1) Let $\mathcal{E}$ be the space of $2\pi$-periodic, continuously differentiable, real-valued functions on $\mathbb{R}$ with $\int_{-\pi}^{\pi} f(x) \, dx = 0$; i.e.,

$$\mathcal{E} = \{ f : \mathbb{R} \to \mathbb{R} \mid f \in C^1 \text{ on } \mathbb{R}, \int_{-\pi}^{\pi} f(x) \, dx = 0, \text{ and } f(x+2\pi) = f(x) \text{ for all } x \in \mathbb{R} \}$$

Let the norm $\| \cdot \|_{1,\infty}$ on $\mathcal{E}$ be defined as

$$\| f \|_{1,\infty} = \sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \in \mathbb{R}} |f'(x)|$$

Let $d_1$ be a metric on $\mathcal{E}$ defined as $d_1(f, g) = \| f - g \|_{1,\infty}$, for $f, g \in \mathcal{E}$.

(a) Recall that for $n \in \mathbb{Z}$ the Fourier coefficient $\hat{f}(n)$ is given as $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx$. Prove that for $f \in \mathcal{E}$, $|\hat{f}(n)| \leq \| f \|_{1,\infty}$ and $|n| \| \hat{f}(n) \| \leq \| f \|_{1,\infty}$ for all $n \in \mathbb{Z}$.

(b) Prove that the space $\mathcal{E}$ equipped with the metric $d_1$ is a complete metric space. (Hint: You may want to use the Fundamental Theorem of Calculus.)

(c) Prove that for every $f \in \mathcal{E}$, its Fourier series converges uniformly to $f$ on $\mathbb{R}$. In particular, $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$.

2) Let $\mathcal{E}$ be the space of functions as given in Problem 1. You are allowed/encouraged to use the results/statements of Problem 1.

Let $R$ be an operation on the functions in $\mathcal{E}$, defined as following: for $f \in \mathcal{E}$ written as the infinite sum $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$ (this is possible for all $f \in \mathcal{E}$ because of Problem 1(c)), let the function $R[f]$ on $\mathbb{R}$ be defined as

$$R[f](x) = \sum_{n=-\infty}^{\infty} |\hat{f}(100n)|^2 e^{inx}$$

Of course, for this definition to work, we need to prove that the series expression converges. Verify this and furthermore prove that $R$ gives a map from $\mathcal{E}$ to $\mathcal{E}$. Namely,

(a) prove that the series $\sum_{n=-\infty}^{\infty} |\hat{f}(100n)|^2 e^{inx}$ converges uniformly on $\mathbb{R}$;

(b) prove that $R[f]$ is a real-valued function on $\mathbb{R}$. (Hint: $f \in \mathcal{E}$ is real-valued);

(c) prove that $R[f]$ is $2\pi$-periodic;

(d) prove that $\int_{-\pi}^{\pi} R[f](x) \, dx = 0$;

(e) prove that $R[f]$ is a continuously differentiable function on $\mathbb{R}$.

3) Let $\mathcal{E}$ be the space of functions as given in Problem 1 and let $K : \mathcal{E} \to \mathcal{E}$ be a mapping that satisfies for all $f, g \in \mathcal{E}$, $d_1(K[f], K[g]) \leq \frac{1}{2} d_1(f, g)$. Here, the metric $d_1$ is given in Problem 1. Assume that $K[0] = 0$. Here, 0 denotes the constant zero function. Let $F : \mathcal{E} \to \mathcal{E}$ be defined as $F[f] = f + K[f]$, for $f \in \mathcal{E}$.

(a) Prove that $F$ is injective, i.e. if $F[f] = F[g]$ then $f = g$.

(b) Prove that for all $g \in \mathcal{E}$, with $\|g\|_{1,\infty} \leq \frac{1}{2}$ , there exists $f \in \mathcal{E}$, with $\|f\|_{1,\infty} \leq 1$, such that $F[f] = g$.

4) Let $f : [0, 1] \to \mathbb{R}$ be a Riemann integrable function, such that $|f| \leq 1$ on $[0, 1]$. Suppose

$$\int_{0}^{1} f(x)x^n \, dx = 0$$

for all $n \in \mathbb{Z}$, with $n \geq 0$. Let $x_0$ be a point in the interval $[0, 1]$. Assume that $f$ is continuous at $x_0$.

Prove that $f(x_0) = 0$.

[Hint: From one of the HW problems, we know that this holds if $f$ is continuous on $[0, 1]$. But, in this problem $f$ is assumed to be continuous only at a fixed point $x_0$.]
6) Let \( i = \sqrt{-1} \) be the pure imaginary number. Let \( \{ f_n \} \) be the sequence of functions \( f_n : \mathbb{R} \to \mathbb{C} \) defined as \( f_n(x) = e^{inx} \) for \( x \in \mathbb{R} \), i.e. \( f_1(x) = e^{ix}, f_2(x) = e^{i2x}, f_3(x) = e^{i3x}, \ldots \). Let \( \{ g_n \} \) be a sequence of \( 2\pi \)-periodic Riemann integrable functions \( g_n : \mathbb{R} \to \mathbb{R} \). Assume that for all \( n \), \( |g_n(x) - g_n(y)| \leq |x - y| \) for all \( x, y \in \mathbb{R} \). Consider the sequence of \( 2\pi \)-periodic functions \( \{ f_n * g_n \} \), where \( f_n * g_n : \mathbb{R} \to \mathbb{C} \) is given by

\[
    f_n * g_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(x-t) g_n(t) \, dt \quad \text{for } x \in \mathbb{R}
\]

(a) Prove that there exists a subsequence \( \{ h_k \} \) of \( \{ f_n * g_n \} \) such that as \( k \to \infty \), \( h_k \) converges uniformly to a function, say \( h_\infty \), on \( \mathbb{R} \).

(Hint: Observe that for \( n \geq 1 \), \( f_n * g_n = f_n * (g_n - g_n(0)) \).

(b) Let \( \{ u_k \} \) be a subsequence of \( \{ f_n * g_n \} \) such that \( \{ u_k \} \) converges uniformly on \( \mathbb{R} \). (Such a uniformly convergent subsequence exists by the result of (a).) Prove that as \( k \to \infty \), \( u_k \to 0 \) uniformly on \( \mathbb{R} \), i.e.

\[
    \limsup_{k \to \infty} |u_k(x)| = 0.
\]

(Hint: this problem could be hard.)

(c) Use (a) (or the solution of (a)) and (b) to prove that in fact \( f_n * g_n \to 0 \) uniformly on \( \mathbb{R} \).

**Extra Problem** (This problem is only for extra marks, and it could be difficult. Do not try this unless you have time left after finishing all the previous problems.)

Let \( \mathcal{E} \) be the space of functions given in Problem 1, and let \( R : \mathcal{E} \to \mathcal{E} \) be the mapping given in Problem 2. Let \( X = \{ f \in \mathcal{E} \mid \| f \|_{1,\infty} \leq 1 \} \), i.e. \( X \) is the subset of \( \mathcal{E} \) consisting of functions \( f \) with \( \| f \|_{1,\infty} \). Prove that \( R \) satisfies that for all \( f, g \in X \),

\[
    d_1(R[f], R[g]) \leq \frac{1}{2} d_1(f, g)
\]

Here, the metric \( d_1 \) is as given in Problem 1. You are allowed/encouraged to use the results/statements of Problem 1 and Problem 2. (Hint: If you can solve Problem 2(e), it is likely that you can solve this problem. You may want to use the identity \( |a|^2 - |b|^2 = (|a| + |b|)(|a| - |b|) \) and the inequality

\[
    |a| - |b| \leq |a - b|
\]

for complex numbers \( a, b \in \mathbb{C} \).