The Omnibus Theorem of (Main) Convergence Tests

Definition 1  Let $\mathcal{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{F}$.

(a) The sequence $(a_n)_{n \in \mathbb{N}}$ converges to $A \in \mathcal{F}$ if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } |a_n - A| < \varepsilon \text{ whenever } n \geq N$$

A sequence which does not converge to any $A$ is said to diverge.

(b) The sequence $(a_n)_{n \in \mathbb{N}}$ is said to be Cauchy if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } |a_m - a_n| < \varepsilon \text{ whenever } m, n \geq N$$

(c) The series $\sum_{n=1}^{\infty} a_n$ is said to converge to $S$ if the sequence $(s_n)_{n \in \mathbb{N}}$ of partial sums $s_n = \sum_{k=1}^{n} a_k$ converges to $S$. A series which does not converge to any $S$ is said to diverge.

(d) The series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if the $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem 2  Let $(a_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}}$ and $(d_n)_{n \in \mathbb{N}}$ be sequences of real or complex numbers and $s_n = \sum_{k=1}^{n} a_k$. Then

(a) (Cauchy Criterion)

$$\sum_{k=1}^{\infty} a_k \text{ converges } \iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } \left| \sum_{k=n}^{m} a_k \right| < \varepsilon \text{ whenever } m \geq n \geq N$$

(b) (Divergence Test) $\sum_{n=1}^{\infty} a_n \text{ converges } \implies \lim_{n \to \infty} a_n = 0$

(c) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then

$$\sum_{k=1}^{\infty} a_k \text{ converges } \iff (s_n)_{n \in \mathbb{N}} \text{ is bounded}$$

(d) absolute convergence $\implies$ convergence

(e) (Comparison Test) Let $N_0 \in \mathbb{N}$.

(i) $|a_n| \leq c_n \ \forall n \geq N_0$ and $\sum_{n=1}^{\infty} c_n \text{ converges } \implies \sum_{n=1}^{\infty} a_n \text{ converges absolutely}$

(ii) $a_n \geq d_n \geq 0 \ \forall n \geq N_0$ and $\sum_{n=1}^{\infty} d_n \text{ diverges } \implies \sum_{n=1}^{\infty} a_n \text{ diverges}$
(f) (Integral Test) Let $c \in \mathbb{R}$. If $f(x)$ is a real valued function which is defined and continuous for all $x \geq c$ and which obeys

(i) $f(x) \geq 0 \ \forall x \geq c$
(ii) $f(x)$ is monotonically decreasing, i.e. $y > x \implies f(y) \leq f(x)$
(iii) $f(n) = a_n \ \forall n \geq c$

then

$$\sum_{n=1}^{\infty} a_n \text{ converges } \iff \int_c^{\infty} f(x) \, dx < \infty$$

Furthermore, if $N \geq c$, the truncation error

$$\left| \sum_{n=1}^{\infty} a_n - s_N \right| \leq \int_N^{\infty} f(x) \, dx$$

(g) (Examples)

(i) (Geometric Series) $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

(ii) (p Test) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

(iii) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

(h) (Root Test) Let $\alpha = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$. Then

$$\alpha < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolutely}$$
$$\alpha > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges}$$
$$\alpha = 1 \implies \text{nothing}$$

(i) (Ratio Test) If $(a_n)_{n \in \mathbb{N}}$ is a sequence of nonzero numbers, then

$$\sum_{n=1}^{\infty} a_n \text{ converges absolutely if } \limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$
$$\text{diverges if } \left| \frac{a_{n+1}}{a_n} \right| \geq 1 \text{ for all } n \text{ bigger than some natural number } N$$
(i) (Alternating Series Test) If

(i) \( a_n \geq 0 \quad \forall n \in \mathbb{N} \)
(ii) \( a_{n+1} \leq a_n \quad \forall n \in \mathbb{N} \)
(iii) \( \lim_{n \to \infty} a_n = 0 \)

then \( s = \sum_{n=1}^{\infty} (-1)^{n-1}a_n \) converges and \( s - s_n \) is between 0 and (the first dropped term) \((-1)^n a_{n+1}\).

Proof: (a) ⇒ Let \( \lim_{n \to \infty} s_n = s \) and let \( \varepsilon > 0 \). Then

\[ \exists \tilde{N} \in \mathbb{N} \text{ s.t. } n \geq \tilde{N} \implies |s_n - s| < \frac{\varepsilon}{2} \]

Set \( N = \tilde{N} + 1 \). Then

\[ m \geq n \geq N \implies m > n - 1 \geq \tilde{N} \implies |s_m - s| < \frac{\varepsilon}{2}, |s_{n-1} - s| < \frac{\varepsilon}{2} \]
\[ \implies |s_m - s_{n-1}| = |(s_m - s) + (s - s_{n-1})| < \varepsilon \]
\[ \implies \left| \sum_{k=n}^{m} a_k \right| < \varepsilon \]

(a) ⇐ By hypothesis

\( \forall \varepsilon > 0 \exists N_{\varepsilon} \in \mathbb{N} \text{ s.t. } m \geq n \geq N_{\varepsilon} \implies |s_m - s_n| < \varepsilon \)

\( \implies (s_n)_{n \in \mathbb{N}} \text{ is bounded} \)

(Choosing \( \varepsilon = 1, n = N_1 \), we see that \( \{ s_m \mid m \geq N_1 \} \) is bounded.)
\( \implies (s_n)_{n \in \mathbb{N}} \text{ has a subsequential limit. Say } \lim_{n \to \infty} s_{i_n} = s. \)
\( \implies \forall \varepsilon > 0 \ n \geq N_{\varepsilon/2} \implies |s_n - s| = |s_n - s_{i_n} + s_{i_n} - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \)

if we choose \( m \) large enough that \( i_m \geq N_{\varepsilon/2} \) and \( |s_{i_m} - s| < \frac{\varepsilon}{2} \).

(b) Apply part (a) with \( m = n. \)

(c) As \( a_n \geq 0 \) for all \( n \in \mathbb{N} \), the sequence \( (s_n)_{n \in \mathbb{N}} \) is monotonically increasing. So it converges if and only if it is bounded.

(d) If \( \sum_{k=1}^{\infty} |a_k| \) converges, then \( \sum_{k=1}^{\infty} |a_k| \) obeys the Cauchy criterion. As \( \left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k|, \)
the series \( \sum_{k=1}^{\infty} a_k \) then also obeys the Cauchy criterion and hence converges.
(e.i) The sequence \( \{ t_n = \sum_{k=1}^{n} |a_k| \} \) is monotonic and bounded above by \( \sum_{k=1}^{N_0} |a_k| + \sum_{k=1}^{\infty} c_k \) and hence converges. So \( \sum_{k=1}^{\infty} a_k \) converges absolutely and hence also converges by part (d).

(e.ii) If \( \sum_{k=1}^{\infty} a_k \) were to converge, then \( \sum_{k=1}^{\infty} d_k \) would also converge by part (e.i). This is a contradiction.

(f) Let \( N \) be any fixed integer bigger than \( c + 1 \). Then \( \sum_{k=1}^{\infty} a_k \) converges if and only if \( \sum_{k=N}^{\infty} a_k \) converges, which in turn is the case if and only if the sequence \( t_n = \sum_{k=1}^{n} a_k \) is bounded, since \( a_k \geq 0 \) for all \( k \geq N > c \). As \( f(x) \) decreases as \( x \) increases, we have that \( f(x) \geq a_k \) when \( x \leq k \) and \( f(x) \leq a_k \) when \( x \geq k \). Thus, for any \( k \geq N \)

\[
\int_{k-1}^{k} f(x) \, dx \leq a_k \leq \int_{k}^{k+1} f(x) \, dx
\]

Summing \( k \) from \( N \) to \( n \),

\[
\int_{N}^{n+1} f(x) \, dx \leq \sum_{k=N}^{n} a_k \leq \int_{N-1}^{n} f(x) \, dx
\]

So the \( t_n \)'s are bounded if and only if \( \int_{c}^{\infty} f(x) \, dx < \infty \).

(g,i) Exercise.

(g.ii) If \( p \leq 0 \), the series \( \sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} n^{-|p|} \) diverges by part (b). The remaining cases are easily proven using the integral test with \( c = 1 \) and \( f(x) = \frac{1}{x^p} \), which is positive and monotonically decreasing for \( p > 0 \). We just have to observe that

\[
\int_{1}^{L} \frac{1}{x^p} \, dx = \begin{cases} 
\ln L & \text{if } p = 1 \\
\frac{L^{1-p} - 1}{1-p} & \text{if } p \neq 1
\end{cases}
\]

grows unboundedly as \( L \to \infty \) when \( p \leq 1 \) and converges to \( \frac{1}{1-p} \) when \( p > 1 \).

Here is a second proof that does not use integrals. If \( p > 0 \), each successive term in the series is smaller than the term before. So each term with index \( n \) between \( 2^m \) and \( 2^{m+1} \) obeys

\[
\frac{1}{2^{(m+1)p}} \leq \frac{1}{n^p} \leq \frac{1}{2^{mp}}
\]

There are \( 2^{m+1} - 2^m \) such terms. Hence

\[
\frac{1}{2} 2^{(m+1)(1-p)} = \frac{2^{m+1} - 2^m}{2^{(m+1)p}} \leq \sum_{2^m \leq n < 2^{m+1}} \frac{1}{n^p} \leq \frac{2^{m+1} - 2^m}{2^{mp}} = \frac{1}{2^{m(p-1)}}
\]
If $0 \leq p \leq 1$, the left hand bound $\frac{1}{2} 2^{(m+1)(1-p)} \geq \frac{1}{2}$. Summing $m$ from 0 to $N-1$, we have $\sum_{n=1}^{2^N-1} \frac{1}{n^p} \geq \frac{N}{2}$ which forces divergence. If $p > 1$, then summing $m$ from 0 to $N-1$ in the right hand bound, we have $\sum_{n=1}^{2^N-1} \frac{1}{n^p} \leq \sum_{m=0}^{N-1} \frac{(1-\frac{1}{2^{p-1}})^m}{p-1} \leq (1 - \frac{1}{2^{p-1}})^{-1}$. So the series is bounded and hence converges by part (c).

(g.iii) If $p \leq 0$, then for all $n \geq 2$, $\frac{1}{n(\ln n)^p} \geq \frac{(\ln 2)^{|p|}}{n}$ and the series diverges by the comparison test. Again, the remaining cases are easily proven using the integral test, this time with $c = 2$ and $f(x) = \frac{1}{x(\ln x)^p}$. To compute the integral, we make the change of variables $y = \ln x$, $dy = \frac{dx}{x}$.

$$\int_2^L \frac{1}{x(\ln x)^p} dx = \int_{\ln 2}^{\ln L} \frac{1}{y^p} dy = \begin{cases} \ln(\ln L) - \ln(\ln 2) & \text{if } p = 1 \\ \frac{(\ln L)^{1-p} - (\ln 2)^{1-p}}{1-p} & \text{if } p \neq 1 \end{cases}$$

$$\to \begin{cases} \infty & \text{if } p \leq 1 \\ \frac{(\ln 2)^{1-p}}{p-1} & \text{if } p > 1 \end{cases}$$ as $L \to \infty$

Again, here is a second proof that does not use the integral test. If $p > 0$, each term with index $n$ between $2^m$ and $2^{m+1}$ obeys

$$\frac{1}{2^{m+1}(m+1)^p(\ln 2)^p} \leq \frac{1}{n(\ln n)^p} \leq \frac{1}{2^m(m\ln 2)^p}$$

Hence

$$\frac{1}{2(\ln 2)^p} \frac{1}{(m+1)^p} = \frac{2^{m+1-2^m}}{2^{m+1}(m+1)^p(\ln 2)^p} \leq \sum_{2^m \leq n < 2^{m+1}} \frac{1}{n(\ln n)^p} \leq \frac{2^{m+1-2^m}}{2^m(m\ln 2)^p} = \frac{1}{(\ln 2)^p} \frac{1}{m^p}$$

Summing $m$ from 1 to $N-1$, we have

$$\frac{1}{2(\ln 2)^p} \sum_{m=1}^{N-1} \frac{1}{(m+1)^p} \leq \sum_{n=2}^{2^N-1} \frac{1}{n(\ln n)^p} \leq \frac{1}{(\ln 2)^p} \sum_{m=1}^{N-1} \frac{1}{m^p}$$

So if $p > 1$, the right hand inequality together with part (g.ii) shows that the series is bounded and hence converges, by part (c), and if $p \leq 1$, the left hand inequality together with part (g.ii) shows that the series diverges.

(h) If $\alpha < 1$, pick any $r$ obeying $\alpha < r < 1$. There is a natural number $N$ such that $\sqrt[n]{|a_n|} < r$, or equivalently, $|a_n| < r^n$, for all $n \geq N$. So convergence follows by comparison with the geometric series. On the other hand, if $\alpha > 1$, there must be a subsequence with $\sqrt[n]{|a_{n_k}|} \geq 1$, or equivalently, $|a_{n_k}| \geq 1$. So the series diverges by part (b).
(i) If \(|\frac{a_{n+1}}{a_n}| \geq 1\) for all \(n \geq N\), then \(|a_n| \geq |a_N| > 0\) for all \(n \geq N\) and the series diverges by part (b). On the other hand, if \(\limsup_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| < 1\), there is a \(r < 1\) and an \(N \in \mathbb{N}\) such that \(|\frac{a_{n+1}}{a_n}| \leq r\) for all \(n \geq N\). Consequently \(|a_{N+1}| \leq r|a_N|, |a_{N+2}| \leq r|a_{N+1}| \leq r^2|a_N|\) and so on. Iterating gives \(|a_{N+p}| \leq r^p|a_N|\) which implies convergence by comparison with the geometric series.

(j) Exercise