

Complex Numbers and Exponentials

Definition and Basic Operations

A complex number is nothing more than a point in the xy -plane. The sum and product of two complex numbers (x_1, y_1) and (x_2, y_2) is defined by

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ (x_1, y_1)(x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\end{aligned}$$

respectively. It is conventional to use the notation $x + iy$ to stand for the complex number (x, y) . In other words, it is conventional to write x in place of $(x, 0)$ and i in place of $(0, 1)$. In this notation, the sum and product of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is given by

$$\begin{aligned}z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2) \\ z_1 z_2 &= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)\end{aligned}$$

The complex number i has the special property

$$i^2 = (0 + 1i)(0 + 1i) = (0 \times 0 - 1 \times 1) + i(0 \times 1 + 1 \times 0) = -1$$

For example, if $z = 1 + 2i$ and $w = 3 + 4i$, then

$$\begin{aligned}z + w &= (1 + 2i) + (3 + 4i) = 4 + 6i \\ zw &= (1 + 2i)(3 + 4i) = 3 + 4i + 6i + 8i^2 = 3 + 4i + 6i - 8 = -5 + 10i\end{aligned}$$

Addition and multiplication of complex numbers obey the familiar algebraic rules

$$\begin{aligned}z_1 + z_2 &= z_2 + z_1 & z_1 z_2 &= z_2 z_1 \\ z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 & z_1(z_2 z_3) &= (z_1 z_2) z_3 \\ 0 + z_1 &= z_1 & 1 z_1 &= z_1 \\ z_1(z_2 + z_3) &= z_1 z_2 + z_1 z_3 & (z_1 + z_2) z_3 &= z_1 z_3 + z_2 z_3\end{aligned}$$

The negative of any complex number $z = x + iy$ is defined by $-z = -x + (-y)i$, and obeys $z + (-z) = 0$.

Other Operations

The complex conjugate of z is denoted \bar{z} and is defined to be $\bar{z} = x - iy$. That is, to take the complex conjugate, one replaces every i by $-i$. Note that

$$z\bar{z} = (x + iy)(x - iy) = x^2 - ixy + ixy + y^2 = x^2 + y^2$$

is always a positive real number. In fact, it is the square of the distance from $x + iy$ (recall that this is the point (x, y) in the xy -plane) to 0 (which is the point $(0, 0)$). The distance from $z = x + iy$ to 0 is denoted $|z|$ and is called the absolute value, or modulus, of z . It is given by

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

Since $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$,

$$\begin{aligned} |z_1 z_2| &= \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2} \\ &= \sqrt{x_1^2 x_2^2 - 2x_1 x_2 y_1 y_2 + y_1^2 y_2^2 + x_1^2 y_2^2 + 2x_1 y_2 x_2 y_1 + x_2^2 y_1^2} \\ &= \sqrt{x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\ &= |z_1| |z_2| \end{aligned}$$

for all complex numbers z_1, z_2 .

Since $|z|^2 = z\bar{z}$, we have $z\left(\frac{\bar{z}}{|z|^2}\right) = 1$ for all complex numbers $z \neq 0$. This says that the multiplicative inverse, denoted z^{-1} or $\frac{1}{z}$, of any nonzero complex number $z = x + iy$ is

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$$

It is easy to divide a complex number by a real number. For example

$$\frac{11+2i}{25} = \frac{11}{25} + \frac{2}{25}i$$

In general, there is a trick for rewriting any ratio of complex numbers as a ratio with a real denominator. For example, suppose that we want to find $\frac{1+2i}{3+4i}$. The trick is to multiply by $1 = \frac{3-4i}{3-4i}$. The number $3-4i$ is the complex conjugate of $3+4i$. Since $(3+4i)(3-4i) = 9-12i+12i+16 = 25$

$$\frac{1+2i}{3+4i} = \frac{1+2i}{3+4i} \frac{3-4i}{3-4i} = \frac{(1+2i)(3-4i)}{25} = \frac{11+2i}{25} = \frac{11}{25} + \frac{2}{25}i$$

The notations $\operatorname{Re} z$ and $\operatorname{Im} z$ stand for the real and imaginary parts of the complex number z , respectively. If $z = x + iy$ (with x and y real) they are defined by

$$\operatorname{Re} z = x \quad \operatorname{Im} z = y$$

Note that both $\operatorname{Re} z$ and $\operatorname{Im} z$ are real numbers. Just subbing in $\bar{z} = x - iy$ gives

$$\operatorname{Re} z = \frac{1}{2}(z + \bar{z}) \quad \operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$$

The Complex Exponential

Definition and Basic Properties. For any complex number $z = x + iy$ the exponential e^z , is defined by

$$e^{x+iy} = e^x \cos y + ie^x \sin y$$

In particular, $e^{iy} = \cos y + i \sin y$. This definition is not as mysterious as it looks. We could also define e^{iy} by the subbing x by iy in the Taylor series expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \frac{(iy)^6}{6!} + \dots$$

The even terms in this expansion are

$$1 + \frac{(iy)^2}{2!} + \frac{(iy)^4}{4!} + \frac{(iy)^6}{6!} + \dots = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots = \cos y$$

and the odd terms in this expansion are

$$iy + \frac{(iy)^3}{3!} + \frac{(iy)^5}{5!} + \cdots = i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots\right) = i \sin y$$

For any two complex numbers z_1 and z_2

$$\begin{aligned} e^{z_1} e^{z_2} &= e^{x_1}(\cos y_1 + i \sin y_1) e^{x_2}(\cos y_2 + i \sin y_2) \\ &= e^{x_1+x_2}(\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2) \\ &= e^{x_1+x_2} \{(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\cos y_1 \sin y_2 + \cos y_2 \sin y_1)\} \\ &= e^{x_1+x_2} \{\cos(y_1 + y_2) + i \sin(y_1 + y_2)\} \\ &= e^{(x_1+x_2)+i(y_1+y_2)} \\ &= e^{z_1+z_2} \end{aligned}$$

so that the familiar multiplication formula also applies to complex exponentials. For any complex number $c = \alpha + i\beta$ and real number t

$$e^{ct} = e^{\alpha t + i\beta t} = e^{\alpha t} [\cos(\beta t) + i \sin(\beta t)]$$

so that the derivative with respect to t

$$\begin{aligned} \frac{d}{dt} e^{ct} &= \alpha e^{\alpha t} [\cos(\beta t) + i \sin(\beta t)] + e^{\alpha t} [-\beta \sin(\beta t) + i\beta \cos(\beta t)] \\ &= (\alpha + i\beta) e^{\alpha t} [\cos(\beta t) + i \sin(\beta t)] \\ &= c e^{ct} \end{aligned}$$

is also the familiar one.

Relationship with sin and cos. When θ is a real number

$$\begin{aligned} e^{i\theta} &= \cos \theta + i \sin \theta \\ e^{-i\theta} &= \cos \theta - i \sin \theta = \overline{e^{i\theta}} \end{aligned}$$

are complex numbers of modulus one. Solving for $\cos \theta$ and $\sin \theta$ (by adding and subtracting the two equations)

$$\begin{aligned} \cos \theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \operatorname{Re} e^{i\theta} \\ \sin \theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \operatorname{Im} e^{i\theta} \end{aligned}$$

These formulae make it easy derive trig identities. For example

$$\begin{aligned} \cos \theta \cos \phi &= \frac{1}{4}(e^{i\theta} + e^{-i\theta})(e^{i\phi} + e^{-i\phi}) \\ &= \frac{1}{4}(e^{i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)} + e^{-i(\theta+\phi)}) \\ &= \frac{1}{4}(e^{i(\theta+\phi)} + e^{-i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)}) \\ &= \frac{1}{2}(\cos(\theta + \phi) + \cos(\theta - \phi)) \end{aligned}$$

and, using $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$,

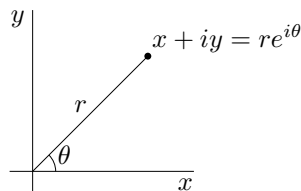
$$\begin{aligned} \sin^3 \theta &= -\frac{1}{8i}(e^{i\theta} - e^{-i\theta})^3 \\ &= -\frac{1}{8i}(e^{i3\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-i3\theta}) \\ &= \frac{3}{4} \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) - \frac{1}{4} \frac{1}{2i}(e^{i3\theta} - e^{-i3\theta}) \\ &= \frac{3}{4} \sin \theta - \frac{1}{4} \sin(3\theta) \end{aligned}$$

and

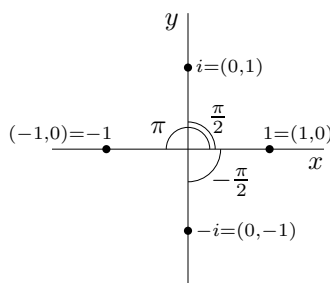
$$\begin{aligned}\cos(2\theta) &= \operatorname{Re} e^{i2\theta} = \operatorname{Re} (e^{i\theta})^2 \\ &= \operatorname{Re} (\cos \theta + i \sin \theta)^2 \\ &= \operatorname{Re} (\cos^2 \theta + 2i \sin \theta \cos \theta - \sin^2 \theta) \\ &= \cos^2 \theta - \sin^2 \theta\end{aligned}$$

Polar Coordinates. Let $z = x + iy$ be any complex number. Writing (x, y) in polar coordinates in the usual way gives $x = r \cos \theta$, $y = r \sin \theta$ and

$$x + iy = r \cos \theta + ir \sin \theta = re^{i\theta}$$



In particular



$$\begin{aligned}1 &= e^{i0} = e^{2\pi i} = e^{2k\pi i} && \text{for } k = 0, \pm 1, \pm 2, \dots \\ -1 &= e^{i\pi} = e^{3\pi i} = e^{(1+2k)\pi i} && \text{for } k = 0, \pm 1, \pm 2, \dots \\ i &= e^{i\pi/2} = e^{\frac{5}{2}\pi i} = e^{(\frac{1}{2}+2k)\pi i} && \text{for } k = 0, \pm 1, \pm 2, \dots \\ -i &= e^{-i\pi/2} = e^{\frac{3}{2}\pi i} = e^{(-\frac{1}{2}+2k)\pi i} && \text{for } k = 0, \pm 1, \pm 2, \dots\end{aligned}$$

The polar coordinate $\theta = \tan^{-1} \frac{y}{x}$ associated with the complex number $z = x + iy$ is also called the argument of z .

The polar coordinate representation makes it easy to find square roots, third roots and so on. Fix any positive integer n . The n^{th} roots of unity are, by definition, all solutions z of

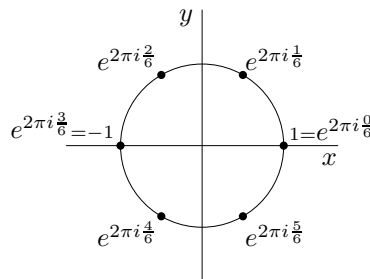
$$z^n = 1$$

Writing $z = re^{i\theta}$

$$r^n e^{n\theta i} = 1e^{0i}$$

The polar coordinates (r, θ) and (r', θ') represent the same point in the xy -plane if and only if $r = r'$ and $\theta = \theta' + 2k\pi$ for some integer k . So $z^n = 1$ if and only if $r^n = 1$, i.e. $r = 1$, and $n\theta = 2k\pi$ for some integer k . The n^{th} roots of unity are all complex numbers $e^{2\pi i \frac{k}{n}}$ with k integer. There are precisely n distinct n^{th} roots of unity because $e^{2\pi i \frac{k}{n}} = e^{2\pi i \frac{k'}{n}}$ if and only if $2\pi \frac{k}{n} - 2\pi i \frac{k'}{n} = 2\pi \frac{k-k'}{n}$ is an integer multiple of 2π . That is, if and only if $k - k'$ is an integer multiple of n . The are n distinct n^{th} roots of unity are

$$1, e^{2\pi i \frac{1}{n}}, e^{2\pi i \frac{2}{n}}, e^{2\pi i \frac{3}{n}}, \dots, e^{2\pi i \frac{n-1}{n}}$$



Exploiting Complex Exponentials in Calculus Computations

Example 1

$$\begin{aligned}\int e^x \cos x \, dx &= \frac{1}{2} \int e^x [e^{ix} + e^{-ix}] \, dx = \frac{1}{2} \int [e^{(1+i)x} + e^{(1-i)x}] \, dx \\ &= \frac{1}{2} \left[\frac{1}{1+i} e^{(1+i)x} + \frac{1}{1-i} e^{(1-i)x} \right] + C\end{aligned}$$

This form of the indefinite integral looks a little wierd because of the i 's. But it is correct and it is purely real, despite the i 's, because $\frac{1}{1-i} e^{(1-i)x}$ is the complex conjugate of $\frac{1}{1+i} e^{(1+i)x}$. We can convert the indefinite integral into a more familiar form just by subbing back in $e^{\pm ix} = \cos x \pm i \sin x$, $\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2}$ and $\frac{1}{1-i} = \frac{1+i}{2}$.

$$\begin{aligned}\int e^x \cos x \, dx &= \frac{1}{2} e^x \left[\frac{1}{1+i} e^{ix} + \frac{1}{1-i} e^{-ix} \right] + C \\ &= \frac{1}{2} e^x \left[\frac{1-i}{2} (\cos x + i \sin x) + \frac{1+i}{2} (\cos x - i \sin x) \right] + C \\ &= \frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + C\end{aligned}$$

Example 2 Using $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$,

$$\begin{aligned}\int \cos^4 x \, dx &= \frac{1}{2^4} \int [e^{ix} + e^{-ix}]^4 \, dx = \frac{1}{2^4} \int [e^{4ix} + 4e^{2ix} + 6 + 4e^{-2ix} + e^{-4ix}] \, dx \\ &= \frac{1}{2^4} \left[\frac{1}{4i} e^{4ix} + \frac{4}{2i} e^{2ix} + 6x + \frac{4}{-2i} e^{-2ix} + \frac{1}{-4i} e^{-4ix} \right] + C \\ &= \frac{1}{2^4} \left[\frac{1}{2} \frac{1}{2i} (e^{4ix} - e^{-4ix}) + \frac{4}{2i} (e^{2ix} - e^{-2ix}) + 6x \right] + C \\ &= \frac{1}{2^4} \left[\frac{1}{2} \sin 4x + 4 \sin 2x + 6x \right] + C \\ &= \frac{1}{32} \sin 4x + \frac{1}{4} \sin 2x + \frac{3}{8} x + C\end{aligned}$$

Example 3 We shall now guess a solution to the differential equation

$$y'' + 2y' + 3y = \cos t \tag{1}$$

Equations like this arise, for example, in the study of the RLC circuit. We shall simplify the computation by exploiting that $\cos t = \operatorname{Re} e^{it}$. First, we shall guess a function $Y(t)$ obeying

$$Y'' + 2Y' + 3Y = e^{it} \tag{2}$$

Then, taking complex conjugates,

$$\bar{Y}'' + 2\bar{Y}' + 3\bar{Y} = e^{-it} \tag{\bar{2}}$$

and, adding $\frac{1}{2}(2)$ and $\frac{1}{2}(\bar{2})$ together will give

$$(\operatorname{Re} Y)'' + 2(\operatorname{Re} Y)' + 3(\operatorname{Re} Y) = \operatorname{Re} e^{it} = \cos t$$

which shows that $\operatorname{Re} Y(t)$ is a solution to (1). Let's try $Y(t) = Ae^{it}$. This is a solution of (2) if and only if

$$\begin{aligned}\frac{d^2}{dt^2} (Ae^{it}) + 2\frac{d}{dt} (Ae^{it}) + 3Ae^{it} &= e^{it} \\ \iff (2+2i)Ae^{it} &= e^{it} \\ \iff A &= \frac{1}{2+2i}\end{aligned}$$

So we have found a solution to (2) and $\operatorname{Re} \frac{e^{it}}{2+2i}$ is a solution to (1). To simplify this, write $2+2i$ in polar coordinates. So

$$2+2i = 2\sqrt{2}e^{i\frac{\pi}{4}} \Rightarrow \frac{e^{it}}{2+2i} = \frac{e^{it}}{2\sqrt{2}e^{i\frac{\pi}{4}}} = \frac{1}{2\sqrt{2}}e^{i(t-\frac{\pi}{4})} \Rightarrow \operatorname{Re} \frac{e^{it}}{2+2i} = \frac{1}{2\sqrt{2}} \cos(t - \frac{\pi}{4})$$