Equicontinuous Functions

Theorem (Arzelà–Ascoli[(1)]) *Let K be a compact metric space, with metric $d_K(p,p')$, and let $\mathcal{C}(K)$ denote the space of real (or complex) valued continuous functions on K. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{C}(K)$ obeying*

- *(H1) for each $p \in K$ there is a $\phi(p) > 0$ such that $|f_n(p)| \le \phi(p)$ for all $p \in K$ (in which case the sequence is said to be pointwise bounded) and*
- *(H2) for each $\varepsilon > 0$ there is a $\delta > 0$ such that $d_K(p,p') < \delta \implies |f_n(p) - f_n(p')| < \varepsilon$ for all $n \in \mathbb{N}$ (in which case the sequence is said to be equicontinuous)*

then

- *(a) there is an $M > 0$ such that $|f_n(p)| \le M$ for all $p \in K$ and all $n \in \mathbb{N}$ (in which case the sequence is said to be uniformly bounded) and*
- *(b) the sequence $\{f_n\}_{n\in\mathbb{N}}$ contains a uniformly convergent subsequence.*

Remark.

(a) In hypothesis (H1), there is no requirement that $\phi(p)$ be a bounded function of p. For example, it is perfectly possible to have a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions on $[0,1]$ with $f_n(0) = \phi(0) = 0$ for all $n \in \mathbb{N}$ and $|f_n(x)| \le \phi(x) = \frac{1}{x}$ for all $n \in \mathbb{N}$ and $x \in (0,1]$. Such a sequence of functions would be pointwise bounded, but not uniformly bounded. Conclusion (a) asserts that, if hypothesis (H2) is also satisfied, then it is possible to choose $\phi(p)$ to be a bounded function.

(b) Any continuous function on any compact metric space, like each f_n, $n \in \mathbb{N}$, is automatically uniformly continuous. This means that for each $\varepsilon > 0$ and for each $n \in \mathbb{N}$, there is a $\delta > 0$ (which, however, may depend on both ε and n) such that $|f_n(p) - f_n(p')| < \varepsilon$ for all $p, p' \in K$ with $d_K(p,p') < \delta$. Hypothesis (H2) further requires the existence of a δ which works simultaneously for all $n \in \mathbb{N}$. For example, set, for each $n \in \mathbb{N}$, $f_n(x) = \sin(nx)$. Then each f_n is uniformly continuous on $K = [0, 2\pi]$ (given any $\varepsilon > 0$, we can choose $\delta = \frac{\varepsilon}{n}$, just because $|f_n'(x)| \le n$), but the sequence $\{f_n\}_{n\in\mathbb{N}}$ is not equicontinuous.

The proof is given in four steps, with the first two steps encapsulated in lemmas.

Lemma 1 *There is a countable subset $\{\, p_\ell \mid \ell \in \mathbb{N} \,\} \subset K$ that is dense in K.*

Proof: We have to find a countable subset $\{\, p_\ell \mid \ell \in \mathbb{N} \,\} \subset K$ with the property that, for each $\varepsilon > 0$ and each $p \in K$, there is some p_ℓ, $\ell \in \mathbb{N}$, whose distance from p is less than ε. Since K is compact, it is totally bounded. This means that for each $\varepsilon > 0$, K can be covered by a finite union of balls of radius ε. In particular, for each $m \in \mathbb{N}$, K is covered by finitely many open balls of radius $\frac{1}{m}$. That is, for each $m \in \mathbb{N}$, there are finitely many points $q_{m,1}$, $\cdots$, q_{m,n_m} in K (the centres of the open balls) such that each point of K is less than distance $\frac{1}{m}$ from at least one of $q_{m,1}$, $\cdots$, q_{m,n_m}. Then

$$\{\, p_1, p_2, p_3, \cdots \,\} = \{\, \overbrace{q_{1,1}, \cdots, q_{1,n_1}}^{m=1}, \overbrace{q_{2,1}, \cdots, q_{2,n_2}}^{m=2}, \overbrace{q_{3,1}, \cdots, q_{3,n_3}}^{m=3}, \cdots \,\}$$

does the job. To see this, pick any $p \in K$ and any $\varepsilon > 0$. Then pick an $m \in \mathbb{N}$ with $\frac{1}{m} < \varepsilon$. There is some $1 \le j \le n_m$, and correspondingly some $q_{m,j}$ in the sequence, such that $d_K(p, q_{m,j}) < \frac{1}{m} < \varepsilon$. ■

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(1) Cesare Arzelà (1847–1912) and Gulio Ascoli (1843–1896) were both Italian mathematicians

Lemma 2 *Let $\big\{f_n\big\}$ be a pointwise bounded sequence of functions defined on the countable set $\big\{\ p_\ell\ \big|\ \ell\in\mathbb{N}\ \big\}$. Then there is a subsequence $\big\{f_{n_k}\big\}_{k\in\mathbb{N}}$ of $\big\{f_n\big\}$ such that $\big\{f_{n_k}(p_\ell)\big\}_{k\in\mathbb{N}}$ converges for each $\ell\in\mathbb{N}$.*

Proof: I'll give the proof assuming that the functions are real valued. For the proof in the complex valued case, just replace $\mathbb{R}$ with $\mathbb{C}$.

- First concentrate on p_1. By hypothesis, there is an $M_1>0$ such that $\big|f_n(p_1)\big|\le M_1$ for all $n\in\mathbb{N}$. Since the closed ball $\big\{\ x\in\mathbb{R}\ \big|\ |x|\le M_1\ \big\}$ is compact, there is a subsequence $\big\{f_{n_j}(p_1)\big\}_{j\in\mathbb{N}}$ that converges in $\mathbb{R}$. Rename n_j to $N_1(j)$. Think of N_1 as a function defined on $\mathbb{N}$. The statement that $\big\{f_{N_1(j)}(p_1)\big\}_{j\in\mathbb{N}}$ is a subsequence of $\big\{f_n(p_1)\big\}_{n\in\mathbb{N}}$ means, firstly, that the range of the function N_1 is a subset of $\mathbb{N}$ and, secondly, that N_1 is strictly increasing.
- Now concentrate on p_2. By hypothesis, there is an $M_2>0$ such that $\big|f_{N_1(j)}(p_2)\big|\le M_2$ for all $j\in\mathbb{N}$. Since the closed ball $\big\{\ x\in\mathbb{R}\ \big|\ |x|\le M_2\ \big\}$ is compact, there is a subsequence $\big\{f_{N_1(j_k)}(p_2)\big\}_{k\in\mathbb{N}}$ of $\big\{f_{N_1(j)}(p_2)\big\}_{j\in\mathbb{N}}$ that converges in $\mathbb{R}$. Rename $N_1(j_k)$ to $N_2(k)$. Again, think of N_2 as a function defined on $\mathbb{N}$. The statement that $\big\{f_{N_2(k)}(p_2)\big\}_{k\in\mathbb{N}}$ is a subsequence of $\big\{f_{N_1(j)}(p_2)\big\}_{j\in\mathbb{N}}$ means, firstly, that the range of the function N_2 is a subset of the range of N_1 and, secondly, that N_2 is strictly increasing.
- Continuing in this way we can construct, for each $\ell\in\mathbb{N}$, a subsequence $\big\{f_{N_\ell(n)}\big\}_{n\in\mathbb{N}}$ of $\big\{f_n\big\}$ for which $\lim\limits_{n\to\infty}f_{N_\ell(n)}(p_\ell)$ exists. Furthermore, for each $\ell\in\mathbb{N}$, $\big\{f_{N_\ell(n)}\big\}_{n\in\mathbb{N}}$ is a subsequence of $\big\{f_{N_{\ell-1}(n)}\big\}_{n\in\mathbb{N}}$. (For $\ell=1$, set $N_{\ell-1}(n)=N_0(n)=n$.) That is, N_ℓ is a strictly increasing function whose range is contained in the range of $N_{\ell-1}$.

Here is a table giving one possible set of $N_\ell(n)$'s for small values of ℓ and n. The top row gives the indices for the original sequence $\big\{f_n\big\}_{n\in\mathbb{N}}$. The second row gives the indices for the first subsequence $\big\{f_{N_1(j)}\big\}_{j\in\mathbb{N}}$. And so on.

1	2	3	4	5	6	7	8	9	10	$\cdots$
	$N_1(1)=2$		$N_1(2)=4$		$N_1(3)=6$	$N_1(4)=7$		$N_1(5)=9$		
			$N_2(1)=4$			$N_2(2)=7$		$N_2(3)=9$		
			$N_3(1)=4$					$N_3(2)=9$		

I claim that $\big\{f_{N_n(n)}\big\}_{n\in\mathbb{N}}$ is a subsequence of $\big\{f_n\big\}_{n\in\mathbb{N}}$ for which $\lim\limits_{n\to\infty}f_{N_n(n)}(p_\ell)$ converges for every $\ell\in\mathbb{N}$.

- To verify that $\big\{f_{N_n(n)}\big\}_{n\in\mathbb{N}}$ is a legitimate subsequence of $\big\{f_n\big\}_{n\in\mathbb{N}}$, I just have to verify that $N_n(n)$ is strictly increasing in n. This is the case because, for each $n\in\mathbb{N}$, $N_{n+1}(n+1)>N_{n+1}(n)\ge N_n(n)$. (The first inequality is true because N_{n+1} is a strictly increasing function. The second inequality is true because $\big\{f_{N_{n+1}(j)}\big\}_{j\in\mathbb{N}}$ is a subsequence of $\big\{f_{N_n(j)}\big\}_{j\in\mathbb{N}}$.)
- Fix any $\ell\in\mathbb{N}$. To verify that $\lim\limits_{n\to\infty}f_{N_n(n)}(p_\ell)$ converges, we now show that $\{f_{N_n(n)}(p_\ell)\}_{n\ge\ell}$ is a subsequence of $\{f_{N_\ell(n)}(p_\ell)\}_{n\in\mathbb{N}}$, which we already know converges. We already know that $N_n(n)$ is strictly increasing in n. So to show that $\{f_{N_n(n)}(p_\ell)\}_{n\ge\ell}$ is a subsequence of $\{f_{N_\ell(n)}(p_\ell)\}_{n\in\mathbb{N}}$, we just have to show that, for each $n\ge\ell$, $N_n(n)$ lies in the range of N_ℓ. But, for any $n\ge\ell$, $N_n(n)$ lies in the range of N_n, which is a subset of the range of N_{n-1}, which is a subset of the range of N_{n-2}, $\cdots$, which is a subset of the range of N_ℓ.

■

Proof of Arzelà–Ascoli (a): Let δ be the δ provided by (H2) for $\varepsilon=1$. Since K is compact, it is totally bounded. Hence there are finitely many points p_1, $\cdots$, p_r in K such that every $p\in K$ is (strictly) within a distance δ of at least one of p_1, $\cdots$, p_r. Let $M=\max\big\{\ \phi(p_i)\ \big|\ 1\le i\le r\ \big\}+1$, where $\phi(p)$ was provided by (H1). Let $p\in K$ and $n\in\mathbb{N}$. Let p be (strictly) within a distance δ of p_i. Then, by (H2),

$$|f_n(p)|\le|f_n(p)-f_n(p_i)|+|f_n(p_i)|<\varepsilon+\phi(p_i)\le M$$

■

Proof of Arzelà–Ascoli (b): Let $\{ \ p_\ell \ \big| \ \ell \in \mathbb{N} \ \}$ be the countable dense subset of K provided by Lemma 1 and let $\{f_{n_k}\}$ be the subsequence provided by Lemma 2. Rename f_{n_k} to g_k. We show that $\{g_k\}_{k\in\mathbb{N}}$ converges uniformly. To do so, it suffices to prove that, for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$i, j \geq N \implies |g_i(p) - g_j(p)| < \varepsilon \text{ for all } p \in K$$

So let $\varepsilon > 0$ and let δ be the δ provided by (H2) for $\frac{\varepsilon}{3}$. Since K is compact, it is totally bounded. Hence there are finitely many points $q_1, \cdots, q_r$ in K such that every $p \in K$ is (strictly) within a distance $\frac{\delta}{2}$ of at least one of $q_1, \cdots, q_r$. Since $\{ \ p_\ell \ \big| \ \ell \in \mathbb{N} \ \}$ is dense in K, there is, for each $1 \leq m \leq r$, a p_{ℓ_m} that is within a distance $\frac{\delta}{2}$ of q_m. So each $p \in K$ is within distance δ of at least one of $p_{\ell_1}, \cdots, p_{\ell_r}$. For each $1 \leq m \leq r$, $\lim\limits_{n\to\infty} g_n(p_{\ell_m})$ converges, by Lemma 2, so there is an $N_m \in \mathbb{N}$ such that

$$i, j \geq N_m \implies |g_i(p_{\ell_m}) - g_j(p_{\ell_m})| < \tfrac{\varepsilon}{3}$$

Hence, for any $p \in K$ and any $i, j \geq N = \max\limits_{1\leq m\leq r} N_m$,

$$|g_i(p) - g_j(p)| \leq |g_i(p) - g_i(p_{\ell_m})| + |g_i(p_{\ell_m}) - g_j(p_{\ell_m})| + |g_j(p_{\ell_m}) - g_j(p)| < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon$$

as desired. Here the m used is one for which $d_K(p, p_{\ell_m}) < \delta$. Hypothesis (H2) was used to bound both $|g_i(p) - g_i(p_{\ell_m})|$ and $|g_j(p_{\ell_m}) - g_j(p)|$ by $\frac{\varepsilon}{3}$. ■

Corollary. *Let* $a < b$ *and* $c \in [a, b]$. *Let* $\{f_n : [a, b] \to \mathbb{R}\}_{n\in\mathbb{N}}$ *be a sequence of differentiable functions obeying*

$$\sup\{ \ |f_n(c)| \ \big| \ n \in \mathbb{N} \ \} < \infty \qquad \text{and} \qquad \sup\{ \ |f_n'(x)| \ \big| \ n \in \mathbb{N}, x \in [a, b] \ \} < \infty$$

Then $\{f_n\}_{n\in\mathbb{N}}$ *has a uniformly convergent subsequence.*

Proof: We apply the Arzelá–Ascoli theorem with $K = [a, b]$. Set

$$M = \sup\{ \ |f_n(c)| \ \big| \ n \in \mathbb{N} \ \} \qquad \text{and} \qquad M' = \sup\{ \ |f_n'(\xi)| \ \big| \ n \in \mathbb{N}, \xi \in [a, b] \ \}$$

To verify hypothesis (H2), observe that, by the mean value theorem,

$$\big|f_n(x) - f_n(y)\big| = \big|f_n'(\xi)\big| \ |x - y| \leq M'|x - y|$$

for all $n \in \mathbb{N}$ and all $x, y \in [a, b]$. So it suffices to take $\delta = \frac{\varepsilon}{M'}$. To verify hypothesis (H1), observe that, again by the mean value theorem,

$$\big|f_n(x)\big| \leq \big|f_n(c)\big| + \big|f_n(x) - f_n(c)\big| \leq \big|f_n(c)\big| + \big|f_n'(\xi)\big| \ |x - c| \leq M + M'|x - c|$$

So it suffices to take $\phi(x) = M + M'|x - c|$. ■