Completion

Theorem. *Let (X, d) be a metric space. Then there exists a metric space (X^*, Δ) and a map $\varphi : X \to X^*$ such that*

(1) φ is one–to–one. That is, $\varphi(x) = \varphi(y)$ if and only if $x = y$.

(2) $\Delta\big(\varphi(x), \varphi(y)\big) = d(x, y)$ for all $x, y \in X$.

(3) X^* is complete.

(4) $\varphi(X)$ is dense in X^*. This means that each element of X^* is a limit of elements of $\varphi(X) = \{\ \varphi(x)\ \big|\ x \in X\ \}$. Equivalently, for each $P \in X^*$ and each $\varepsilon > 0$, there is a $p \in X$ with $\Delta\big(\varphi(p), P\big) < \varepsilon$.

(5) If X is complete, then $\varphi(X) = X^*$.

Remark.

(a) X^* is called the completion of X.

(b) φ gives a one-to–one correspondance between elements of X and elements of $\varphi(X)$. So we can think of φ as giving the new name $\varphi(x)$ to each $x \in X$ and we can think of $\varphi(X)$ as being the same as X but with the names of the elements changed. Thus, we can think of X as being $\varphi(X) \subset X^*$. Using this point of view, the above theorem says that any metric space can be extended to a complete metric space. I.e. can have its holes filled in.

(c) If $X = \mathbb{Q}$, then we can take $X^* = \mathbb{R}$, $\Delta(x, y) = |x - y|$ and $\varphi(x) = x$. It is easy to check that (1)-(4) are obeyed.

Motivation.

The hard part of the proof is to make a guess as to what X^* should be. That's what we'll do now. A good strategy is to work backwards. Suppose that, somehow, we have found a suitable X^* with $X \subset X^*$. If we can find a way to describe each element of X^* purely in terms of elements of X, then we can turn around and take that as the definition of X^*.

Consider, for example $X = \mathbb{Q}$ and $X^* = \mathbb{R}$. Each element of $\mathbb{R}$ may be written as the limit of a sequence of rational numbers, and each such sequence is Cauchy. For example, $\sqrt{2}$ is the limit of the Cauchy sequence 1, 1.4, 1.41, 1.414, 1.4142, $\cdots$. Thus specifying a real number is equivalent to specifying a Cauchy sequence of rational numbers. But there is not a one to one correspondance between real numbers and Cauchy sequences of rational numbers, because many different Cauchy sequences of rational numbers converge to the same real number. For example

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ \cdots$$
$$0,\ 1.3,\ 1.40,\ 1.413,\ 1.4141,\ \cdots$$
$$2,\ 1.5,\ 1.42,\ 1.415,\ 1.4143,\ \cdots$$

all converge to $\sqrt{2}$. (For the second sequence, I just subtracted one from the last nonzero decimal place. For the third sequence, I added one to the last nonzero decimal place.) To get a one to one correspondance, we can identify each real number P with the set of all Cauchy sequences in $\mathbb{Q}$ that converge to P.

Outline of Proof. First define

$$X' = \left\{ \{p_n\}_{n\in\mathbb{N}} \;\middle|\; \{p_n\}_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } X \right\}$$

That is, X' is the set of all Cauchy sequences in X. Next we define two Cauchy sequences $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ in X to be "equivalent", written $\{p_n\}_{n\in\mathbb{N}} \sim \{q_n\}_{n\in\mathbb{N}}$, if and only if

$$\lim_{n\to\infty} d(p_n, q_n) = 0$$

This definition is rigged so that any two convergent sequences have the same limit if and only if they are equivalent. Next, if $\{p_n\}_{n\in\mathbb{N}}$, we define the "equivalence class of $\{p_n\}_{n\in\mathbb{N}}$" to be the set

$$\left[\{p_n\}_{n\in\mathbb{N}}\right] = \left\{ \{q_n\}_{n\in\mathbb{N}} \;\middle|\; \{q_n\}_{n\in\mathbb{N}} \sim \{p_n\}_{n\in\mathbb{N}} \right\}$$

of all Cauchy sequences that are equivalent to $\{p_n\}_{n\in\mathbb{N}}$. We shall shortly prove

Lemma 1 *If* $\{p_n\}_{n\in\mathbb{N}},\ \{q_n\}_{n\in\mathbb{N}} \in X'$ *then*

$$\textit{either } \left[\{p_n\}_{n\in\mathbb{N}}\right] = \left[\{q_n\}_{n\in\mathbb{N}}\right] \quad \textit{or} \quad \left[\{p_n\}_{n\in\mathbb{N}}\right] \cap \left[\{q_n\}_{n\in\mathbb{N}}\right] = \emptyset$$

If you think of a Cauchy sequence as one person and an equivalence class of Cauchy sequences as a "family" of related people, then the above Lemma says, that the whole world is divided into a collection of nonoverlapping families. Next, we define

$$X^* = \left\{ \left[\{p_n\}_{n\in\mathbb{N}}\right] \;\middle|\; \{p_n\}_{n\in\mathbb{N}} \in X' \right\}$$

as the set of all "families" and prove

Lemma 2 *If* $\{p_n\}_{n\in\mathbb{N}},\ \{q_n\}_{n\in\mathbb{N}} \in X'$ *then* $\lim\limits_{n\to\infty} d(p_n, q_n)$ *exists.*

Lemma 3 *If* $\{p_n\}_{n\in\mathbb{N}}$ *and* $\{p'_n\}_{n\in\mathbb{N}}$ *are two representatives of* $P \in X^*$ *(this means that* $P = \left[\{p_n\}_{n\in\mathbb{N}}\right] = \left[\{p'_n\}_{n\in\mathbb{N}}\right]$*) and* $\{q_n\}_{n\in\mathbb{N}}$ *and* $\{q'_n\}_{n\in\mathbb{N}}$ *are two representatives of* $Q \in X^*$*, then*

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n)$$

This allows us to define $\Delta(P, Q)$. If $P, Q \in X^*$ and $\{p_n\}_{n\in\mathbb{N}}$ is a representative of P and $\{q_n\}_{n\in\mathbb{N}}$ is a representative of $Q \in X^*$, we define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n) \tag{1}$$

By Lemma 3, we always get the same value for $\Delta(P, Q)$ no matter which representatives for P and Q are used in (1). Finally, we define $\varphi : X \to X^*$ by

$$\varphi(p) = \left[\{p, p, p, \cdots\}\right]$$

The conclusions of the Theorem are now proven as a series of Lemmata.

Lemma 4 Δ *is a metric on* X^**.*

Lemma 5 X^* *is complete*

Lemma 6 $\Delta\big(\varphi(p), \varphi(q)\big) = d(p,q)$ *for all* $p, q \in X$*.*

Lemma 7 φ *is one–to–one.*

Lemma 8 $\varphi(X)$ *is dense in* X^**.*

Lemma 9 *If* X *is complete, then* $\varphi(X) = X^*$*.*

■

So now we just have to prove all of the Lemmata.

Lemma 1 *If* $\{p_n\}_{n\in\mathbb{N}},\ \{q_n\}_{n\in\mathbb{N}} \in X'$ *then*

$$\textit{either}\ \big[\{p_n\}_{n\in\mathbb{N}}\big] = \big[\{q_n\}_{n\in\mathbb{N}}\big] \qquad \textit{or} \qquad \big[\{p_n\}_{n\in\mathbb{N}}\big] \cap \big[\{q_n\}_{n\in\mathbb{N}}\big] = \emptyset$$

Proof: Suppose that the conclusion is false. Then $\big[\{p_n\}_{n\in\mathbb{N}}\big] \cap \big[\{q_n\}_{n\in\mathbb{N}}\big]$ must be nonempty so that there is a Cauchy sequence $\{a_n\}$ that is in both $\big[\{p_n\}_{n\in\mathbb{N}}\big]$ and $\big[\{q_n\}_{n\in\mathbb{N}}\big]$. Consequently $\{a_n\}$ is equivalent to both $\{p_n\}$ and $\{q_n\}$ so that

$$\lim_{n\to\infty} d(a_n, p_n) = 0 \text{ and } \lim_{n\to\infty} d(a_n, q_n) = 0$$

On the other hand $\big[\{p_n\}_{n\in\mathbb{N}}\big]$ and $\big[\{q_n\}_{n\in\mathbb{N}}\big]$ must be different, so that there is a Cauchy sequence $\{b_n\}$ that is in one of them (say $\big[\{p_n\}_{n\in\mathbb{N}}\big]$) and not in the other (say $\big[\{q_n\}_{n\in\mathbb{N}}\big]$). Since $\{b_n\}$ is in $\big[\{p_n\}_{n\in\mathbb{N}}\big]$,

$$\lim_{n\to\infty} d(b_n, p_n) = 0$$

By the triangle inequality

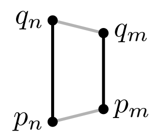
$$d(b_n, q_n) \le d(b_n, p_n) + d(p_n, a_n) + d(a_n, q_n)$$

But this forces $\lim_{n\to\infty} d(b_n, q_n) = 0$, which contradicts the assumption that $\{b_n\}$ is not in $\big[\{q_n\}_{n\in\mathbb{N}}\big]$.
■

Lemma 2 *If* $\{p_n\}_{n\in\mathbb{N}},\ \{q_n\}_{n\in\mathbb{N}} \in X'$ *then* $\lim_{n\to\infty} d(p_n, q_n)$ *exists.*

Proof: We first prove

$$\big|d(p_n,q_n)-d(p_m,q_m)\big| \le d(p_n,p_m)+d(q_m,q_n) \tag{2}$$



by combining the two right hand inequalities of

$$\begin{array}{ll} d(p_n,q_n) \le d(p_n,p_m)+d(p_m,q_m)+d(q_m,q_n) & \Rightarrow\ d(p_n,q_n)-d(p_m,q_m) \le d(p_n,p_m)+d(q_m,q_n) \\ d(p_m,q_m) \le d(p_m,p_n)+d(p_n,q_n)+d(q_n,q_m) & \Rightarrow\ d(p_m,q_m)-d(p_n,q_n) \le d(p_n,p_m)+d(q_n,q_m) \end{array}$$

Since $\{p_n\}$ and $\{q_n\}$ are both Cauchy, both $d(p_n,p_m)$ and $d(q_m,q_n)$ converge to zero as $m,n\to\infty$. So the same is true for $\big|d(p_n,q_n)-d(p_m,q_m)\big|$. So $\{d(p_n,q_n)\}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, this Cauchy sequence converges. ■

Lemma 3 *If $\{p_n\}_{n\in\mathbb{N}}$ and $\{p'_n\}_{n\in\mathbb{N}}$ are two representatives of $P\in X^*$ (this means that $P=\big[\{p_n\}_{n\in\mathbb{N}}\big]=\big[\{p'_n\}_{n\in\mathbb{N}}\big]$) and $\{q_n\}_{n\in\mathbb{N}}$ and $\{q'_n\}_{n\in\mathbb{N}}$ are two representatives of $Q\in X^*$, then*

$$\lim_{n\to\infty} d(p_n,q_n)=\lim_{n\to\infty} d(p'_n,q'_n)$$

Proof: Replace, in (2), p_m and q_m by p'_n and q'_n respectively. This gives

$$\big|d(p_n,q_n)-d(p'_n,q'_n)\big| \le d(p_n,p'_n)+d(q'_n,q_n)$$

Since $\{p'_n\}\in\big[\{p_n\}_{n\in\mathbb{N}}\big]$, we have that $\lim_{n\to\infty} d(p_n,p'_n)=0$. Since $\{q'_n\}\in\big[\{q_n\}_{n\in\mathbb{N}}\big]$, we have that $\lim_{n\to\infty} d(q_n,q'_n)=0$. Consequently, $\lim_{n\to\infty}\big|d(p_n,q_n)-d(p'_n,q'_n)\big|=0$. ■

Lemma 4 *Δ is a metric on X^*.*

Proof: Let

$$P=\big[\{p_n\}_{n\in\mathbb{N}}\big]\in X^* \qquad Q=\big[\{q_n\}_{n\in\mathbb{N}}\big]\in X^* \qquad R=\big[\{r_n\}_{n\in\mathbb{N}}\big]\in X^*$$

(i) Since every $d(p_n,q_n)\ge 0$, we have $\Delta(P,Q)=\lim_{n\to\infty} d(p_n,q_n)\ge 0$. On the other hand, if $\Delta(P,Q)=0$, then $\lim_{n\to\infty} d(p_n,q_n)=0$ so that $\{p_n\}\sim\{q_n\}$ and $P=Q$.

(ii) $\Delta(P,Q)=\lim_{n\to\infty} d(p_n,q_n)=\lim_{n\to\infty} d(q_n,p_n)=\Delta(Q,P)$.

(iii) $\Delta(P,Q)=\lim_{n\to\infty} d(p_n,q_n)\le\lim_{n\to\infty}\Big(d(p_n,r_n)+d(r_n,q_n)\Big)=\Delta(P,R)+\Delta(R,Q)$.

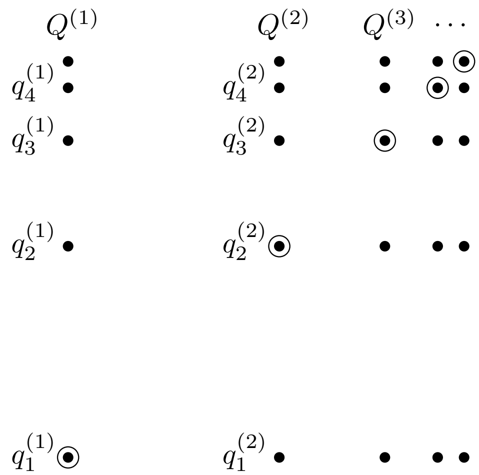
■

Lemma 5 X^* *is complete*

Proof: Let $\{Q^{(n)} \in X^*\}_{n\in\mathbb{N}}$ be a Cauchy sequence. We must prove that it has a limit, $\mathrm{Q} \in X^*$, as $n \to \infty$. Each $Q^{(n)}$ is an equivalence class of Cauchy sequences in X. Say $Q^{(n)} = \big[\{q_\ell^{(n)}\}_{\ell\in\mathbb{N}}\big]$. We shall guess $\mathrm{Q} = \big[\{\mathrm{q}_n\}_{n\in\mathbb{N}}\big]$ by choosing each q_n to be a $q_\ell^{(n)}$ with ℓ chosen larger and larger as n increases. Here we go:

$$\begin{aligned}
&\{q_\ell^{(1)}\}_{\ell\in\mathbb{N}} \text{ Cauchy} \Rightarrow \exists\,\ell_1 \in \mathbb{N} \text{ s.t. } \ell \geq \ell_1 \Rightarrow d(q_\ell^{(1)}, q_{\ell_1}^{(1)}) < 1. \text{ Pick } \mathrm{q}_1 = q_{\ell_1}^{(1)}.\\
&\{q_\ell^{(2)}\}_{\ell\in\mathbb{N}} \text{ Cauchy} \Rightarrow \exists\,\ell_2 > \ell_1 \text{ s.t. } \ell \geq \ell_2 \Rightarrow d(q_\ell^{(2)}, q_{\ell_2}^{(2)}) < \tfrac{1}{2}. \text{ Pick } \mathrm{q}_2 = q_{\ell_2}^{(2)}.\\
&\{q_\ell^{(3)}\}_{\ell\in\mathbb{N}} \text{ Cauchy} \Rightarrow \exists\,\ell_3 > \ell_2 \text{ s.t. } \ell \geq \ell_3 \Rightarrow d(q_\ell^{(3)}, q_{\ell_3}^{(3)}) < \tfrac{1}{3}. \text{ Pick } \mathrm{q}_3 = q_{\ell_3}^{(3)}.\\
&\qquad\vdots\\
&\{q_\ell^{(n)}\}_{\ell\in\mathbb{N}} \text{ Cauchy} \Rightarrow \exists\,\ell_n > \ell_{n-1} \text{ s.t. } \ell \geq \ell_n \Rightarrow d(q_\ell^{(n)}, q_{\ell_n}^{(n)}) < \tfrac{1}{n}. \text{ Pick } \mathrm{q}_n = q_{\ell_n}^{(n)}.\\
&\qquad\vdots
\end{aligned}$$

In the example sketched below, the $q_{\ell_n}^{(n)}$'s are circled.



Proof that $\{\mathrm{q}_n\}$ *is Cauchy:* Let $\varepsilon > 0$. By the triangle inequality

$$\begin{aligned}
d(\mathrm{q}_n, \mathrm{q}_m) &\leq d\big(q_{\ell_n}^{(n)}, q_\ell^{(n)}\big) + d\big(q_\ell^{(n)}, q_\ell^{(m)}\big) + d\big(q_\ell^{(m)}, q_{\ell_m}^{(m)}\big)\\
&= d\big(q_{\ell_n}^{(n)}, q_\ell^{(n)}\big) + \big[d\big(q_\ell^{(n)}, q_\ell^{(m)}\big) - \Delta(Q^{(n)}, Q^{(m)})\big] + \Delta(Q^{(n)}, Q^{(m)}) + d\big(q_\ell^{(m)}, q_{\ell_m}^{(m)}\big)
\end{aligned} \tag{3}$$

for any $\ell \in \mathbb{N}$.

- ∘ If $\ell \geq \ell_n$, the first term is smaller than $\frac{1}{n}$.
- ∘ By definition, $\Delta(Q^{(n)}, Q^{(m)}) = \lim_{\ell\to\infty} d\big(q_\ell^{(n)}, q_\ell^{(m)}\big)$. So there is a natural number $N_{n,m}$ such that the second term is smaller than $\frac{\varepsilon}{4}$ whenever $\ell \geq N_{n,m}$.
- ∘ By hypothesis, the sequence $\{Q^{(n)}\}$ is Cauchy. So there is a natural number $\tilde{N}$ such that the third term is smaller than $\frac{\varepsilon}{4}$ whenever $n, m \geq \tilde{N}$.
- ∘ Finally, if $\ell \geq \ell_m$, the last term is smaller than $\frac{1}{m}$.

Choose any natural number $N \geq \max\big\{\tilde{N}, \frac{4}{\varepsilon}\big\}$. I claim that $d(\mathrm{q}_n, \mathrm{q}_m) < \varepsilon$ whenever $n, m \geq N$. So let $n, m \geq N$. Now choose ℓ to be any natural number bigger than $\max\big\{N_{n,m}, \ell_n, \ell_m\big\}$. Then the four terms in (3) are each smaller than $\frac{\varepsilon}{4}$.

Proof that $\mathrm{Q} = \lim_{n\to\infty} Q^{(n)}$: Let $\varepsilon > 0$. By definition

$$\Delta(\mathrm{Q}, Q^{(n)}) = \lim_{m\to\infty} d(\mathrm{q}_m, q_m^{(n)}) = \lim_{m\to\infty} d\big(q_{\ell_m}^{(m)}, q_m^{(n)}\big)$$

By the triangle inequality

$$d\big(q_{\ell_m}^{(m)}, q_m^{(n)}\big) \le d\big(q_{\ell_m}^{(m)}, q_{\ell_n}^{(n)}\big) + d\big(q_{\ell_n}^{(n)}, q_m^{(n)}\big) \tag{3}$$

- Since the sequence $\big\{\mathrm{q}_n = q_{\ell_n}^{(n)}\big\}$ is Cauchy, there is a natural number N' such that the first term is smaller than $\frac{\varepsilon}{2}$ whenever $n, m \ge N'$.
- By the construction of ℓ_n, the second term is smaller than $\frac{1}{n}$ whenever $m \ge \ell_n$.

Choose any natural number $N \ge \max\big\{N', \frac{2}{\varepsilon}\big\}$. I claim that $\Delta(\mathrm{Q}, Q^{(n)}) < \varepsilon$ whenever $n \ge N$. So let $n \ge N$. Now choose m to be any natural number bigger than $\max\big\{N', \ell_n\big\}$. Then the two terms in (4) are each smaller than $\frac{\varepsilon}{2}$.

■

Lemma 6 $\Delta\big(\varphi(p), \varphi(q)\big) = d(p,q)$ *for all* $p, q \in X$.

Proof: $\Delta\big(\varphi(p), \varphi(q)\big) = \lim_{n\to\infty} d\big(\varphi(p)_n, \varphi(q)_n\big) = \lim_{n\to\infty} d(p,q) = d(p,q)$ ■

Lemma 7 φ *is one–to–one.*

Proof: $\varphi(p) = \varphi(q) \implies \Delta\big(\varphi(p), \varphi(q)\big) = 0 \implies d(p,q) = 0 \implies p = q.$ ■

Lemma 8 $\varphi(X)$ *is dense in* X^**.*

Proof: Let $P = \big[\{p_n\}_{n\in\mathbb{N}}\big] \in X^*$. Then I claim that the sequence $\big\{\varphi(p_m)\big\}_{m\in\mathbb{N}}$ converges in X^* to P. To check this, it suffices to observe that

$$\Delta\big(P, \varphi(p_m)\big) = \lim_{n\to\infty} d\big(p_n, \varphi(p_m)_n\big) = \lim_{n\to\infty} d(p_n, p_m)$$

Since $\{p_n\}_{n\in\mathbb{N}}$ is Cauchy, this converges to zero as $m \to \infty$. ■

Lemma 9 *If* X *is complete, then* $\varphi(X) = X^*$*.*

Proof: Let $Q \in X^*$. We must find $q \in X$ with $\varphi(q) = Q$. By Lemma 8

$$\begin{aligned}
&\exists\ \big\{q_n \in X\big\}_{n\in\mathbb{N}}\ \text{ s.t. }\ Q = \lim_{n\to\infty} \varphi(q_n) \\
&\implies \big\{\varphi(q_n)\big\} \text{ is Cauchy in } X^* \\
&\implies \big\{q_n\big\} \text{ is Cauchy in } X\text{, by Lemma 6} \\
&\implies q = \lim_{n\to\infty} q_n \text{ exists, since } X \text{ is complete} \\
&\implies \varphi(q) = \lim_{n\to\infty} \varphi(q_n) \text{ by Lemma 6} \\
&\implies \varphi(q) = Q
\end{aligned}$$

■