1) Below $a, b \in \mathbb{R}$ with $a < b$.
(a) Define carefully what it means for a function $f$ to be of bounded variation on $[a, b]$.
(b) When do we say that a function sequence $\{f_n\}_{n \in \mathbb{N}}$ is uniformly bounded on $[a, b]$?
(c) When do we say that a function sequence $\{f_n\}_{n \in \mathbb{N}}$ is equicontinuous on $[a, b]$?

Solution. (a) The function $f : [a, b] \to \mathbb{R}$ is said to be of bounded variation on $[a, b]$ if and only if there is a constant $M > 0$ such that
\[
\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq M
\]
for all partitions $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$.

(b) The sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions $f_n : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, is said to be uniformly bounded on $[a, b]$ if and only if there is a constant $M > 0$ such that
\[
|f_n(x)| \leq M \quad \text{for all } x \in [a, b] \text{ and } n \in \mathbb{N}
\]

(c) The sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions $f_n : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, is said to be equicontinuous on $[a, b]$ if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that
\[
x, y \in [a, b], |x - y| < \delta, n \in \mathbb{N} \implies |f_n(x) - f_n(y)| < \varepsilon
\]

2) Let $\alpha$ be increasing and $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Denote by $m$ and $M$ the infimum and supremum of \{ $f(x) : x \in [a, b]$ \} respectively.
(a) Show that there exists $c \in [m, M]$ such that
\[
\int_{a}^{b} f(x) \, d\alpha = c[\alpha(b) - \alpha(a)]
\]

(b) If, in addition, $f$ is continous on $[a, b]$, prove that there exists $x_0 \in [a, b]$ such that
\[
\int_{a}^{b} f(x) \, d\alpha = f(x_0)[\alpha(b) - \alpha(a)]
\]

Solution. (a) Because $\alpha$ is increasing, we have that
\[
f_1(x) \leq f_2(x) \text{ for all } a \leq x \leq b \implies \int_{a}^{b} f_1(x) \, d\alpha \leq \int_{a}^{b} f_2(x) \, d\alpha
\]
Consequently
\[
m \leq f(x) \leq M \implies \int_{a}^{b} m \, d\alpha \leq \int_{a}^{b} f(x) \, d\alpha \leq \int_{a}^{b} M \, d\alpha
\]
\[
\implies m[\alpha(b) - \alpha(a)] \leq \int_{a}^{b} f(x) \, d\alpha \leq M[\alpha(b) - \alpha(a)]
\]
\[
\implies m \leq \frac{\int_{a}^{b} f(x) \, d\alpha}{\alpha(b) - \alpha(a)} \leq M
\]
and it suffices to set $c = \frac{\int_{a}^{b} f(x) \, d\alpha}{\alpha(b) - \alpha(a)}$. 

1
(b) Since $f$ is a continuous function on a compact interval, there are $a \leq x_m, x_M \leq b$ with $f(x_m) = m$ and $f(x_M) = M$. By the intermediate value theorem, there is an $x_0$ between $x_m$ and $x_M$, and in particular, in $[a, b]$, with $f(x_0) = c$.

3) Let $\alpha$ be increasing and assume that $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Define $F$ as

$$F(x) := \int_a^x f \, d\alpha \quad \text{for } x \in [a, b]$$

(a) Show that $F$ is of bounded variation on $[a, b]$. [Hint: The mean value theorem that you proved in the previous problem may be useful.]

(b) For $F$ defined above, show that $F$ is continuous at every point at which $\alpha$ is continuous.

**Solution.** As in question 2, denote by $m$ and $M$ the infimum and supremum of $\{f(x) : x \in [a, b]\}$, respectively. Observe that, for each $c \in [m, M]$ we have $|c| \leq \max \{|m|, |M|\}$.

(a) Let $x_0 = a \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$. Then

$$\sum_{j=1}^n |F(x_j) - F(x_{j-1})| = \sum_{j=1}^n \left| \int_{x_{j-1}}^{x_j} f \, d\alpha \right| = \sum_{j=1}^n |c_j[\alpha(x_j) - \alpha(x_{j-1})]|$$

with each $c_j \in [m, M]$. As $\alpha$ is increasing,

$$\sum_{j=1}^n |F(x_j) - F(x_{j-1})| \leq \sum_{j=1}^n \max \{|m|, |M|\} |\alpha(x_j) - \alpha(x_{j-1})| = \max \{|m|, |M|\} |\alpha(b) - \alpha(a)|$$

This is bounded uniformly in $n$, so $F(x)$ is of bounded variation.

(b) Let $\alpha$ be continuous at $x$. Then, for any $a \leq y \leq b$, there is a $c \in [m, M]$ such that

$$|F(y) - F(x)| = \left| \int_x^y f \, d\alpha \right| = |c(\alpha(y) - \alpha(x))| \leq \max \{|m|, |M|\} |\alpha(y) - \alpha(x)|$$

As $\alpha$ is continuous at $x$,

$$\lim_{y \to x} |F(y) - F(x)| \leq \lim_{y \to x} \max \{|m|, |M|\} |\alpha(y) - \alpha(x)| = 0$$

4) Give examples of each of the following together with a brief explanation. Make your examples as simple as possible. Sketch the graphs of the functions involved if feasible.

(a) $f_n \to f$ in the mean (i.e., in $L^2$), but not pointwise or uniformly.

(b) $f_n \to f$ uniformly, but not in $L^2$.

(c) $f_n \to f$ uniformly, all of the $f_n$ and $f$ are integrable, but $\int_{-\infty}^\infty f_n(x) \, dx$ does not converge to $\int_{-\infty}^\infty f(x) \, dx$.

(d) A bounded function $f$ and an increasing function $\alpha$ defined on $[0, 1]$ such that $|f| \in \mathcal{R}(\alpha)$ but for which $\int_1^1 f \, d\alpha$ does not exist.

**Solution.** (a) Define, on the interval $[-1, 1]$ the functions $f(x) = 0$ and

$$f_n(x) = \begin{cases} \frac{n^{1/4}}{2n} & \text{if } -\frac{1}{2n} \leq x \leq \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

Since

$$\|f_n - f\|_{\infty} = n^{1/4} \to 0 \text{ as } n \to \infty$$

$$\|f_n - f\|_2 = \left[ \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n^{1/2} \, dx \right]^{1/2} = \frac{1}{n^{1/2}} \to 0 \text{ as } n \to \infty$$


we have convergence in the mean, but not uniformly. We do not have pointwise convergence since

\[ f_n(0) = \frac{1}{n^{1/4}} \]

diverges as \( n \to \infty \).

(b) Define the functions \( f_n, f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = 0 \) and

\[
  f_n(x) = \begin{cases} 
  \frac{1}{\sqrt{n}} & \text{if } -\frac{n}{2} \leq x \leq \frac{n}{2} \\
  0 & \text{otherwise}
  \end{cases}
\]

Since

\[
  \|f_n - f\|_\infty = \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty
\]

\[
  \|f_n - f\|_2 = \left[ \int_{-\frac{n}{2}}^{\frac{n}{2}} \frac{1}{n} \, dx \right]^{1/2} = 1 \to 0 \text{ as } n \to \infty
\]

we have uniform, but not \( L^2 \) convergence.

(c) Define the functions \( f_n, f : \mathbb{R} \to \mathbb{R} \), by \( f(x) = 0 \) and

\[
  f_n(x) = \begin{cases} 
  \frac{1}{n} & \text{if } -\frac{n}{2} \leq x \leq \frac{n}{2} \\
  0 & \text{otherwise}
  \end{cases}
\]

We have

\[
  \|f - f_n\|_\infty = \frac{1}{n} \to 0 \text{ as } n \to \infty
\]

\[
  \int_{-\infty}^{\infty} f_n(x) \, dx = 1 \to 0 = \int_{-\infty}^{\infty} f(x) \, dx
\]

(d) Define \( \alpha(x) = x \) and

\[
  f(x) = \begin{cases} 
  1 & \text{if } x \in \mathbb{Q} \\
  -1 & \text{if } x \notin \mathbb{Q}
  \end{cases}
\]

Then \( |f(x)| \) is the constant function 1 and is continuous so that \( f \in \mathcal{R}(\alpha) \). But \( \int_0^1 f \, d\alpha \) does not exist, because for any partition \( P, f \) takes both the values 1 and \(-1\) on every subinterval of the partition, so that

\[
  U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta \alpha_i = \sum_{i=1}^n (1) \Delta \alpha_i = 1
\]

\[
  L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta \alpha_i = \sum_{i=1}^n (-1) \Delta \alpha_i = -1
\]

5) Prove that if \( f : [0, 1] \to \mathbb{R} \) is a continuous function that obeys

\[
  \int_0^1 f(x)x^n \, dx = 0
\]

for all \( n \in \mathbb{Z} \) with \( n \geq 0 \), then \( f(x) \) is identically zero. [Hint: use the Weierstrass approximation theorem to prove that \( \int_0^1 f(x)^2 \, dx = 0 \).]

Solution. Let \( \varepsilon > 0 \). Since \( f(x) \) is continuous there is an \( M \in \mathbb{R} \) such \( |f(x)| \leq M \) for all \( x \in [0, 1] \) and, furthermore, the Weierstrass approximation theorem assures us that there is a polynomial \( P(x) \) such that \( |f(x) - P(x)| < \frac{\varepsilon}{M} \) for all \( 0 \leq x \leq 1 \). So

\[
  0 \leq \int_0^1 f(x)^2 \, dx = \int_0^1 f(x)P(x) \, dx + \int_0^1 f(x)[f(x) - P(x)] \, dx = \int_0^1 f(x)[f(x) - P(x)] \, dx \\
  \leq \int_0^1 M \times \frac{\varepsilon}{M} \, dx = \varepsilon
\]
As this is the case for all $\varepsilon > 0$, we have $\int_0^1 f(x)^2 \, dx = 0$. As $f$ is continuous, we have that $f(x) = 0$ for all $0 \leq x \leq 1$.

6) Let $f$ be $2\pi$–periodic function which on $[0, 2\pi)$ satisfies

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \pi, \\ 0 & \text{if } \pi < t < 2\pi. \end{cases}$$

(a) For $k \in \mathbb{Z}$, let $\hat{f}(k)$ denote the $k$th Fourier coefficient of $f$, i.e., $\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} \, dt$. Show that

$$\hat{f}(k) = \begin{cases} \frac{1}{i\pi k} & \text{if } k \text{ is an odd integer} \\ \frac{1}{2} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(b) Use (1) and Parseval’s identity to show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

(c) For $a, b, c \in \mathbb{R}$, define

$$G(a, b, c) = \int_0^{2\pi} |f(t) - (a + be^{it} + ce^{-it})|^2 \, dt$$

where $f$ is as in part (a). Find $a_0, b_0, c_0$ such that

$$G(a_0, b_0, c_0) \leq G(a, b, c) \quad \forall a, b, c \in \mathbb{R}$$

(Justify your answer, i.e., identify all theorems that you use.)

**Solution.** (a) By definition

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} \, dt = \frac{1}{2\pi} \int_0^{\pi} e^{-ikt} \, dt = \frac{1}{2\pi} \left\{ \begin{array}{ll} \pi & \text{if } k = 0 \\ \frac{1}{i\pi} [e^{-ikt}]_{t=0}^{t=\pi} & \text{if } k \neq 0 \end{array} \right\}$$

$$= \frac{1}{2\pi} \left\{ \begin{array}{ll} \pi & \text{if } k = 0 \\ \frac{1}{i\pi} [e^{-ik\pi} - 1] & \text{if } k \neq 0 \end{array} \right\} = \frac{1}{2\pi} \left\{ \begin{array}{ll} \pi & \text{if } k = 0 \\ \frac{1}{i\pi} [-2] & \text{if } k \neq 0 \text{ is odd} \\ 0 & \text{if } k \neq 0 \text{ is even} \end{array} \right\}$$

That is the desired answer.

(b) Parseval’s identity says that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2$$

So, for our $f$,

$$\frac{1}{2\pi} \int_{0}^{\pi} dt = \sum_{k=0}^{k>0} \frac{\pi}{k \pi} \quad \Rightarrow \quad \frac{1}{4} = \sum_{k \text{ odd}} \frac{\pi}{k \pi} \quad \Rightarrow \quad \sum_{k \text{ odd}} \frac{1}{k} = \frac{\pi^2}{4} \quad \Rightarrow \quad 2 \sum_{k \text{ odd}} \frac{1}{k} = \frac{\pi^2}{2}$$

4
As \( n \) runs over all natural numbers, \( 2n - 1 \) runs over all positive odd integers, so
\[
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}
\]
as desired.

(c) For any \( 2\pi \)-periodic Riemann integrable function \( f(t) \), the function \( g(t) = f(t) - (a + be^{it} + ce^{-it}) \) is again \( 2\pi \)-periodic and Riemann integrable and has
\[
\hat{g}(n) = \begin{cases} 
\hat{f}(0) - a & \text{if } n = 0 \\
\hat{f}(1) - b & \text{if } n = 1 \\
\hat{f}(-1) - c & \text{if } n = -1 \\
\hat{f}(n) & \text{otherwise}
\end{cases}
\]
So again, by Parseval,
\[
G(a, b, c) = \int_{0}^{2\pi} |f(t) - (a + be^{it} + ce^{-it})|^2 dt 
= 2\pi \left\{ |\hat{f}(0) - a|^2 + |\hat{f}(1) - b|^2 + |\hat{f}(-1) - c|^2 + \sum_{n \notin \{0, 1, -1\}} |\hat{f}(n)|^2 \right\}
\]
so that,
\[
G(\text{Re}\hat{f}(0), \text{Re}\hat{f}(1), \text{Re}\hat{f}(-1)) \leq G(a, b, c) \quad \forall a, b, c \in \mathbb{R}
\]
(Recall that, for any complex number \( z \), \( |z|^2 = (\text{Re} z)^2 + (\text{Im} z)^2 \).) For our \( f \), we want
\[
a_0 = \text{Re}\hat{f}(0) = \frac{1}{2} \quad b_0 = \text{Re}\hat{f}(1) = 0 \quad c_0 = \text{Re}\hat{f}(-1) = 0
\]
7) In this problem, we will prove Fejér’s theorem in several steps. If you are stuck with one part, move on to the next (and feel free to use the results from earlier parts).
(a) Let \( x \in \mathbb{R} \) that is not an integer multiple of \( \pi \). Show that
\[
\sum_{k=0}^{n-1} e^{i(2k+1)x} = \frac{\sin(nx)}{\sin x}
\]
Using (2) prove that
\[
\sum_{k=0}^{n-1} \sin [(2k + 1)x] = \frac{\sin^2(nx)}{\sin x}
\]
(b) Let \( D_n \) be the Dirichlet kernel given on \([-\pi, \pi]\) by
\[
D_n(x) = \begin{cases} 
\frac{\sin [(n + \frac{1}{2})x]}{\sin \frac{x}{2}} & \text{if } x \neq 0 \\
2n + 1 & \text{if } x = 0
\end{cases}
\]
Define, now, \( F_n \) on \([-\pi, \pi]\) via
\[
F_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(x)
\]
Show that
\[
F_n(x) = \begin{cases} 
\frac{1}{n} \sin^2 \left( \frac{x}{2} \right) & \text{if } x \neq 0 \\
\frac{1}{n} & \text{if } x = 0
\end{cases}
\]

(c) Let \( f \) be a \( 2\pi \)-periodic function that is Riemann integrable on \([-\pi, \pi]\). Define
\[
\sigma_n(f; x) := \frac{1}{n} \left( s_0(f; x) + s_1(f; x) + \cdots + s_{n-1}(f; x) \right)
\]
where \( s_j(f; x) \) is the \( j \)th partial sum of the Fourier series of \( f \). Show that
\[
\sigma_n(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) F_n(t) \, dt
\]
\[\text{[Hint: recall that } s_n(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t) \, dt.\]

(d) Show that for any positive integer \( n \),
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x) \, dx = 1
\]
\[\text{[Hint: Apply (c) to the constant function } f = 1.\]

(e) Suppose that \( f \) is a \( 2\pi \)-periodic function that is continuous on \([-\pi, \pi]\). Show that \( \sigma_n(f; x) \) converges to \( f(x) \) pointwise. Moreover, show that the convergence is uniform. \[\text{[Hint: Define } g_x(t) := f(x-t) - f(x)\]
and write \( f(x) - \sigma_n(f; x) \) in terms of \( g_x.\]

**Solution.** (a) Applying \( \sum_{k=0}^{N} a r^k = a \frac{r^{N+1} - 1}{1 - r} \) with \( a = e^{ix}, r = e^{i(2x)} \) and \( N = n - 1 \) gives
\[
\sum_{k=0}^{n-1} e^{i(2k+1)x} = \sum_{k=0}^{n-1} e^{ix} e^{2kx} = e^{ix} \frac{e^{2inx} - 1}{e^{2ix} - 1} = e^{inx} \frac{e^{inx} - e^{-inx}}{e^{ix} - e^{-ix}} = e^{inx} \left( \frac{\sin(nx)}{\sin x} \right)
\]
Taking the complex conjugate of both sides gives
\[
\sum_{k=0}^{n-1} e^{-i(2k+1)x} = e^{-inx} \frac{\sin(nx)}{\sin x}
\]
so that
\[
\sum_{k=0}^{n-1} \sin [(2k + 1)x] = \frac{1}{2i} \sum_{k=0}^{n-1} [e^{i(2k+1)x} - e^{-i(2k+1)x}] = \frac{1}{2i} \frac{\sin(nx)}{\sin x} [e^{inx} - e^{-inx}] = \frac{\sin^2(nx)}{\sin x}
\]
\[\text{[b) For } x = 0, \]
\[
\frac{1}{n} \sum_{k=0}^{n-1} D_k(x = 0) = \frac{1}{n} \sum_{k=0}^{n-1} (2k + 1) = 1 + \frac{2}{n} \sum_{k=0}^{n-1} k = 1 + \frac{2}{n} \frac{(n - 1)n}{2} = n
\]
For \( x \neq 0 \), using the second result of part (a), but with \( x \) replaced by \( \frac{x}{2} \),
\[
\frac{1}{n} \sum_{k=0}^{n-1} D_k(x) = \frac{1}{n} \sum_{k=0}^{n-1} \sin \left( \frac{(k + \frac{1}{2})x}{2} \right) = \frac{1}{n} \frac{1}{\sin \frac{x}{2}} \sin^2 \frac{x}{2} = \frac{1}{n} \frac{1}{\sin \frac{x}{2}} \frac{\sin^2 \left( \frac{nx}{2} \right)}{\sin^2 \left( \frac{x}{2} \right)}
\]


(c) By definition

\[\sigma_n(f; x) = \frac{1}{n} \sum_{k=0}^{n-1} s_k(f; x) = \frac{1}{2\pi n} \sum_{k=0}^{n-1} \int_{-\pi}^{\pi} f(x-t) D_k(t) \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) F_n(t) \, dt\]

(d) For the function \(f(x) = 1\), all Fourier series partial sums \(s_j(1; x)\) are also 1 so that \(\sigma_n(1; x)\) is 1 too. So when \(f(x) = 1\), part (c) reduces to

\[1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(t) \, dt\]

(e) By parts (d) and (c),

\[f(x) - \sigma_n(f; x) = f(x) \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(t) \, dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) F_n(t) \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x) - f(x-t)] F_n(t) \, dt\]

Now pick any \(\varepsilon > 0\). Since \(f\) is continuous and periodic, there is a \(\delta > 0\) such that \(|f(x) - f(x-t)| < \frac{\varepsilon}{2}\) for all \(-\pi \leq x \leq \pi\) and \(|t| < \delta\) and there is also a \(\Phi > 0\) such that \(|f(x)| \leq \Phi\) for all \(x \in \mathbb{R}\). Now having fixed \(\varepsilon > 0\) and \(\delta > 0\) chose

\[N = \frac{8\Phi}{\varepsilon \sin^2 \left(\frac{\delta}{2}\right)}\]

Since \(\sin \left(\frac{\pi}{2}\right)\) is increasing on \(0 \leq x \leq \pi\) we have, for all \(\delta \leq |t| \leq \pi\) and \(n \geq N\)

\[0 \leq F_n(t) = \frac{1}{n} \frac{\sin^2 \left(\frac{\pi t}{2}\right)}{\sin^2 \left(\frac{n \pi t}{2}\right)} \leq \frac{1}{N} \frac{1}{\sin^2 \left(\frac{\delta}{2}\right)} = \frac{\varepsilon}{8\Phi}\]

so that, for all \(n \geq N\),

\[|f(x) - \sigma_n(f; x)| \leq \frac{1}{2\pi} \left\{ \int_{\substack{-\pi \leq t \leq \pi \\mid t \leq \delta \\mid t \geq \delta}} |f(x) - f(x-t)| F_n(t) \, dt + \int_{\substack{-\pi \leq t \leq \pi \\mid t \geq \delta}} |f(x) - f(x-t)| F_n(t) \, dt \right\} \leq \frac{1}{2\pi} \left\{ \int_{\substack{-\pi \leq t \leq \pi \\mid t \leq \delta \\mid t \geq \delta}} \frac{\varepsilon}{2} F_n(t) \, dt + \int_{\substack{-\pi \leq t \leq \pi \\mid t \geq \delta}} 2\Phi \frac{\varepsilon}{8\Phi} \, dt \right\} \leq \frac{3\varepsilon}{4}\]

This proves that \(\sigma_n(f; x)\) converges uniformly to \(f(x)\) as \(n \to \infty\).