ and $\vec{y}^*$ are two solutions. Then

$$\begin{aligned} \vec{x}^* = \vec{g}(\vec{x}^*),\ \vec{y}^* = \vec{g}(\vec{y}^*) \quad &\implies \quad \|\vec{x}^* - \vec{y}^*\| = \|\vec{g}(\vec{x}^*) - \vec{g}(\vec{y}^*)\| \\ &\overset{\text{(H1)}}{\implies} \quad \|\vec{x}^* - \vec{y}^*\| \le G\|\vec{x}^* - \vec{y}^*\| \\ &\implies \quad (1 - G)\|\vec{x}^* - \vec{y}^*\| = 0 \end{aligned}$$

As $G < 1$, $1 - G$ is nonzero and $\|\vec{x}^* - \vec{y}^*\|$ must be zero. That is, $\vec{x}^*$ and $\vec{y}^*$ must be the same.

Proof that there is at least one solution: Set

$$\vec{x}_0 = 0 \quad \vec{x}_1 = \vec{g}(\vec{x}_0) \quad \vec{x}_2 = \vec{g}(\vec{x}_1) \quad \cdots \quad \vec{x}_n = \vec{g}(\vec{x}_{n-1}) \quad \cdots$$

We showed in "Significance of hypothesis (H2)" that $\vec{g}(\vec{x})$ is in B_a for all $\vec{x}$ in B_a. So $\vec{x}_0,\ \vec{x}_1,\ \vec{x}_2,\ \cdots$ are all in B_a. So the definition $\vec{x}_n = \vec{g}(\vec{x}_{n-1})$ is legitimate. We shall show that the sequence $\vec{x}_0,\ \vec{x}_1,\ \vec{x}_2,\ \cdots$ converges to some vector $\vec{x}^*$. Since $\vec{g}$ is continuous, this vector will obey

$$\vec{x}^* = \lim_{n\to\infty} \vec{x}_n = \lim_{n\to\infty} \vec{g}(\vec{x}_{n-1}) = \vec{g}\big(\lim_{n\to\infty} \vec{x}_{n-1}\big) = \vec{g}(\vec{x}^*)$$

In other words, $\vec{x}^*$ is a solution of $\vec{x} = \vec{g}(\vec{x})$.

To prove that the sequence converges, we first observe that, applying (H1) numerous times,

$$\begin{aligned}
&\|\vec{x}_m - \vec{x}_{m-1}\| &&= \big\|\vec{g}(\vec{x}_{m-1}) - \vec{g}(\vec{x}_{m-2})\big\| \\
&\le G\big\|\vec{x}_{m-1} - \vec{x}_{m-2}\big\| &&= G\big\|\vec{g}(\vec{x}_{m-2}) - \vec{g}(\vec{x}_{m-3})\big\| \\
&\le G^2\big\|\vec{x}_{m-2} - \vec{x}_{m-3}\big\| &&= G^2\big\|\vec{g}(\vec{x}_{m-3}) - \vec{g}(\vec{x}_{m-4})\big\| \\
&\ \ \vdots \\
&\le G^{m-1}\big\|\vec{x}_1 - \vec{x}_0\big\| &&= G^{m-1}\|\vec{g}(\vec{0})\|
\end{aligned}$$

Remember that $G < 1$. So the distance $\|\vec{x}_m - \vec{x}_{m-1}\|$ between the $(m-1)^{\rm st}$ and $m^{\rm th}$ entries in the sequence gets really small for m large. As

$$\vec{x}_n = \vec{x}_0 + \big(\vec{x}_1 - \vec{x}_0\big) + \big(\vec{x}_2 - \vec{x}_1\big) + \cdots + \big(\vec{x}_n - \vec{x}_{n-1}\big) = \sum_{m=1}^{n} \big(\vec{x}_m - \vec{x}_{m-1}\big)$$

(recall that $\vec{x}_0 = \vec{0}$) it suffices to prove that $\sum\limits_{m=1}^{n} \big(\vec{x}_m - \vec{x}_{m-1}\big)$ converges as $n \to \infty$. To do so it suffices to prove that $\sum\limits_{m=1}^{n} \big\|\vec{x}_m - \vec{x}_{m-1}\big\|$ converges as $n \to \infty$, which we do now.

$$\sum_{m=1}^{n} \big\|\vec{x}_m - \vec{x}_{m-1}\big\| \le \sum_{m=1}^{n} G^{m-1}\|\vec{g}(\vec{0})\| = \frac{1 - G^n}{1 - G}\|\vec{g}(\vec{0})\|$$

As n tends to ∞, G^n converges to zero (because $0 \le G < 1$) and $\frac{1-G^n}{1-G}\|\vec{g}(\vec{0})\|$ converges to $\frac{1}{1-G}\|\vec{g}(\vec{0})\|$. ■

Generalization: The same argument proves the following generalization:

Let X be a complete metric space, with metric d, and $g : X \to X$. If there is a constant $0 \le G < 1$ such that

$$d\big(g(x), g(y)\big) \le G\, d(x, y) \qquad \text{for all } x, y \in X$$

then there exists a unique $x \in X$ obeying $g(x) = x$.

The Implicit Function Theorem: As an application of the contraction mapping theorem, we now prove the implicit function theorem. Consider some function $\vec{f}(\vec{x}, \vec{y})$ with $\vec{x}$ running over $\mathbb{R}^n$, $\vec{y}$ running over $\mathbb{R}^d$ and $\vec{f}$ taking values in $\mathbb{R}^d$. Suppose that we have one point $(\vec{x}_0, \vec{y}_0)$ on the surface $\vec{f}(\vec{x}, \vec{y}) = 0$. In other words, suppose that $\vec{f}(\vec{x}_0, \vec{y}_0) = 0$. And suppose that we wish to solve $\vec{f}(\vec{x}, \vec{y}) = 0$ for $\vec{y}$ as a function of $\vec{x}$ near $(\vec{x}_0, \vec{y}_0)$. First observe that for each fixed $\vec{x}$, $\vec{f}(\vec{x},\vec{y})=0$ is a system of d equations in d unknowns. So at least the number of unknowns matches the number of equations. Denote by A the $d\times d$ matrix $\big[\frac{\partial f_i}{\partial y_j}(\vec{x}_0,\vec{y}_0)\big]_{1\le i,j\le d}$ of first partial $\vec{y}$ derivatives at $(\vec{x}_0,\vec{y}_0)$. Assume that this matrix exists and has an inverse. When $d=1$, A is invertible if and only if $\frac{\partial f}{\partial y}(x_0,\vec{y}_0)\ne 0$. For $d>1$, A is invertible if and only if 0 is not an eigenvalue of A. Also, A is invertible if and only if $\det A\ne 0$. In any event, assuming that A^{-1} exists

$$\vec{f}(\vec{x},\vec{y})=0 \quad\iff\quad A^{-1}\vec{f}(\vec{x},\vec{y})=0 \quad\iff\quad \vec{y}-\vec{y}_0-A^{-1}\vec{f}(\vec{x},\vec{y})=\vec{y}-\vec{y}_0$$

Rename $\vec{y}-\vec{y}_0=\vec{z}$ and define $\vec{g}(\vec{x},\vec{z})=\vec{z}-A^{-1}\vec{f}(\vec{x},\vec{z}+\vec{y}_0)$. Then

$$\vec{f}(\vec{x},\vec{y})=0 \quad\iff\quad \vec{y}=\vec{y}_0+\vec{z}\text{ and }\vec{g}(\vec{x},\vec{z})=\vec{z}$$

Now apply the Contraction Mapping Theorem with $\vec{x}$ viewed as a fixed parameter and $\vec{z}$ viewed as the variable. That is, fix any $\vec{x}$ sufficiently near $\vec{x}_0$. Then $\vec{g}(\vec{x},\vec{z})$ is a function of $\vec{z}$ only and one may apply the Contraction Mapping Theorem to it.

We must of course check that the hypotheses are satisfied. Observe first, that when $\vec{z}=\vec{0}$ and $\vec{x}=\vec{x}_0$, the matrix $\big[\frac{\partial g_i}{\partial z_j}(\vec{x}_0,\vec{0})\big]_{1\le i,j\le d}$ of first derivatives of $\vec{g}$ is exactly $\mathbb{1}-A^{-1}A$, where $\mathbb{1}$ is the identity matrix. The identity $\mathbb{1}$ arises from differentiating the term $\vec{z}$ in $\vec{g}(\vec{x}_0,\vec{z})=\vec{z}-A^{-1}\vec{f}(\vec{x}_0,\vec{z}+\vec{y}_0)$ and $-A^{-1}A$ arises from differentiating $-A^{-1}\vec{f}(\vec{x}_0,\vec{z}+\vec{y}_0)$. So $\big[\frac{\partial g_i}{\partial z_j}(\vec{x}_0,\vec{0})\big]_{1\le i,j\le d}$ is exactly the zero matrix. For $(\vec{x},\vec{z})$ sufficiently close to $(\vec{x}_0,\vec{0})$, the matrix $\big[\frac{\partial g_i}{\partial z_j}(\vec{x},\vec{z})\big]_{1\le i,j\le d}$ will, by continuity, be small enough that (H1) is satisfied. This is because, for any $\vec{u}$, $\vec{v}\in\mathbb{R}^d$, and any $1\le i\le d$,

$$g_i(\vec{x},\vec{u})-g_i(\vec{x},\vec{v})=\int_0^1 \tfrac{d}{dt}g_i\big(\vec{x},t\vec{u}+(1-t)\vec{v}\big)\,dt=\sum_{j=1}^d\int_0^1 \big(u_j-v_j\big)\tfrac{\partial g_i}{\partial z_j}\big(\vec{x},t\vec{u}+(1-t)\vec{v}\big)\,dt$$

so that

$$\big|g_i(\vec{x},\vec{u})-g_i(\vec{x},\vec{v})\big|\le d\ \|\vec{u}-\vec{v}\|\max_{\substack{0\le t\le 1\\1\le j\le d}}\ \big|\tfrac{\partial g_i}{\partial z_j}\big(\vec{x},t\vec{u}+(1-t)\vec{v}\big)\big|$$

Also observe that $\vec{g}(\vec{x}_0,\vec{0})=-A^{-1}\vec{f}(\vec{x}_0,\vec{y}_0)=\vec{0}$. So, once again, by continuity, if $\vec{x}$ is sufficiently close to $\vec{x}_0$, $\vec{g}(\vec{x},\vec{0})$ will be small enough that (H2) is satisfied.

We conclude from the Contraction Mapping Theorem that, assuming A is invertible, $\vec{f}(\vec{x},\vec{y})=0$ has exactly one solution, $\vec{y}(\vec{x})$, near $\vec{y}_0$ for each $\vec{x}$ sufficiently near $\vec{x}_0$. That's the existence and uniqueness part of the

Theorem (Implicit Function Theorem) *Let $n,d\in\mathbb{N}$ and let $U\subset\mathbb{R}^{n+d}$ be an open set. Let $\vec{f}:U\to\mathbb{R}^d$ be C^∞ with $\vec{f}(\vec{x}_0,\vec{y}_0)=0$ for some $\vec{x}_0\in\mathbb{R}^n$, $\vec{y}_0\in\mathbb{R}^d$ with $(\vec{x}_0,\vec{y}_0)\in U$.*

Assume that $\det\left[\frac{\partial f_i}{\partial y_j}(\vec{x}_0,\vec{y}_0)\right]_{1\le i,j\le d}\ne 0$. *Then there exist open sets* $V\subset \mathbb{R}^{n+d}$ *and* $W\subset\mathbb{R}^n$ *with* $\vec{x}_0\in W$ *and* $(\vec{x}_0,\vec{y}_0)\in V$ *such that*

$$\textit{for each } \vec{x}\in W, \textit{ there is a unique } (\vec{x},\vec{y})\in V \textit{ with } \vec{f}(\vec{x},\vec{y})=0.$$

If the $\vec{y}$ *above is denoted* $\vec{Y}(\vec{x})$, *then* $\vec{Y}:W\to\mathbb{R}^d$ *is* C^∞, $\vec{Y}(\vec{x}_0)=\vec{y}_0$ *and* $\vec{f}\big(\vec{x},\vec{Y}(\vec{x})\big)=0$ *for all* $\vec{x}\in W$. *Furthermore*

$$\frac{\partial\vec{Y}}{\partial\vec{x}}(\vec{x})=-\left[\frac{\partial\vec{f}}{\partial\vec{y}}\big(\vec{x},\vec{Y}(\vec{x})\big)\right]^{-1}\frac{\partial\vec{f}}{\partial\vec{x}}\big(\vec{x},\vec{Y}(\vec{x})\big) \tag{1}$$

where $\frac{\partial\vec{Y}}{\partial\vec{x}}$ *denotes the* $d\times n$ *matrix* $\left[\frac{\partial Y_i}{\partial x_j}\right]_{\substack{1\le i\le d\\1\le j\le n}}$, $\frac{\partial\vec{f}}{\partial\vec{x}}$ *denotes the* $d\times n$ *matrix of first partial derivatives of* $\vec{f}$ *with respect to* $\vec{x}$ *and* $\frac{\partial\vec{f}}{\partial\vec{y}}$ *denotes the* $d\times d$ *matrix of first partial derivatives of* $\vec{f}$ *with respect to* $\vec{y}$.

Proof: We have already proven the existence and uniqueness part of the theorem.

The rest will follow once we know that $\vec{Y}(\vec{x})$ has one continuous derivative, because then differentiating $\vec{f}\big(\vec{x},\vec{Y}(\vec{x})\big)=0$ with respect to $\vec{x}$ gives

$$\frac{\partial\vec{f}}{\partial\vec{x}}\big(\vec{x},\vec{Y}(\vec{x})\big)+\frac{\partial\vec{f}}{\partial\vec{y}}\big(\vec{x},\vec{Y}(\vec{x})\big)\frac{\partial\vec{Y}}{\partial\vec{x}}(\vec{x})=\vec{0}$$

which implies (1). (The inverse of the matrix $\frac{\partial\vec{f}}{\partial\vec{y}}\big(\vec{x},\vec{Y}(\vec{x})\big)$ exists, for all $\vec{x}$ close enough to $\vec{x}_0$, because the determinant of $\frac{\partial\vec{f}}{\partial\vec{y}}(\vec{x},\vec{y})$ is nonzero for all $(\vec{x},\vec{y})$ close enough to $(\vec{x}_0,\vec{y}_0)$, by continuity.) Once we have (1), the existence of, and formulae for, all higher derivatives follow by repeatedly differentiating (1). For example, if we know that $\vec{Y}(\vec{x})$ is C^1, then the right hand side of (1) is C^1, so that $\frac{\partial\vec{Y}}{\partial\vec{x}}(\vec{x})$ is C^1 and $\vec{Y}(\vec{x})$ is C^2.

We have constructed $\vec{Y}(\vec{x})$ as the limit of the sequence of approximations $\vec{Y}_n(\vec{x})$ determined by $\vec{Y}_0(\vec{x})=\vec{y}_0$ and

$$\vec{Y}_{n+1}(\vec{x})=\vec{Y}_n(\vec{x})-A^{-1}\vec{f}\big(\vec{x},\vec{Y}_n(\vec{x})\big) \tag{2}$$

Since $\vec{Y}_0(\vec{x})$ is C^∞ (it's a constant) and $\vec{f}$ is C^∞ by hypothesis, all of the $\vec{Y}_n(\vec{x})$'s are C^∞ by induction and the chain rule. We could prove that $\vec{Y}(\vec{x})$ is C^1 by differentiating (2) to get an inductive formula for $\frac{\partial\vec{Y}_n}{\partial\vec{x}}(\vec{x})$ and then proving that the sequence $\left\{\frac{\partial\vec{Y}_n}{\partial\vec{x}}(\vec{x})\right\}_{n\in\mathbb{N}}$ of derivatives converges uniformly.

Instead, we shall pick any unit vector $\hat{e}\in\mathbb{R}^d$ and prove that the directional derivative of $\vec{Y}(\vec{x})$ in direction $\hat{e}$ exists and is given by formula (1) multiplying the vector $\hat{e}$. Since the right hand side of (1) is continuous in $\vec{x}$, this will prove that $\vec{Y}(\vec{x})$ is C^1. We have

$\vec{f}\big(\vec{x}+h\hat{e},\vec{Y}(\vec{x}+h\hat{e})\big)=0$ for all sufficiently small $h\in\mathbb{R}$. Hence

$$\begin{aligned}
0&=\vec{f}\big(\,\vec{x}+h\hat{e}\,,\,\vec{Y}(\vec{x}+h\hat{e})\,\big)-\vec{f}\big(\,\vec{x}\,,\,\vec{Y}(\vec{x})\,\big)\\
&=\vec{f}\big(\,\vec{x}+th\hat{e}\,,\,t\vec{Y}(\vec{x}+h\hat{e})+(1-t)\vec{Y}(\vec{x})\,\big)\Big|_{t=0}^{t=1}\\
&=\int_0^1 \tfrac{d}{dt}\vec{f}\big(\,\vec{x}+th\hat{e}\,,\,t\vec{Y}(\vec{x}+h\hat{e})+(1-t)\vec{Y}(\vec{x})\,\big)\;dt\\
&=h\int_0^1 \tfrac{\partial\vec{f}}{\partial\vec{x}}\hat{e}\;dt+\int_0^1 \tfrac{\partial\vec{f}}{\partial\vec{y}}[\vec{Y}(\vec{x}+h\hat{e})-\vec{Y}(\vec{x})]\;dt
\end{aligned}$$

where the arguments of both $\frac{\partial\vec{f}}{\partial\vec{x}}$ and $\frac{\partial\vec{f}}{\partial\vec{y}}$ are $\big(\,\vec{x}+th\hat{e}\,,\,t\vec{Y}(\vec{x}+h\hat{e})+(1-t)\vec{Y}(\vec{x})\,\big)$. Note that $[\vec{Y}(\vec{x}+h\hat{e})-\vec{Y}(\vec{x})]$ is independent of t and hence can be factored out of the second integral. Dividing by h gives

$$\tfrac{1}{h}[\vec{Y}(\vec{x}+h\hat{e})-\vec{Y}(\vec{x})]=-\bigg[\int_0^1 \tfrac{\partial\vec{f}}{\partial\vec{y}}\;dt\bigg]^{-1}\int_0^1 \tfrac{\partial\vec{f}}{\partial\vec{x}}\,\hat{e}\;dt \tag{3}$$

Since

$$\lim_{h\to 0}\big(\,\vec{x}+th\hat{e}\,,\,t\vec{Y}(\vec{x}+h\hat{e})+(1-t)\vec{Y}(\vec{x})\,\big)=\big(\,\vec{x}\,,\,\vec{Y}(\vec{x})\,\big)$$

uniformly in $t\in[0,1]$, the right hand side of (3) — and hence the left hand side of (3) — converges to

$$-\big[\tfrac{\partial\vec{f}}{\partial\vec{y}}\big(\vec{x},\vec{Y}(\vec{x})\big)\big]^{-1}\tfrac{\partial\vec{f}}{\partial\vec{x}}\big(\vec{x},\vec{Y}(\vec{x})\big)\,\hat{e}$$

as $h\to 0$, as desired.

■