The Omnibus Theorem of Convergence Tests

Theorem. *Let* $\{a_n\}_{n\in\mathbb{N}}$, $\{c_n\}_{n\in\mathbb{N}}$ *and* $\{d_n\}_{n\in\mathbb{N}}$ *be sequences of real or complex numbers and* $s_n = \sum_{k=1}^{n} a_k$. *Then*

(1) (Cauchy Criterion)

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ s.t. } m \geq n \geq N \implies \Big| \sum_{k=n}^{m} a_k \Big| < \varepsilon$$

(2) $\sum_{n=1}^{\infty} a_n$ *converges* $\implies \lim_{n\to\infty} a_n = 0$

(3) If $a_n \geq 0$ *for all* $n \in \mathbb{N}$, *then*

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \{s_n\} \text{ is bounded}$$

(4) absolute convergence $\implies$ *convergence*

(5) (Comparison Test) Let $N_0 \in \mathbb{N}$.

(a) $|a_n| \leq c_n\ \forall n \geq N_0$ *and* $\sum_{n=1}^{\infty} c_n$ *converges* $\implies \sum_{n=1}^{\infty} a_n$ *converges absolutely*

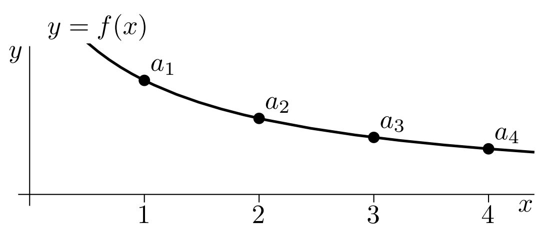
(b) $a_n \geq d_n \geq 0\ \forall n \geq N_0$ *and* $\sum_{n=1}^{\infty} d_n$ *diverges* $\implies \sum_{n=1}^{\infty} a_n$ *diverges*

(6) (Integral Test) Let $c \in \mathbb{R}$. *If* $f(x)$ *is a real valued function which is defined and continuous for all* $x \geq c$ *and which obeys*

$$(i)\ f(x) \geq 0\ \forall x \geq c \quad (ii)\ f(x) \text{ is monotonically decreasing} \quad (iii)\ f(n) = a_n\ \forall n \geq c$$

then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_c^{\infty} f(x)\, dx < \infty$$



(7) (Examples)

(a) $\sum_{n=0}^{\infty} r^n$ *converges to* $\frac{1}{1-r}$ *if* $|r| < 1$ *and diverges if* $|r| \geq 1$.

(b) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

(c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

(8) (Root Test) Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. *Then*

$$\alpha < 1 \implies \text{absolute convergence}$$

$$\alpha > 1 \implies \text{divergence}$$

$$\alpha = 1 \implies \text{nothing}$$

(9) (Ratio Test) If $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of nonzero numbers, then

$$\sum_{n=1}^{\infty} a_n \text{ converges absolutely if } \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$\text{diverges if } \left|\frac{a_{n+1}}{a_n}\right| \ge 1 \text{ for all } n \text{ bigger than some natural number } N$$

(10) (Alternating Series Test) If

$$a_n \ge 0\ \forall n\in\mathbb{N} \quad a_{n+1}\le a_n\ \forall n\in\mathbb{N} \quad \lim_{n\to\infty} a_n = 0$$

then $s=\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges and $s-s_n$ is between 0 and (the first dropped term) $(-1)^n a_{n+1}$.

Proof: (1) $\Rightarrow$ Let $\lim_{n\to\infty} s_n = s$. Then

$$\begin{aligned}\forall\varepsilon>0\ \exists N\in\mathbb{N} \text{ s.t. } m>n-1\ge N &\implies |s_m-s|<\tfrac{\varepsilon}{2},\ |s_{n-1}-s|<\tfrac{\varepsilon}{2}\\ &\implies |s_m-s_{n-1}| = |(s_m-s)+(s-s_{n-1})|<\varepsilon\\ &\implies \left|\sum_{k=n}^{m} a_k\right| < \varepsilon\end{aligned}$$

(1) $\Leftarrow$ By hypothesis

$$\forall\varepsilon>0\ \exists N_\varepsilon\in\mathbb{N} \text{ s.t. } m>n\ge N_\varepsilon \implies |s_m-s_n|<\varepsilon$$

$$\implies \{s_n\} \text{ is bounded (Choosing } \varepsilon=1,\ n=N_1, \text{ we see that } \{\ s_m \mid m>N_1\ \} \text{ is bounded.)}$$

$$\implies \{s_n\} \text{ has a subsequential limit. Say } \lim_{n\to\infty} s_{i_n}=s.$$

$$\implies \forall\varepsilon>0\ \ n\ge N_{\varepsilon/2} \implies |s_n-s| = \left|s_n-s_{i_m}+s_{i_m}-s\right| < \tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2}=\varepsilon$$

if we choose m large enough that $i_m\ge N_{\varepsilon/2}$ and $\left|s_{i_m}-s\right|<\frac{\varepsilon}{2}$.

(2) Apply (1) with $m=n$.

(3) As $a_n\ge 0$ for all $n\in\mathbb{N}$, the sequence $\{s_n\}$ is monotonically increasing. So it converges if and only if it is bounded.

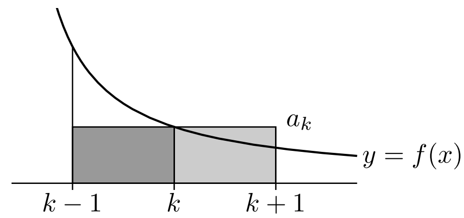
(4) If $\sum_{k=1}^{\infty}|a_k|$ converges, then $\sum_{k=1}^{\infty}|a_k|$ obeys the Cauchy criterion. As $\left|\sum_{k=n}^{m}a_k\right|\le\sum_{k=n}^{m}|a_k|$, the series $\sum_{k=1}^{\infty}a_k$ then also obeys the Cauchy criterion and hence converges.

(5a) The sequence $\left\{s_n=\sum_{k=1}^{n}|a_k|\right\}$ is monotonic and bounded above by $\sum_{k=1}^{N_0}|a_k|+\sum_{k=1}^{\infty}c_k$ and hence converges. So $\sum_{k=1}^{\infty}a_k$ converges absolutely and hence also converges by part (4).

(5b) If $\sum_{k=1}^{\infty}a_k$ were to converge, then $\sum_{k=1}^{\infty}d_k$ would also converge by part (5a). This is a contradiction.

(6) Let N be any fixed integer bigger than $c+1$. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=N}^{\infty} a_k$ converges, which in turn is the case if and only if the sequence $t_n = \sum_{k=N}^{n} a_k$ is bounded, since $a_k \geq 0$ for all $k \geq N > c$. As $f(x)$ decreases as x increases, we have that $f(x) \geq a_k$ when $x \leq k$ and $f(x) \leq a_k$ when $x \geq k$. Thus, for any $k \geq N$

$$\int_k^{k+1} f(x)\ dx \leq a_k \leq \int_{k-1}^{k} f(x)\ dx$$



Summing k from N to n,

$$\int_N^{n+1} f(x)\ dx \leq \sum_{k=N}^{n} a_k \leq \int_{N-1}^{n} f(x)\ dx$$

So the t_n's are bounded if and only if $\int_c^\infty f(x)\ dx < \infty$.

(7a) If $r = 1$, then $\sum_{n=0}^{N} r^n = N+1$ obviously diverges. If $r \neq 1$, then $(1-r)(1+r+r^2+\cdots+r^N) = 1-r^{N+1}$ (just multiply out the left hand side) so that

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$

This converges to $\frac{1}{1-r}$ if $|r| < 1$ and diverges if $r = -1$ or $|r| > 1$.

(7b) If $p \leq 0$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} n^{|p|}$ diverges by part (2). The remaining cases are easily proven using the integral test with $c = 1$ and $f(x) = \frac{1}{x^p}$, which is positive and monotonically decreasing for $p > 0$. We just have to observe that

$$\int_1^L \tfrac{1}{x^p}\ dx = \begin{cases} \ln L & \text{if } p = 1 \\ \frac{L^{1-p}-1}{1-p} & \text{if } p \neq 1 \end{cases}$$

grows unboundedly as $L \to \infty$ when $p \leq 1$ and converges to $\frac{1}{p-1}$ when $p > 1$.

Here is a second proof that does not use integrals. If $p > 0$, each successive term in the series is smaller than the term before. So each term with index n between 2^m and 2^{m+1} obeys

$$\tfrac{1}{2^{(m+1)p}} \leq \tfrac{1}{n^p} \leq \tfrac{1}{2^{mp}}$$

Hence

$$\tfrac{1}{2}2^{(m+1)(1-p)} = \tfrac{2^{m+1}-2^m}{2^{(m+1)p}} \leq \sum_{2^m \leq n < 2^{m+1}} \tfrac{1}{n^p} \leq \tfrac{2^{m+1}-2^m}{2^{mp}} = \tfrac{1}{2^{m(p-1)}}$$

If $0 \leq p \leq 1$, the left hand bound $\frac{1}{2}2^{(m+1)(1-p)} \geq \frac{1}{2}$. Summing m from 0 to $N-1$, we have $\sum_{n=1}^{2^N-1} \frac{1}{n^p} \geq \frac{N}{2}$ which forces divergence. If $p > 1$, then summing m from 0 to $N-1$ in the right hand bound, we have $\sum_{n=1}^{2^N} \frac{1}{n^p} \leq \sum_{m=0}^{N-1} \left(\frac{1}{2^{p-1}}\right)^m \leq \left(1 - \frac{1}{2^{p-1}}\right)^{-1}$. So the series is bounded and hence converges by part (3).

(7c) If $p \leq 0$, then for all $n \geq 2$, $\frac{1}{n(\ln n)^p} \geq \frac{(\ln 2)^{|p|}}{n}$ and the series diverges by the comparison test. Again, the remaining cases are easily proven using the integral test, this time with $c = 2$ and $f(x) = \frac{1}{x(\ln x)^p}$. To compute the integral, we make the change of variables $y = \ln x$, $dy = \frac{dx}{x}$.

$$\int_2^L \tfrac{1}{x(\ln x)^p}\ dx = \int_{\ln 2}^{\ln L} \tfrac{1}{y^p}\ dy = \left\{ \begin{array}{ll} \ln(\ln L) - \ln(\ln 2) & \text{if } p = 1 \\ \frac{(\ln L)^{1-p}-(\ln 2)^{1-p}}{1-p} & \text{if } p \neq 1 \end{array} \right\} \to \left\{ \begin{array}{ll} \infty & \text{if } p \leq 1 \\ \frac{(\ln 2)^{1-p}}{p-1} & \text{if } p > 1 \end{array} \right\} \text{as } L \to \infty$$

Again, here is a second proof that does not use the integral test. If $p > 0$, each term with index n between 2^m and 2^{m+1} obeys

$$\frac{1}{2^{m+1}(m+1)^p(\ln 2)^p} \le \frac{1}{n(\ln n)^p} \le \frac{1}{2^m(m\ln 2)^p}$$

Hence

$$\frac{1}{2(\ln 2)^p}\frac{1}{(m+1)^p} = \frac{2^{m+1}-2^m}{2^{m+1}(m+1)^p(\ln 2)^p} \le \sum_{2^m\le n<2^{m+1}} \frac{1}{n(\ln n)^p} \le \frac{2^{m+1}-2^m}{2^m(m\ln 2)^p} = \frac{1}{(\ln 2)^p}\frac{1}{m^p}$$

Summing m from 1 to $N-1$, we have

$$\frac{1}{2(\ln 2)^p}\sum_{m=1}^{N-1}\frac{1}{(m+1)^p} \le \sum_{n=2}^{2^N-1}\frac{1}{n(\ln n)^p} \le \frac{1}{(\ln 2)^p}\sum_{m=1}^{N-1}\frac{1}{m^p}$$

So if $p > 1$, the right hand inequality together with part (7b) shows that the series is bounded and hence converges, by part (3), and if $p \le 1$, the left hand inequality together with part (7b) shows that the series diverges.

(8) If $\alpha < 1$, pick any β obeying $\alpha < \beta < 1$. There is a natural number N such that $\sqrt[n]{|a_n|} < \beta$, or equivalently, $|a_n| < \beta^n$, for all $n \ge N$. So convergence follows by comparison with the geometric series. On the other hand, if $\alpha > 1$, there must be a subsequence with $\sqrt[n_k]{|a_{n_k}|} \ge 1$, or equivalently, $|a_{n_k}| \ge 1$. So the series diverges by part (2).

(9) If $\left|\frac{a_{n+1}}{a_n}\right| \ge 1$ for all $n \ge N$, then $|a_n| \ge |a_N| > 0$ for all $n \ge N$ and the series diverges by part (2). On the other hand, if $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, there is a $\beta < 1$ and an $N \in \mathbb{N}$ such that $\left|\frac{a_{n+1}}{a_n}\right| \le \beta$ for all $n \ge N$. Consequently $|a_{N+1}| \le \beta|a_N|$, $|a_{N+2}| \le \beta|a_{N+1}| \le \beta^2|a_N|$ and so on. Iterating gives $|a_{N+p}| \le \beta^p|a_N|$ which implies convergence by comparison with the geometric series.

(10) First observe that for every even integer n,

$$s_n = \overbrace{a_1 - a_2}^{\ge 0} + \overbrace{a_3 - a_4}^{\ge 0} + \cdots + \overbrace{a_{n-1} - a_n}^{\ge 0} \ge 0 \quad \text{and} \quad s_{n+1} = \overbrace{s_n}^{\ge 0} + \overbrace{a_{n+1}}^{\ge 0} \ge 0$$

and that for every odd integer n,

$$s_n = a_1 - \overbrace{(a_2 - a_3)}^{\ge 0} - \overbrace{(a_4 - a_5)}^{\ge 0} - \cdots - \overbrace{a_{n-1} - a_n}^{\ge 0} \le a_1 \quad \text{and} \quad s_{n+1} = \overbrace{s_n}^{\le a_1} - \overbrace{a_{n+1}}^{\ge 0} \le a_1$$

Thus $0 \le s_n \le a_1$ for all $n \in \mathbb{N}$. Applying this with a_k replaced by $b_k = a_{n+k}$ gives that, for any natural numbers $n < m$

$$\sum_{k=n+1}^{m}(-1)^{k-1}a_k = (-1)^n\sum_{k=1}^{m-n}(-1)^{k-1}b_k$$

is between 0 and $(-1)^n b_1 = (-1)^n a_{n+1}$. As $\lim_{n\to\infty} a_{n+1} = 0$, the series $\sum_{k=1}^{\infty}(-1)^{k-1}a_k$ converges by the Cauchy criterion. Furthermore, the truncation error

$$s - s_n = \sum_{k=n+1}^{\infty}(-1)^{k-1}a_k = \lim_{m\to\infty}\sum_{k=n+1}^{m}(-1)^{k-1}a_k$$

is also between 0 and $(-1)^n a_{n+1}$. ■