The Omnibus Theorem of (Main) Convergence Tests

Definition 1 Let $\mathcal{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $\big(a_n\big)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$.

(a) The sequence $\big(a_n\big)_{n\in\mathbb{N}}$ converges to $A \in \mathcal{F}$ if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \ \text{ s.t. } \big|a_n - A\big| < \varepsilon \text{ whenever } n \geq N$$

A sequence which does not converge to any A is said to diverge.

(b) The sequence $\big(a_n\big)_{n\in\mathbb{N}}$ is said to be Cauchy if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \ \text{ s.t. } \big|a_m - a_n\big| < \varepsilon \text{ whenever } m, n \geq N$$

(c) The series $\sum_{n=1}^{\infty} a_n$ is said to converge to S if the sequence $\big(s_n\big)_{n\in\mathbb{N}}$ of partial sums $s_n = \sum\limits_{k=1}^{n} a_k$ converges to S. A series which does not converge to any S is said to diverge.

(d) The series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if the $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem 2 *Let $\big(a_n\big)_{n\in\mathbb{N}}$, $\big(c_n\big)_{n\in\mathbb{N}}$ and $\big(d_n\big)_{n\in\mathbb{N}}$ be sequences of real or complex numbers and $s_n = \sum\limits_{k=1}^{n} a_k$. Then*

(a) (Cauchy Criterion)

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \ \text{ s.t. } \Big|\sum_{k=n}^{m} a_k\Big| < \varepsilon \text{ whenever } m \geq n \geq N$$

(b) (Divergence Test) $\sum\limits_{n=1}^{\infty} a_n$ *converges* $\implies \lim\limits_{n\to\infty} a_n = 0$

(c) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \big(s_n\big)_{n\in\mathbb{N}} \text{ is bounded}$$

(d) absolute convergence $\implies$ convergence

(e) (Comparison Test) Let $N_0 \in \mathbb{N}$.

(i) $|a_n| \leq c_n\ \forall n \geq N_0$ *and* $\sum\limits_{n=1}^{\infty} c_n$ *converges* $\implies \sum\limits_{n=1}^{\infty} a_n$ *converges absolutely*

(ii) $a_n \geq d_n \geq 0\ \forall n \geq N_0$ *and* $\sum\limits_{n=1}^{\infty} d_n$ *diverges* $\implies \sum\limits_{n=1}^{\infty} a_n$ *diverges*

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(f) (Integral Test) Let $c \in \mathbb{R}$. If $f(x)$ is a real valued function which is defined and continuous for all $x \geq c$ and which obeys

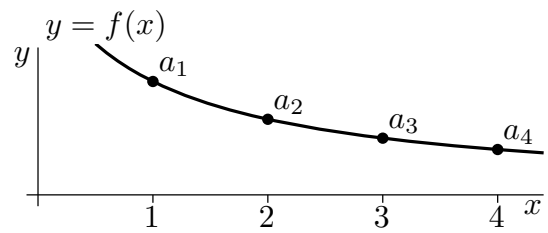
(i) $f(x) \geq 0 \quad \forall x \geq c$

(ii) *$f(x)$ is monotonically decreasing, i.e.* $y > x \implies f(y) \leq f(x)$

(iii) $f(n) = a_n \quad \forall n \geq c$

then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_c^{\infty} f(x)\, dx < \infty$$



Furthermore, if $N \geq c$, the truncation error

$$\left| \sum_{n=1}^{\infty} a_n - s_N \right| \leq \int_N^{\infty} f(x)\, dx$$

(g) (Examples)

(i) (Geometric Series) $\sum\limits_{n=0}^{\infty} r^n$ *converges to* $\frac{1}{1-r}$ *if* $|r| < 1$ *and diverges if* $|r| \geq 1$.

(ii) (p Test) $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

(iii) $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

(h) (Root Test) Let $\alpha = \limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}$. *Then*

$$\alpha < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolutely}$$

$$\alpha > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges}$$

$$\alpha = 1 \implies \text{nothing}$$

(i) (Ratio Test) If $\big(a_n\big)_{n\in\mathbb{N}}$ *is a sequence of nonzero numbers, then*

$$\sum_{n=1}^{\infty} a_n \text{ converges absolutely if } \limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$\text{diverges if } \left| \frac{a_{n+1}}{a_n} \right| \geq 1 \text{ for all } n \text{ bigger than some natural number } N$$

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(j) (Alternating Series Test) If

(i) $a_n \geq 0 \quad \forall n \in \mathbb{N}$

(ii) $a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$

(iii) $\lim_{n\to\infty} a_n = 0$

then $s = \sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ *converges and* $s - s_n$ *is between* 0 *and (the first dropped term)* $(-1)^n a_{n+1}$.

Proof: (a) $\Rightarrow$ Let $\lim\limits_{n\to\infty} s_n = s$ and let $\varepsilon > 0$. Then

$$\exists \tilde{N} \in \mathbb{N} \text{ s.t. } n \geq \tilde{N} \implies |s_n - s| < \tfrac{\varepsilon}{2}$$

Set $N = \tilde{N} + 1$. Then

$$\begin{aligned} m \geq n \geq N \implies m > n-1 \geq \tilde{N} &\implies |s_m - s| < \tfrac{\varepsilon}{2},\ |s_{n-1} - s| < \tfrac{\varepsilon}{2} \\ &\implies |s_m - s_{n-1}| = |(s_m - s) + (s - s_{n-1})| < \varepsilon \\ &\implies \left| \sum_{k=n}^{m} a_k \right| < \varepsilon \end{aligned}$$

(a) $\Leftarrow$ By hypothesis

$$\begin{aligned} &\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N} \text{ s.t. } m \geq n \geq N_\varepsilon \implies |s_m - s_n| < \varepsilon \\ &\implies \big(s_n\big)_{n\in\mathbb{N}} \text{ is bounded} \\ &\qquad \text{(Choosing } \varepsilon = 1,\ n = N_1\text{, we see that } \{\ s_m \mid m \geq N_1\ \} \text{ is bounded.)} \\ &\implies \big(s_n\big)_{n\in\mathbb{N}} \text{ has a subsequential limit. Say } \lim_{n\to\infty} s_{i_n} = s. \\ &\implies \forall \varepsilon > 0 \quad n \geq N_{\varepsilon/2} \implies |s_n - s| = \left|s_n - s_{i_m} + s_{i_m} - s\right| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon \end{aligned}$$

if we choose m large enough that $i_m \geq N_{\varepsilon/2}$ and $\left|s_{i_m} - s\right| < \frac{\varepsilon}{2}$.

(b) Apply part (a) with $m = n$.

(c) As $a_n \geq 0$ for all $n \in \mathbb{N}$, the sequence $\big(s_n\big)_{n\in\mathbb{N}}$ is monotonically increasing. So it converges if and only if it is bounded.

(d) If $\sum\limits_{k=1}^{\infty} |a_k|$ converges, then $\sum\limits_{k=1}^{\infty} |a_k|$ obeys the Cauchy criterion. As $\left| \sum\limits_{k=n}^{m} a_k \right| \leq \sum\limits_{k=n}^{m} |a_k|$, the series $\sum\limits_{k=1}^{\infty} a_k$ then also obeys the Cauchy criterion and hence converges.

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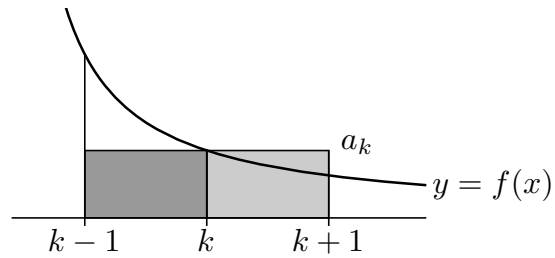
(e.i) The sequence $\left\{t_n = \sum\limits_{k=1}^{n} |a_k|\right\}$ is monotonic and bounded above by $\sum\limits_{k=1}^{N_0} |a_k| + \sum\limits_{k=1}^{\infty} c_k$ and hence converges. So $\sum\limits_{k=1}^{\infty} a_k$ converges absolutely and hence also converges by part (d).

(e.ii) If $\sum\limits_{k=1}^{\infty} a_k$ were to converge, then $\sum\limits_{k=1}^{\infty} d_k$ would also converge by part (e.i). This is a contradiction.

(f) Let N be any fixed integer bigger than $c+1$. Then $\sum\limits_{k=1}^{\infty} a_k$ converges if and only if $\sum\limits_{k=N}^{\infty} a_k$ converges, which in turn is the case if and only if the sequence $t_n = \sum\limits_{k=N}^{n} a_k$ is bounded, since $a_k \geq 0$ for all $k \geq N > c$. As $f(x)$ decreases as x increases, we have that $f(x) \geq a_k$ when $x \leq k$ and $f(x) \leq a_k$ when $x \geq k$. Thus, for any $k \geq N$

$$\int_k^{k+1} f(x)\ dx \leq a_k \leq \int_{k-1}^{k} f(x)\ dx$$



Summing k from N to n,

$$\int_N^{n+1} f(x)\ dx \leq \sum_{k=N}^{n} a_k \leq \int_{N-1}^{n} f(x)\ dx$$

So the t_n's are bounded if and only if $\int_c^{\infty} f(x)\ dx < \infty$.

(g,i) Exercise.

(g.ii) If $p \leq 0$, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p} = \sum\limits_{n=1}^{\infty} n^{|p|}$ diverges by part (b). The remaining cases are easily proven using the integral test with $c = 1$ and $f(x) = \frac{1}{x^p}$, which is positive and monotonically decreasing for $p > 0$. We just have to observe that

$$\int_1^L \frac{1}{x^p}\ dx = \begin{cases} \ln L & \text{if } p = 1 \\ \frac{L^{1-p}-1}{1-p} & \text{if } p \neq 1 \end{cases}$$

grows unboundedly as $L \to \infty$ when $p \leq 1$ and converges to $\frac{1}{p-1}$ when $p > 1$.

Here is a second proof that does not use integrals. If $p > 0$, each successive term in the series is smaller than the term before. So each term with index n between 2^m and 2^{m+1} obeys

$$\frac{1}{2^{(m+1)p}} \leq \frac{1}{n^p} \leq \frac{1}{2^{mp}}$$

There are $2^{m+1} - 2^m$ such terms. Hence

$$\tfrac{1}{2} 2^{(m+1)(1-p)} = \frac{2^{m+1}-2^m}{2^{(m+1)p}} \leq \sum_{2^m \leq n < 2^{m+1}} \frac{1}{n^p} \leq \frac{2^{m+1}-2^m}{2^{mp}} = \frac{1}{2^{m(p-1)}}$$

If $0 \le p \le 1$, the left hand bound $\frac{1}{2}2^{(m+1)(1-p)} \ge \frac{1}{2}$. Summing m from 0 to $N-1$, we have $\sum\limits_{n=1}^{2^N-1} \frac{1}{n^p} \ge \frac{N}{2}$ which forces divergence. If $p > 1$, then summing m from 0 to $N-1$ in the right hand bound, we have $\sum\limits_{n=1}^{2^N} \frac{1}{n^p} \le \sum\limits_{m=0}^{N-1} \left(\frac{1}{2^{p-1}}\right)^m \le \left(1 - \frac{1}{2^{p-1}}\right)^{-1}$. So the series is bounded and hence converges by part (c).

(g.iii) If $p \le 0$, then for all $n \ge 2$, $\frac{1}{n(\ln n)^p} \ge \frac{(\ln 2)^{|p|}}{n}$ and the series diverges by the comparison test. Again, the remaining cases are easily proven using the integral test, this time with $c = 2$ and $f(x) = \frac{1}{x(\ln x)^p}$. To compute the integral, we make the change of variables $y = \ln x$, $dy = \frac{dx}{x}$.

$$\int_2^L \frac{1}{x(\ln x)^p}\, dx = \int_{\ln 2}^{\ln L} \frac{1}{y^p}\, dy = \left\{ \begin{array}{ll} \ln(\ln L) - \ln(\ln 2) & \text{if } p = 1 \\ \frac{(\ln L)^{1-p} - (\ln 2)^{1-p}}{1-p} & \text{if } p \ne 1 \end{array} \right\}$$
$$\to \left\{ \begin{array}{ll} \infty & \text{if } p \le 1 \\ \frac{(\ln 2)^{1-p}}{p-1} & \text{if } p > 1 \end{array} \right\} \text{as } L \to \infty$$

Again, here is a second proof that does not use the integral test. If $p > 0$, each term with index n between 2^m and 2^{m+1} obeys

$$\frac{1}{2^{m+1}(m+1)^p(\ln 2)^p} \le \frac{1}{n(\ln n)^p} \le \frac{1}{2^m(m \ln 2)^p}$$

Hence

$$\frac{1}{2(\ln 2)^p}\frac{1}{(m+1)^p} = \frac{2^{m+1}-2^m}{2^{m+1}(m+1)^p(\ln 2)^p} \le \sum_{2^m \le n < 2^{m+1}} \frac{1}{n(\ln n)^p} \le \frac{2^{m+1}-2^m}{2^m(m \ln 2)^p} = \frac{1}{(\ln 2)^p}\frac{1}{m^p}$$

Summing m from 1 to $N-1$, we have

$$\frac{1}{2(\ln 2)^p} \sum_{m=1}^{N-1} \frac{1}{(m+1)^p} \le \sum_{n=2}^{2^N-1} \frac{1}{n(\ln n)^p} \le \frac{1}{(\ln 2)^p} \sum_{m=1}^{N-1} \frac{1}{m^p}$$

So if $p > 1$, the right hand inequality together with part (g.ii) shows that the series is bounded and hence converges, by part (c), and if $p \le 1$, the left hand inequality together with part (g.ii) shows that the series diverges.

(h) If $\alpha < 1$, pick any r obeying $\alpha < r < 1$. There is a natural number N such that $\sqrt[n]{|a_n|} < r$, or equivalently, $|a_n| < r^n$, for all $n \ge N$. So convergence follows by comparison with the geometric series. On the other hand, if $\alpha > 1$, there must be a subsequence with $\sqrt[n_k]{|a_{n_k}|} \ge 1$, or equivalently, $|a_{n_k}| \ge 1$. So the series diverges by part (b).

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(i) If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq N$, then $|a_n| \geq |a_N| > 0$ for all $n \geq N$ and the series diverges by part (b). On the other hand, if $\limsup\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, there is a $r < 1$ and an $N \in \mathbb{N}$ such that $\left|\frac{a_{n+1}}{a_n}\right| \leq r$ for all $n \geq N$. Consequently $|a_{N+1}| \leq r|a_N|$, $|a_{N+2}| \leq r|a_{N+1}| \leq r^2|a_N|$ and so on. Iterating gives $|a_{N+p}| \leq r^p|a_N|$ which implies convergence by comparison with the geometric series.

(j) Exercise ■

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