Complex Numbers and Exponentials

Definition and Basic Operations

A complex number is nothing more than a point in the xy–plane. The first component, x, of the complex number (x, y) is called its real part and the second component, y, is called its imaginary part, even though there is nothing imaginary[1] about it. The sum and product of two complex numbers (x_1, y_1) and (x_2, y_2) are defined by

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ (x_1, y_1)\,(x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\end{aligned}$$

respectively. We'll get an effective memory aid for the definition of multiplication shortly. It is conventional to use the notation $x + iy$ (or in electrical engineering $x + jy$) to stand for the complex number (x, y). In other words, it is conventional to write x in place of $(x, 0)$ and i in place of $(0, 1)$. In this notation, the sum and product of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is given by

$$\begin{aligned}z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2)\\ z_1z_2 &= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)\end{aligned}$$

Addition and multiplication of complex numbers obey the familiar algebraic rules

$$\begin{aligned}z_1 + z_2 &= z_2 + z_1 & z_1z_2 &= z_2z_1\\ z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 & z_1(z_2z_3) &= (z_1z_2)z_3\\ 0 + z_1 &= z_1 & 1z_1 &= z_1\\ z_1(z_2 + z_3) &= z_1z_2 + z_1z_3 & (z_1 + z_2)z_3 &= z_1z_3 + z_2z_3\end{aligned}$$

The negative of any complex number $z = x + iy$ is defined by $-z = -x + (-y)i$, and obeys $z + (-z) = 0$. The inverse, z^{-1} or $\frac{1}{z}$, of any complex number $z = x + iy$, other than 0, is defined by $\frac{1}{z}z = 1$. We shall see below that it is given by the formula $\frac{1}{z} = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}i$. The complex number i has the special property

$$i^2 = (0 + 1i)(0 + 1i) = (0 \times 0 - 1 \times 1) + i(0 \times 1 + 1 \times 0) = -1$$

To remember how to multiply complex numbers, you just have to supplement the familiar rules of the real number system with $i^2 = -1$. For example, if $z = 1 + 2i$ and $w = 3 + 4i$, then

$$\begin{aligned}z + w &= (1 + 2i) + (3 + 4i) = 4 + 6i\\ zw &= (1 + 2i)(3 + 4i) \quad = 3 + 4i + 6i + 8i^2 = 3 + 4i + 6i - 8 = -5 + 10i\end{aligned}$$

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[1] Don't attempt to attribute any special significance to the word "complex" in "complex number", or to the word "real" in "real number" and "real part", or to the word "imaginary" in "imaginary part". All are *just names*.

Other Operations

The complex conjugate of z is denoted $\bar{z}$ and is defined to be $\bar{z} = x - iy$. That is, to take the complex conjugate, one replaces every i by $-i$. Note that

$$z\bar{z} = (x+iy)(x-iy) = x^2 - ixy + ixy + y^2 = x^2 + y^2$$

is always a positive real number. In fact, it is the square of the distance from $x + iy$ (recall that this is the point (x, y) in the xy–plane) to 0 (which is the point $(0, 0)$). The distance from $z = x + iy$ to 0 is denoted $|z|$ and is called the absolute value, or modulus, of z. It is given by

$$|z| \;=\; \sqrt{x^2+y^2} \;=\; \sqrt{z\bar{z}}$$

Since $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$,

$$\begin{aligned}
|z_1z_2| &= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2} \\
&= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1y_2x_2y_1 + x_2^2y_1^2} \\
&= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2} = \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)} \\
&= |z_1||z_2|
\end{aligned}$$

for all complex numbers z_1, z_2.

Since $|z|^2 = z\bar{z}$, we have $z\left(\frac{\bar{z}}{|z|^2}\right) = 1$ for all complex numbers $z \neq 0$. This says that the multiplicative inverse, $\frac{1}{z}$, of any nonzero complex number $z = x + iy$ is

$$z^{-1} = \tfrac{\bar{z}}{|z|^2} = \tfrac{x-iy}{x^2+y^2} = \tfrac{x}{x^2+y^2} - \tfrac{y}{x^2+y^2}i$$

This is the formula for $\frac{1}{z}$ given above. It is easy to divide a complex number by a real number. For example

$$\tfrac{11+2i}{25} = \tfrac{11}{25} + \tfrac{2}{25}i$$

In general, there is a trick for rewriting any ratio of complex numbers as a ratio with a real denominator. For example, suppose that we want to find $\frac{1+2i}{3+4i}$. The trick is to multiply by $1 = \frac{3-4i}{3-4i}$. The number $3 - 4i$ is the complex conjugate of the denominator $3 + 4i$. Since $(3 + 4i)(3 - 4i) = 9 - 12i + 12i + 16 = 25$

$$\tfrac{1+2i}{3+4i} = \tfrac{1+2i}{3+4i}\tfrac{3-4i}{3-4i} = \tfrac{(1+2i)(3-4i)}{25} = \tfrac{11+2i}{25} = \tfrac{11}{25} + \tfrac{2}{25}i$$

The notations $\operatorname{Re} z$ and $\operatorname{Im} z$ stand for the real and imaginary parts of the complex number z, respectively. If $z = x + iy$ (with x and y real) they are defined by

$$\operatorname{Re} z = x \qquad \operatorname{Im} z = y$$

Note that both $\operatorname{Re} z$ and $\operatorname{Im} z$ are real numbers. Just subbing in $\bar{z} = x - iy$ gives

$$\operatorname{Re} z = \tfrac{1}{2}(z + \bar{z}) \qquad \operatorname{Im} z = \tfrac{1}{2i}(z - \bar{z})$$

The Complex Exponential

Definition and Basic Properties. For any complex number $z = x + iy$ the exponential e^z, is defined by

$$e^{x+iy} \;=\; e^x \cos y + ie^x \sin y$$

In particular, $e^{iy} \;=\; \cos y + i \sin y$. This definition is not as mysterious as it looks. We could also define e^{iy} by replacing x with iy in the Taylor series expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$$e^{iy} = 1 + iy + \tfrac{(iy)^2}{2!} + \tfrac{(iy)^3}{3!} + \tfrac{(iy)^4}{4!} + \tfrac{(iy)^5}{5!} + \tfrac{(iy)^6}{6!} + \cdots$$

The even terms in this expansion are

$$1 + \tfrac{(iy)^2}{2!} + \tfrac{(iy)^4}{4!} + \tfrac{(iy)^6}{6!} + \cdots = 1 - \tfrac{y^2}{2!} + \tfrac{y^4}{4!} - \tfrac{y^6}{6!} + \cdots = \cos y$$

and the odd terms in this expansion are

$$iy + \tfrac{(iy)^3}{3!} + \tfrac{(iy)^5}{5!} + \cdots = i\Big(y - \tfrac{y^3}{3!} + \tfrac{y^5}{5!} + \cdots\Big) = i \sin y$$

For any two complex numbers z_1 and z_2

$$\begin{aligned}
e^{z_1} e^{z_2} &= e^{x_1}(\cos y_1 + i \sin y_1)\; e^{x_2}(\cos y_2 + i \sin y_2) \\
&= e^{x_1+x_2}(\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2) \\
&= e^{x_1+x_2}\left\{(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\cos y_1 \sin y_2 + \sin y_1 \cos y_2)\right\} \\
&= e^{x_1+x_2}\left\{\cos(y_1 + y_2) + i \sin(y_1 + y_2)\right\} \\
&= e^{(x_1+x_2)+i(y_1+y_2)} \\
&= e^{z_1+z_2}
\end{aligned}$$

so that the familiar multiplication formula for real exponentials also applies to complex exponentials. For any complex number $c = \alpha + i\beta$ and real number t

$$e^{ct} = e^{\alpha t + i\beta t} = e^{\alpha t}[\cos(\beta t) + i \sin(\beta t)]$$

so that the derivative with respect to t

$$\begin{aligned}
\tfrac{d}{dt} e^{ct} &= \alpha e^{\alpha t}[\cos(\beta t) + i \sin(\beta t)] + e^{\alpha t}[-\beta \sin(\beta t) + i\beta \cos(\beta t)] \\
&= (\alpha + i\beta) e^{\alpha t}[\cos(\beta t) + i \sin(\beta t)] \\
&= ce^{ct}
\end{aligned}$$

is also the familiar one.

Relationship with sin **and** cos**.** When θ is a real number

$$\begin{aligned}
e^{i\theta} &= \cos\theta + i \sin\theta \\
e^{-i\theta} &= \cos\theta - i \sin\theta = \overline{e^{i\theta}}
\end{aligned} \tag{1}$$

are complex numbers of modulus one. Solving for $\cos\theta$ and $\sin\theta$ (by adding and subtracting the two equations)

$$\begin{aligned}
\cos\theta &= \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \;= \operatorname{Re} e^{i\theta} \\
\sin\theta &= \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \operatorname{Im} e^{i\theta}
\end{aligned}$$

These formulae make it easy derive trig identities. For example

$$\begin{aligned}
\cos\theta\cos\phi &= \tfrac{1}{4}(e^{i\theta}+e^{-i\theta})(e^{i\phi}+e^{-i\phi}) \\
&= \tfrac{1}{4}(e^{i(\theta+\phi)}+e^{i(\theta-\phi)}+e^{i(-\theta+\phi)}+e^{-i(\theta+\phi)}) \\
&= \tfrac{1}{4}(e^{i(\theta+\phi)}+e^{-i(\theta+\phi)}+e^{i(\theta-\phi)}+e^{i(-\theta+\phi)}) \\
&= \tfrac{1}{2}\big[\cos(\theta+\phi)+\cos(\theta-\phi)\big]
\end{aligned}$$

and, using $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$,

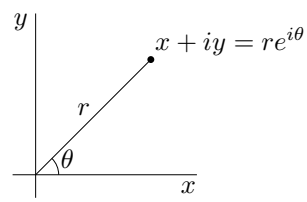
$$\begin{aligned}
\sin^3\theta &= -\tfrac{1}{8i}\big(e^{i\theta}-e^{-i\theta}\big)^3 \\
&= -\tfrac{1}{8i}\big(e^{i3\theta}-3e^{i\theta}+3e^{-i\theta}-e^{-i3\theta}\big) \\
&= \tfrac{3}{4}\tfrac{1}{2i}\big(e^{i\theta}-e^{-i\theta}\big)-\tfrac{1}{4}\tfrac{1}{2i}\big(e^{i3\theta}-e^{-i3\theta}\big) \\
&= \tfrac{3}{4}\sin\theta-\tfrac{1}{4}\sin(3\theta)
\end{aligned}$$

and

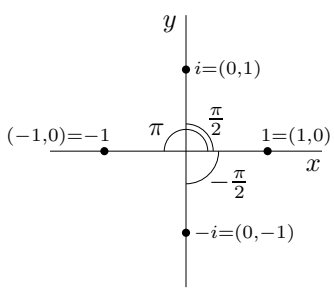
$$\begin{aligned}
\cos(2\theta) &= \operatorname{Re} e^{2\theta i} = \operatorname{Re}\big(e^{i\theta}\big)^2 \\
&= \operatorname{Re}\big(\cos\theta+i\sin\theta\big)^2 \\
&= \operatorname{Re}\big(\cos^2\theta+2i\sin\theta\cos\theta-\sin^2\theta\big) \\
&= \cos^2\theta-\sin^2\theta
\end{aligned}$$

Polar Coordinates. Let $z = x+iy$ be any complex number. Writing (x,y) in polar coordinates in the usual way gives $x = r\cos\theta$, $y = r\sin\theta$ and

$$x+iy = r\cos\theta + ir\sin\theta = re^{i\theta}$$



In particular



$$\begin{array}{rclclcll}
1 &=& e^{i0} &=& e^{2\pi i} &=& e^{2k\pi i} & \text{for } k=0,\pm1,\pm2,\cdots \\
-1 &=& e^{i\pi} &=& e^{3\pi i} &=& e^{(1+2k)\pi i} & \text{for } k=0,\pm1,\pm2,\cdots \\
i &=& e^{i\pi/2} &=& e^{\frac{5}{2}\pi i} &=& e^{(\frac{1}{2}+2k)\pi i} & \text{for } k=0,\pm1,\pm2,\cdots \\
-i &=& e^{-i\pi/2} &=& e^{\frac{3}{2}\pi i} &=& e^{(-\frac{1}{2}+2k)\pi i} & \text{for } k=0,\pm1,\pm2,\cdots
\end{array}$$

The polar coordinate $\theta = \tan^{-1}\frac{y}{x}$ associated with the complex number $z = x+iy$, i.e. the point (x,y) in the xy–plane, is also called the argument of z.

The polar coordinate representation makes it easy to find square roots, third roots and so on. Fix any positive integer n. The nth roots of unity are, by definition, all solutions z of

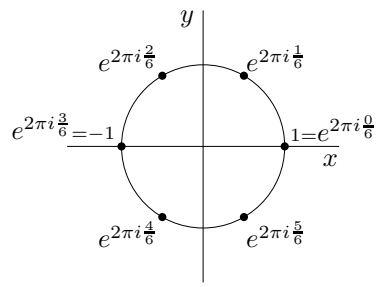
$$z^n = 1$$

Writing $z = re^{i\theta}$

$$r^n e^{n\theta i} \;=\; 1e^{0i}$$

The polar coordinates (r,θ) and (r',θ') represent the same point in the xy–plane if and only if $r = r'$ and $\theta = \theta' + 2k\pi$ for some integer k. So $z^n = 1$ if and only if $r^n = 1$, i.e. $r = 1$, and $n\theta = 2k\pi$ for some integer k. The nth roots of unity are all the complex numbers $e^{2\pi i\frac{k}{n}}$ with k integer. There are precisely n distinct n^{th} roots of unity because $e^{2\pi i\frac{k}{n}} = e^{2\pi i\frac{k'}{n}}$ if and only if $2\pi\frac{k}{n} - 2\pi\frac{k'}{n} = 2\pi\frac{k-k'}{n}$ is an integer multiple of 2π. That is, if and only if $k - k'$ is an integer multiple of n. The n distinct nth roots of unity are

$$1\ ,\ e^{2\pi i\frac{1}{n}}\ ,\ e^{2\pi i\frac{2}{n}}\ ,\ e^{2\pi i\frac{3}{n}}\ ,\ \cdots\ ,\ e^{2\pi i\frac{n-1}{n}}$$



Exploiting Complex Exponentials in Calculus Computations

You have learned how to evaluate integrals involving trigonometric functions by using integration by parts, various trigonometric identities and various substitutions. It is often much easier to just use (1). Here are two examples

Example 1

$$\begin{aligned}\int e^x\cos x\ dx &= \tfrac{1}{2}\int e^x\left[e^{ix}+e^{-ix}\right]\ dx = \tfrac{1}{2}\int\left[e^{(1+i)x}+e^{(1-i)x}\right]\ dx\\ &= \tfrac{1}{2}\left[\tfrac{1}{1+i}e^{(1+i)x}+\tfrac{1}{1-i}e^{(1-i)x}\right]+C\end{aligned}$$

This form of the indefinite integral looks a little weird because of the i's. But it is correct and it is purely real, despite the i's, because $\frac{1}{1-i}e^{(1-i)x}$ is the complex conjugate of $\frac{1}{1+i}e^{(1+i)x}$. We can convert the indefinite integral into a more familiar form just by subbing back in $e^{\pm ix} = \cos x \pm i\sin x$, $\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2}$ and $\frac{1}{1-i} = \overline{\frac{1}{1+i}} = \frac{1+i}{2}$.

$$\begin{aligned}\int e^x\cos x\ dx &= \tfrac{1}{2}e^x\left[\tfrac{1}{1+i}e^{ix}+\tfrac{1}{1-i}e^{-ix}\right]+C\\ &= \tfrac{1}{2}e^x\left[\tfrac{1-i}{2}(\cos x+i\sin x)+\tfrac{1+i}{2}(\cos x-i\sin x)\right]+C\\ &= \tfrac{1}{2}e^x\cos x+\tfrac{1}{2}e^x\sin x+C\end{aligned}$$

Example 2 Using $(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$,

$$\begin{aligned}\int\cos^4 x\ dx &= \tfrac{1}{2^4}\int\left[e^{ix}+e^{-ix}\right]^4\ dx = \tfrac{1}{2^4}\int\left[e^{4ix}+4e^{2ix}+6+4e^{-2ix}+e^{-4ix}\right]\ dx\\ &= \tfrac{1}{2^4}\left[\tfrac{1}{4i}e^{4ix}+\tfrac{4}{2i}e^{2ix}+6x+\tfrac{4}{-2i}e^{-2ix}+\tfrac{1}{-4i}e^{-4ix}\right]+C\\ &= \tfrac{1}{2^4}\left[\tfrac{1}{2}\tfrac{1}{2i}(e^{4ix}-e^{-4ix})+\tfrac{4}{2i}(e^{2ix}-e^{-2ix})+6x\right]+C\\ &= \tfrac{1}{2^4}\left[\tfrac{1}{2}\sin 4x+4\sin 2x+6x\right]+C\\ &= \tfrac{1}{32}\sin 4x+\tfrac{1}{4}\sin 2x+\tfrac{3}{8}x+C\end{aligned}$$

Complex exponentials are also widely used to simplify the process of guessing solutions to ordinary differential equations. Here are two examples.

Example 3 The general solution to the ordinary differential equation

$$y''(t) + 4y'(t) + 5y(t) = 0 \tag{2}$$

is of the form $C_1u_1(t) + C_2u_2(t)$ with $u_1(t)$ and $u_2(t)$ being two (independent) solutions to (2) and with C_1 and C_2 being arbitrary constants. The easiest way to find $u_1(t)$ and $u_2(t)$ is to guess them. And the easiest way to guess them is to try[(2)] $y(t) = e^{rt}$, with r being a constant to be determined. Substituting $y(t) = e^{rt}$ into (2) gives

$$r^2e^{rt} + 4re^{rt} + 5e^{rt} = 0 \iff \big(r^2 + 4r + 5\big)e^{rt} = 0 \iff r^2 + 4r + 5 = 0$$

This quadratic equation for r can be solved either by using the high school formula or by rewriting it

$$r^2 + 4r + 5 = 0 \iff (r+2)^2 + 1 = 0 \iff (r+2)^2 = -1 \iff r + 2 = \pm i \iff r = -2 \pm i$$

So the general solution to (2) is

$$y(t) = C_1e^{(-2+i)t} + C_2e^{(-2-i)t}$$

This is one way to write the general solution, but there are many others. In particular there are quite a few people in the world who are (foolishly) afraid of complex exponentials. We can hide them by using (1).

$$\begin{aligned} y(t) &= C_1e^{(-2+i)t} + C_2e^{(-2-i)t} = C_1e^{-2t}e^{it} + C_2e^{-2t}e^{-it} \\ &= C_1e^{-2t}\big(\cos t + i\sin t\big) + C_2e^{-2t}\big(\cos t - i\sin t\big) \\ &= (C_1 + C_2)e^{-2t}\cos t + (iC_1 - iC_2)e^{-2t}\sin t \\ &= D_1e^{-2t}\cos t + D_2e^{-2t}\sin t \end{aligned}$$

with $D_1 = C_1 + C_2$ and $D_2 = iC_1 - iC_2$ being two other arbitrary constants. Don't make the mistake of thinking that D_2 must be complex because i appears in the formula $D_2 = iC_1 - iC_2$ relating D_2 and C_1, C_2. Noone said that C_1 and C_2 are real numbers. In fact, in typical applications, the arbitrary constants are determined by initial conditions and often D_1 and D_2 turn out to be real and C_1 and C_2 turn out to be complex. For example, the initial conditions $y(0) = 0$, $y'(0) = 2$ force

$$\begin{aligned} 0 &= y(0) = C_1 + C_2 \\ 2 &= y'(0) = (-2+i)C_1 + (-2-i)C_2 \end{aligned}$$

The first equation gives $C_2 = -C_1$ and then the second equation gives

$$(-2+i)C_1 - (-2-i)C_1 = 2 \iff 2iC_1 = 2 \iff iC_1 = 1 \iff C_1 = -i,\ C_2 = i$$

and

$$D_1 = C_1 + C_2 = 0 \qquad D_2 = iC_1 - iC_2 = 2$$

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(2) The reason that $y(t) = e^{rt}$ is a good guess is that, with this guess, all of $y(t)$, $y'(t)$ and $y''(t)$ are constants times e^{rt}. So the left hand side of the differential equation is also a constant, that depends on r, times e^{rt}. So we just have to choose r so that the constant is zero.

Example 4 We shall now guess one solution (i.e. a particular solution) to the differential equation

$$y''(t) + 2y'(t) + 3y(t) = \cos t \tag{3}$$

Equations like this arise, for example, in the study of the RLC circuit. We shall simplify the computation by exploiting that $\cos t = \operatorname{Re} e^{it}$. First, we shall guess a function $Y(t)$ obeying

$$Y'' + 2Y' + 3Y = e^{it} \tag{4}$$

Then, taking complex conjugates,

$$\bar{Y}'' + 2\bar{Y}' + 3\bar{Y} = e^{-it} \tag{$\bar{4}$}$$

and, adding $\frac{1}{2}(2)$ and $\frac{1}{2}(\bar{2})$ together will give

$$(\operatorname{Re} Y)'' + 2(\operatorname{Re} Y)' + 3(\operatorname{Re} Y) = \operatorname{Re} e^{it} = \cos t$$

which shows that $\operatorname{Re} Y(t)$ is a solution to (1). Let's try $Y(t) = Ae^{it}$. This is a solution of (2) if and only if

$$\begin{aligned} & \tfrac{d^2}{dt^2}\big(Ae^{it}\big) + 2\tfrac{d}{dt}\big(Ae^{it}\big) + 3Ae^{it} = e^{it} \\ \iff \quad & (2+2i)Ae^{it} = e^{it} \\ \iff \quad & A = \tfrac{1}{2+2i} \end{aligned}$$

So we have found a solution to (2) and $\operatorname{Re} \frac{e^{it}}{2+2i}$ is a solution to (1). To simplify this, write $2+2i$ in polar coordinates. So

$$2+2i = 2\sqrt{2}e^{i\frac{\pi}{4}} \Rightarrow \tfrac{e^{it}}{2+2i} = \tfrac{e^{it}}{2\sqrt{2}e^{i\frac{\pi}{4}}} = \tfrac{1}{2\sqrt{2}}e^{i(t-\frac{\pi}{4})} \Rightarrow \operatorname{Re} \tfrac{e^{it}}{2+2i} = \tfrac{1}{2\sqrt{2}}\cos(t-\tfrac{\pi}{4})$$