1) Define
(a) \( f_a^b f(x) \, d\alpha(x) \)
(b) a self–adjoint algebra of functions
(c) the Fourier series of a function

**Solution.** (a) Let \( a < b \) and \( f, \alpha : [a, b] \to \mathbb{R} \). \( P = \{x_0, x_1, \ldots, x_n\} \) is partition of \([a, b] \) if \( a = x_0 < x_1 < \cdots < x_n = b \). \( T = \{t_1, \ldots, t_n\} \) is a choice for the partition \( P \) if \( x_{i-1} < t_i \leq x_i \) for all \( 1 \leq i \leq n \). For each partition \( P \) of \([a, b] \) and each choice \( T \) for \( P \), set

\[
S(P, T, f, \alpha) = \sum_{i=1}^{n} f(t_i) [\alpha(x_i) - \alpha(x_{i-1})]
\]

Then \( f_a^b f(x) \, d\alpha(x) = I \) if for each \( \varepsilon > 0 \) there is a partition \( P_\varepsilon \) such that

\[
|S(P, T, f, \alpha) - I| < \varepsilon
\]

for all partitions \( P \) for \([a, b] \) obeying \( P \supset P_\varepsilon \) and all choices \( T \) for \( P \).

(b) A set \( \mathcal{A} \) of functions that are defined on a set \( E \) and take values in \( \mathfrak{C} \) is a self–adjoint algebra if

(i) \( f, g \in \mathcal{A} \Rightarrow fg, f + g \in \mathcal{A} \)
(ii) \( c \in \mathfrak{C}, f \in \mathcal{A} \Rightarrow cf \in \mathcal{A} \)
(iii) \( f \in \mathcal{A} \Rightarrow \overline{f} \in \mathcal{A} \)

(c) Let \( f : \mathbb{R} \to \mathfrak{C} \) be period of period \( 2\pi \) and be integrable on \([-\pi, \pi] \). The Fourier series for \( f \) is

\[
\sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \quad \text{for all} \quad n \in \mathbb{N}
\]

2) Give an example of each of the following, together with a brief explanation of your example. If an example does not exist, explain why not.

(a) A differentiable function which is not monotonic but whose derivative obeys \( |f'(x)| \geq 1 \).
(b) Two functions \( f, \alpha : [0, 1] \to \mathbb{R} \) with \( f \) continuous, but \( f \notin \mathcal{R}(\alpha) \) on \([0, 1] \).
(c) A continuous function \( f : (-1, 1) \to \mathbb{R} \) that cannot be uniformly approximated by a polynomial.
(d) A monotonically decreasing sequence of functions \( f_n : [0, 1] \to \mathbb{R} \) which converges pointwise, but not uniformly to zero.

**Solution.** (a) Assuming that \( f \) is to defined and differentiable on \( \mathbb{R} \) or on an interval, there is no example. Since \( f \) is not monotonic, its derivative must take both signs. Since \( |f'(x)| \geq 1 \), \( f' \) takes values larger than 1 and smaller than \(-1 \) but no values between \( \pm 1 \). This violates the “intermediate value theorem for derivatives”.

On the other hand if we allow the domain of \( f \) to be not connected, then we could have \( f(x) = |x| \) with domain \( \mathbb{R} \setminus \{0\} \).

(b) \( f(x) = x \) and \( \alpha(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases} \). By integration by parts \( f \in \mathcal{R}(\alpha) \) if and only if \( \alpha \in \mathcal{R}(f) \), but \( \alpha \notin \mathcal{R}(f) \) since

\[
\int_{0}^{1} \alpha(x) \, dx = 1 \quad \int_{0}^{1} \alpha(x) \, dx = 0
\]

(c) \( f(x) = \frac{1}{x^2} : (-1, 1) \to \mathbb{R} \). This function is defined but unbounded on \((-1, 1) \). If \( f \) could be approximated uniformly within \( \varepsilon \) by a polynomial, we would have \( ||f||_{\infty} \leq \varepsilon + ||P||_{\infty} < \infty \).

(d) Let

\[
f_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \leq \frac{1}{n} \\ 0 & \text{if } x > \frac{1}{n} \end{cases}
\]
3) Let $f$ be a continuous function on $\mathbb{R}$. Suppose that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists? You must justify your conclusion.

**Solution.** Yes. It follows that $f$ is differentiable at 0 and that $f'(0) = 3$. To prove this, let $\varepsilon > 0$. Since $f'(x) \to 3$ as $x \to 0$, there is a $\delta > 0$ such that $|f'(x) - 3| < \varepsilon$ for all $0 < |x| < \delta$. Then, by the mean value theorem, there is, for each $h > 0$, a $c_h$ strictly between 0 and $h$ such that $f(h) - f(0) = f'(c_h)h$. Hence, if $0 < |h| < \delta$, we have $0 < |c_h| < \delta$ and

$$\left| \frac{f(h) - f(0)}{h} - 3 \right| = |f'(c_h) - 3| < \varepsilon$$

This is the definition of $\lim_{h \to 0} \frac{f(h) - f(0)}{h} = 3$

4) Suppose that the function $f : [a, b] \to \mathbb{R}$ is differentiable and that there is a number $D$ such that $|f'(x)| \leq D$

for all $x \in [a, b]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, $T = \{t_1, \ldots, t_n\}$ be a choice for $P$ and $S(P, T, f) = \sum_{i=1}^{n} f(t_i)[x_i - x_{i-1}]$ be the corresponding Riemann sum. Prove that

$$\left| S(P, T, f) - \int_{a}^{b} f(x) \, dx \right| \leq D\|P\|(b - a)$$

where $\|P\| = \max_{1 \leq i \leq n} |x_i - x_{i-1}|$

**Solution.** We have

$$S(P, T, f) - \int_{a}^{b} f(x) \, dx = \sum_{i=1}^{n} f(t_i)[x_i - x_{i-1}] - \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x) \, dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} [f(t_i) - f(x)] \, dx$$

By the mean value theorem, $|f(t_i) - f(x)| = |f'(c)||t_i - x|$ for some $c$ between $t_i$ and $x$. Since $x_i - x_{i-1} \leq \|P\|$ and $x_{i-1} \leq t_i \leq x_i$ for all $1 \leq i \leq n$, we have $|f(t_i) - f(x)| \leq D|t_i - x| \leq D\|P\|$ for all $1 \leq i \leq n$. Hence

$$\left| S(P, T, f) - \int_{a}^{b} f(x) \, dx \right| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t_i) - f(x)| \, dx \leq D\|P\| \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} dx = D\|P\| \int_{a}^{b} dx = D\|P\|(b - a)$$

5) Let $\{f_n : [0, 1] \to \mathbb{R}\}_{n \in \mathbb{N}}$ be a sequence of continuous functions that obey $|f_n(y)| \leq 1$ for all $n \in \mathbb{N}$ and all $y \in [0, 1]$. Let $T : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous and define, for each $n \in \mathbb{N}$,

$$g_n(x) = \int_{0}^{1} T(x, y) f_n(y) \, dy$$

Prove that the sequence $\{g_n\}_{n \in \mathbb{N}}$ has a uniformly convergent subsequence.

**Solution.** We apply the Arzelà–Ascoli theorem. The three hypotheses are:

(i) The domain of interest is compact. In this case the domain is $[0, 1]$, which is certainly compact.

(ii) The set of functions $\{g_n\}_{n \in \mathbb{N}}$ is pointwise bounded. Since $[0, 1] \times [0, 1]$ is compact and $T$ is continuous, $T$ is bounded. So there is a $t_0$ such that $|T(x, y)| \leq t_0$ for all $0 \leq x, y \leq 1$ and

$$|g_n(x)| \leq \int_{0}^{1} |T(x, y) f_n(y)| \, dy \leq t_0 \int_{0}^{1} dy = t_0$$
So in fact \( \{g_n\}_{n \in \mathbb{N}} \) is uniformly bounded by \( t_0 \).

(iii) The set of functions \( \{g_n\}_{n \in \mathbb{N}} \) is equicontinuous. Let \( \varepsilon > 0 \). Since \([0,1] \times [0,1]\) is compact and \( T \) is continuous, \( T \) is uniformly continuous. So there is a \( \delta > 0 \) such that \( |T(x,y) - T(x',y')| < \varepsilon \) for all \( 0 \leq x, x', y, y' \leq 1 \) with \( |(x,y) - (x',y')| < \delta \). In particular, if \( 0 \leq x, x' \leq 1 \) with \( |x - x'| < \delta \)

\[
|g_n(x) - g_n(x')| = \left| \int_0^1 [T(x,y) - T(x',y)] f_n(y) \, dy \right| 
\leq \int_0^1 |T(x,y) - T(x',y)||f_n(y)| \, dy = t_0 < \varepsilon \int_0^1 \, dy = \varepsilon
\]

So \( \{g_n\}_{n \in \mathbb{N}} \) is indeed equicontinuous.

All hypotheses of the Arzelà–Ascoli theorem are satisfied. So that theorem gives the prescribed result.

6 (a) Let \( H = \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1 \} \). Prove that for any \( \varepsilon > 0 \) and any continuous function \( f : H \to \mathbb{R} \) there exists a function \( g(x,y) \) of the form

\[
g(x,y) = \sum_{m=0}^N \sum_{n=0}^N a_{m,n} x^{2m} y^{2n} \quad N \in \mathbb{Z}, \ N \geq 0, \ a_{m,n} \in \mathbb{R}
\]

such that

\[
\sup_{(x,y) \in H} |f(x,y) - g(x,y)| < \varepsilon
\]

(b) Does the result in (a) hold if \( H \) is replaced by the disk \( \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \} \)?

**Solution.** (a) Define

\[
\mathcal{A} = \left\{ \sum_{m=0}^N \sum_{n=0}^N a_{m,n} x^{2m} y^{2n} \mid N \in \mathbb{Z}, \ N \geq 0, \ a_{m,n} \in \mathbb{R} \right\}
\]

Then \( H \) is compact and \( \mathcal{A} \)

- is an algebra of continuous functions on \([0,1]\)
- vanishes nowhere, since the function \( g(x,y) = 1 \in \mathcal{A} \)
- separates points, since if \((x,y) \neq (x',y')\), then either \( x \neq x' \) or \( y \neq y' \). In the former case \((x,y)\) and \((x',y')\) are separated by \( g(x,y) = x^2 \in \mathcal{A} \) and in the latter case, they are separated by \( g(x,y) = y^2 \in \mathcal{A} \).

So, by the Stone–Weierstrass theorem, the uniform closure of \( \mathcal{A} \) is the set of all continuous functions on \( H \).

(b) No, \( \mathcal{A} \) no longer separates \((x,y)\) and, for example, \((x',y') = (-x,y)\) for \( x \neq 0 \). In fact every \( g(x,y) \in \mathcal{A} \) obeys \( g(-x,-y) = g(-x,y) = g(x,-y) = g(x,y) \). These equations are preserved under uniform (and also under pointwise) limits and so are obeyed by every \( g \in \overline{\mathcal{A}} \). So, for example, \( f(x,y) = x \notin \overline{\mathcal{A}} \).

7) The Legendre polynomials \( P_n(x) : [-1,1] \to \mathbb{R}, n \in \mathbb{N}, n \geq 0, \) are polynomials obeying

(i) \( P_n \) is of degree \( n \) with the coefficient of \( x^n \) strictly greater than zero and

(ii) \( \int_{-1}^1 P_n(x) P_m(x) \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2}{2n+1} & \text{if } n = m \end{cases} \)

Let \( f : [-1,1] \to \mathbb{R} \) be continuous and set \( a_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) \, dx \). Prove that

(a) \( \sum_{n=0}^\infty \frac{2}{2n+1} |a_n|^2 \leq \int_{-1}^1 f(x)^2 \, dx \) with equality if and only if \( \sum_{n=0}^N a_n P_n(x) \) converges to \( f \) in the mean as \( N \to \infty \).

(b) \( \sum_{n=0}^\infty a_n P_n(x) \) converges in the mean to \( f(x) \).
Solution. (a)

\[
\int_{-1}^{1} \left| f(x) - \sum_{n=0}^{N} a_n P_n(x) \right|^2 \, dx = \int_{-1}^{1} |f(x)|^2 \, dx - 2 \sum_{n=0}^{N} a_n \int_{-1}^{1} f(x)P_n(x) \, dx \\
+ \sum_{m=0}^{N} \sum_{n=0}^{N} a_m a_n \int_{-1}^{1} P_m(x)P_n(x) \, dx
\]

The middle integral is \(\frac{2}{2n+1} a_n\) and the last integral is \(\frac{2}{2n+1}\) when \(m = n\) and zero otherwise. So

\[
\int_{-1}^{1} \left| f(x) - \sum_{n=0}^{N} a_n P_n(x) \right|^2 \, dx = \int_{-1}^{1} |f(x)|^2 \, dx - \frac{2}{2n+1} a_n^2
\]

This is positive, for all \(N\), which gives the desired inequality. Furthermore this tends to zero in the limit \(N \to \infty\) if and only if the left hand side tends to zero, which is the case precisely when \(f\) is the limit in the mean of \(\sum_{n=0}^{N} a_n P_n(x)\).

(b) The same calculations in part (a), and as done in class, show that the choice \(c_n = a_n\) minimizes \(\int_{-1}^{1} \left| f(x) - \sum_{n=0}^{N} c_n P_n(x) \right|^2 \, dx \). Furthermore, by Stone–Weierstrass, there is, for each \(\varepsilon > 0\), a polynomial \(Q\) which approximates \(f\) uniformly to within \(\varepsilon\) and hence in the mean to within \(\varepsilon \sqrt{2}\). As any polynomial can be written as a finite linear combination of Legendre polynomials (it suffices to prove this for each \(x^m\) and this is done easily by induction), there is a \(\sum_{n=0}^{N} c_n P_n(x)\) such that

\[
\int_{-1}^{1} \left| f(x) - \sum_{n=0}^{N} c_n P_n(x) \right|^2 \, dx < 2\varepsilon^2
\]

So if \(N\) is large enough \(\int_{-1}^{1} \left| f(x) - \sum_{n=0}^{N} a_n P_n(x) \right|^2 \, dx < 2\varepsilon^2\).