1) Define
(a) \( f_a^b f(x) \, dx \)
(b) a self-adjoint algebra of functions
(c) the Fourier series of a function

2) Give an example of each of the following, together with a brief explanation of your example. If an example does not exist, explain why not.
(a) A differentiable function which is not monotonic but whose derivative obeys \( |f'(x)| \geq 1 \).
(b) Two functions \( f, \alpha : [0, 1] \to \mathbb{R} \) with \( f \) continuous, but \( f \notin \mathcal{R}(\alpha) \) on \([0, 1]\).
(c) A continuous function \( f : (-1, 1) \to \mathbb{R} \) that cannot be uniformly approximated by a polynomial.
(d) A monotonically decreasing sequence of functions \( f_n : [0, 1] \to \mathbb{R} \) which converges pointwise, but not uniformly to zero.

3) Let \( f \) be a continuous function on \( \mathbb{R} \). Suppose that \( f'(x) \) exists for all \( x \neq 0 \) and that \( f'(x) \to 3 \) as \( x \to 0 \). Does it follow that \( f'(0) \) exists? You must justify your conclusion.

4) Suppose that the function \( f : [a, b] \to \mathbb{R} \) is differentiable and that there is a number \( D \) such that

\[ |f'(x)| \leq D \]

for all \( x \in [a, b] \). Let \( P = \{x_0, x_1, \ldots, x_n\} \) be a partition of \([a, b]\), \( T = \{t_1, \ldots, t_n\} \) be a choice for \( P \) and \( S(P, T, f) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \) be the corresponding Riemann sum. Prove that

\[ \left| S(P, T, f) - \int_{a}^{b} f(x) \, dx \right| \leq D\|P\|(b-a) \quad \text{where } \|P\| = \max_{1 \leq i \leq n} |x_i - x_{i-1}| \]

5) Let \( \{f_n : [0, 1] \to \mathbb{R}\}_{n \in \mathbb{N}} \) be a sequence of continuous functions that obey \( |f_n(y)| \leq 1 \) for all \( n \in \mathbb{N} \) and all \( y \in [0, 1] \). Let \( T : [0, 1] \times [0, 1] \to \mathbb{R} \) be continuous and define, for each \( n \in \mathbb{N} \),

\[ g_n(x) = \int_{0}^{1} T(x, y) f_n(y) \, dy \]

Prove that the sequence \( \{g_n\}_{n \in \mathbb{N}} \) has a uniformly convergent subsequence.

6) (a) Let \( H = \{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1 \} \). Prove that for any \( \varepsilon > 0 \) and any continuous function \( f : H \to \mathbb{R} \) there exists a function \( g(x, y) \) of the form

\[ g(x, y) = \sum_{m=0}^{N} \sum_{n=0}^{N} a_{m,n} x^{2m} y^{2n} \quad N \in \mathbb{Z}, \ N \geq 0, \ a_{m,n} \in \mathbb{R} \]

such that

\[ \sup_{(x,y) \in H} |f(x,y) - g(x,y)| < \varepsilon \]

(b) Does the result in (a) hold if \( H \) is replaced by the disk \( \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \} \)?

7) The Legendre polynomials \( P_n(x) : [-1, 1] \to \mathbb{R}, \ n \in \mathbb{Z}, \ n \geq 0 \), are polynomials obeying

(i) \( P_n \) is of degree \( n \) with the coefficient of \( x^n \) strictly greater than zero and

(ii) \( \int_{-1}^{1} P_n(x)P_m(x) \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2}{2n+1} & \text{if } n = m \end{cases} \)

Let \( f : [-1, 1] \to \mathbb{R} \) be continuous and set \( a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x) \, dx \). Prove that

(a) \( \sum_{n=0}^{\infty} \frac{2}{2n+1} |a_n|^2 \leq \int_{-1}^{1} f(x)^2 \, dx \) with equality if and only if \( \sum_{n=0}^{N} a_n P_n(x) \) converges to \( f \) in the mean as \( N \to \infty \).

(b) \( \sum_{n=0}^{\infty} a_n P_n(x) \) converges in the mean to \( f(x) \).