Completion

Theorem. *Let $(\mathcal{M}, d)$ be a metric space. Then there exists a metric space $(\mathcal{M}^*, \Delta)$ and a map $\varphi : \mathcal{M} \to \mathcal{M}^*$ such that*

(1) φ is one–to–one. That is, $\varphi(x) = \varphi(y)$ if and only if $x = y$.

(2) $\Delta\big(\varphi(x), \varphi(y)\big) = d(x, y)$ for all $x, y \in \mathcal{M}$.

(3) $\mathcal{M}^*$ is complete.

(4) $\varphi(\mathcal{M})$ is dense in $\mathcal{M}^*$. This means that each element of $\mathcal{M}^*$ is a limit of elements of

$$\varphi(\mathcal{M}) = \big\{\ \varphi(x) \ \big|\ x \in \mathcal{M}\ \big\}$$

Equivalently, for each $P \in \mathcal{M}^*$ and each $\varepsilon > 0$, there is a $p \in \mathcal{M}$ with $\Delta\big(\varphi(p), P\big) < \varepsilon$.

(5) If $\mathcal{M}$ is complete, then $\varphi(\mathcal{M}) = \mathcal{M}^*$.

$$\mathcal{M}, d \xrightarrow[d(p,q)=\Delta(\varphi(p),\varphi(q))]{\varphi \quad 1\text{–}1} \varphi(\mathcal{M}), \Delta \underset{\text{dense}}{\subset} \underset{\text{complete}}{\mathcal{M}^*, \Delta}$$

Remark.

(a) $\mathcal{M}^*$ is called the completion of $\mathcal{M}$.

(b) φ gives a one-to–one correspondance between elements of $\mathcal{M}$ and elements of $\varphi(\mathcal{M})$. So we can think of φ as giving the new name $\varphi(x)$ to each $x \in \mathcal{M}$ and we can think of $\varphi(\mathcal{M})$ as being the same as $\mathcal{M}$ but with the names of the elements changed. Thus, we can think of $\mathcal{M}$ as being $\varphi(\mathcal{M}) \subset \mathcal{M}^*$. Using this point of view, the above theorem says that any metric space can be extended to a complete metric space. I.e. can have its holes filled in.

(c) If $\mathcal{M} = \mathbb{Q}$, then we can take $\mathcal{M}^* = \mathbb{R}$, $\Delta(x, y) = |x - y|$ and $\varphi(x) = x$. It is easy to check that (1)–(4) are obeyed.

Motivation.

The hard part of the proof is to make a guess as to what $\mathcal{M}^*$ should be. That's what we'll do now. A good strategy is to work backwards. Suppose that, somehow, we have found a suitable $\mathcal{M}^*$ with $\mathcal{M} \subset \mathcal{M}^*$. If we can find a way to describe each element of $\mathcal{M}^*$ purely in terms of elements of $\mathcal{M}$, then we can turn around and take that as the definition of $\mathcal{M}^*$.

Consider, for example $\mathcal{M} = \mathbb{Q}$ and $\mathcal{M}^* = \mathbb{R}$. Each element of $\mathbb{R}$ may be written as the limit of a sequence of rational numbers, and each such sequence is Cauchy. For example, $\sqrt{2}$ is the limit of the Cauchy sequence 1, 1.4, 1.41, 1.414, 1.4142, $\cdots$. Thus specifying a real number is equivalent to specifying a Cauchy sequence of rational numbers. But there is not a one–to–one correspondance between real numbers and Cauchy sequences of rational numbers, because many different Cauchy sequences of rational numbers converge to the same real number. For example

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ \cdots$$
$$0,\ 1.3,\ 1.40,\ 1.413,\ 1.4141,\ \cdots$$
$$2,\ 1.5,\ 1.42,\ 1.415,\ 1.4143,\ \cdots$$

all converge to $\sqrt{2}$. (For the second sequence, I just subtracted one from the last nonzero decimal place. For the third sequence, I added one to the last nonzero decimal place.) To get a one–to–one correspondance, we can identify each real number P with the set of all Cauchy sequences in $\mathbb{Q}$ that converge to P.

Outline of Proof. First define

$$\mathcal{M}' = \left\{ \left(p_n\right)_{n\in\mathbb{N}} \;\middle|\; \left(p_n\right)_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } \mathcal{M} \right\}$$

That is, $\mathcal{M}'$ is the set of all Cauchy sequences in $\mathcal{M}$. Next we define two Cauchy sequences $\left(p_n\right)_{n\in\mathbb{N}}$ and $\left(q_n\right)_{n\in\mathbb{N}}$ in $\mathcal{M}$ to be "equivalent", written $\left(p_n\right)_{n\in\mathbb{N}} \sim \left(q_n\right)_{n\in\mathbb{N}}$, if and only if

$$\lim_{n\to\infty} d(p_n, q_n) = 0$$

This definition is rigged so that any two convergent sequences have the same limit if and only if they are equivalent. Next, if $\left(p_n\right)_{n\in\mathbb{N}}$, we define the "equivalence class of $\left(p_n\right)_{n\in\mathbb{N}}$" to be the set

$$\left[\left(p_n\right)_{n\in\mathbb{N}}\right] = \left\{ \left(q_n\right)_{n\in\mathbb{N}} \in \mathcal{M}' \;\middle|\; \left(q_n\right)_{n\in\mathbb{N}} \sim \left(p_n\right)_{n\in\mathbb{N}} \right\}$$

of all Cauchy sequences that are equivalent to $\left(p_n\right)_{n\in\mathbb{N}}$. We shall shortly prove

Lemma 1 *If $\left(p_n\right)_{n\in\mathbb{N}}$, $\left(q_n\right)_{n\in\mathbb{N}} \in \mathcal{M}'$ then*

$$\textit{either} \quad \left[\left(p_n\right)_{n\in\mathbb{N}}\right] = \left[\left(q_n\right)_{n\in\mathbb{N}}\right] \qquad \textit{or} \qquad \left[\left(p_n\right)_{n\in\mathbb{N}}\right] \cap \left[\left(q_n\right)_{n\in\mathbb{N}}\right] = \emptyset$$

If you think of a Cauchy sequence as one person and an equivalence class of Cauchy sequences as a "family" of related people, then the above Lemma says, that the whole world is divided into a collection of nonoverlapping families. Next, we define

$$\mathcal{M}^* = \left\{ \left[\left(p_n\right)_{n\in\mathbb{N}}\right] \;\middle|\; \left(p_n\right)_{n\in\mathbb{N}} \in \mathcal{M}' \right\}$$

as the set of all "families" and prove

Lemma 2 *If $\left(p_n\right)_{n\in\mathbb{N}}$, $\left(q_n\right)_{n\in\mathbb{N}} \in \mathcal{M}'$ then $\lim\limits_{n\to\infty} d(p_n, q_n)$ exists.*

Lemma 3 *If $\left(p_n\right)_{n\in\mathbb{N}}$ and $\left(p'_n\right)_{n\in\mathbb{N}}$ are two representatives of $P \in \mathcal{M}^*$ (this means that $P = \left[\left(p_n\right)_{n\in\mathbb{N}}\right] = \left[\left(p'_n\right)_{n\in\mathbb{N}}\right]$) and $\left(q_n\right)_{n\in\mathbb{N}}$ and $\left(q'_n\right)_{n\in\mathbb{N}}$ are two representatives of $Q \in \mathcal{M}^*$, then*

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n)$$

This allows us to define $\Delta(P, Q)$. If $P, Q \in \mathcal{M}^*$ and $\left(p_n\right)_{n\in\mathbb{N}}$ is a representative of P and $\left(q_n\right)_{n\in\mathbb{N}}$ is a representative of $Q \in \mathcal{M}^*$, we define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n) \tag{1}$$

By Lemma 3, we always get the same value for $\Delta(P, Q)$ no matter which representatives for P and Q are used in (1). Finally, we define $\varphi : \mathcal{M} \to \mathcal{M}^*$ by

$$\varphi(p) = \left[\left(p, p, p, \cdots\right)\right]$$

The conclusions of the Theorem are now proven as a series of Lemmata.

Lemma 4 *Δ is a metric on $\mathcal{M}^*$.*

Lemma 5 *$\mathcal{M}^*$ is complete*

Lemma 6 *$\Delta\big(\varphi(p), \varphi(q)\big) = d(p,q)$ for all $p, q \in \mathcal{M}$.*

Lemma 7 *φ is one–to–one.*

Lemma 8 *$\varphi(\mathcal{M})$ is dense in $\mathcal{M}^*$.*

Lemma 9 *If $\mathcal{M}$ is complete, then $\varphi(\mathcal{M}) = \mathcal{M}^*$.*

■

So now we just have to prove all of the Lemmata.

Lemma 1 *If $\big(p_n\big)_{n\in\mathbb{N}},\ \big(q_n\big)_{n\in\mathbb{N}} \in \mathcal{M}'$ then*

$$\textit{either} \quad \big[\big(p_n\big)_{n\in\mathbb{N}}\big] = \big[\big(q_n\big)_{n\in\mathbb{N}}\big] \qquad \textit{or} \qquad \big[\big(p_n\big)_{n\in\mathbb{N}}\big] \cap \big[\big(q_n\big)_{n\in\mathbb{N}}\big] = \emptyset$$

Proof: Suppose that the conclusion is false. Then $\big[\big(p_n\big)_{n\in\mathbb{N}}\big] \cap \big[\big(q_n\big)_{n\in\mathbb{N}}\big]$ must be nonempty so that there is a Cauchy sequence $\big(a_n\big)_{n\in\mathbb{N}}$ that is in both $\big[\big(p_n\big)_{n\in\mathbb{N}}\big]$ and $\big[\big(q_n\big)_{n\in\mathbb{N}}\big]$. Consequently (a_n) is equivalent to both (p_n) and (q_n) so that

$$\lim_{n\to\infty} d(a_n, p_n) = 0 \text{ and } \lim_{n\to\infty} d(a_n, q_n) = 0$$

On the other hand $\big[\big(p_n\big)_{n\in\mathbb{N}}\big]$ and $\big[\big(q_n\big)_{n\in\mathbb{N}}\big]$ must be different, so that there is a Cauchy sequence (b_n) that is in one of them (say $\big[\big(p_n\big)_{n\in\mathbb{N}}\big]$) and not in the other (say $\big[\big(q_n\big)_{n\in\mathbb{N}}\big]$). Since (b_n) is in $\big[\big(p_n\big)_{n\in\mathbb{N}}\big]$,

$$\lim_{n\to\infty} d(b_n, p_n) = 0$$

By the triangle inequality

$$d(b_n, q_n) \le d(b_n, p_n) + d(p_n, a_n) + d(a_n, q_n)$$

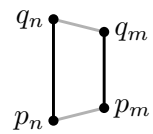
But this forces $\lim\limits_{n\to\infty} d(b_n, q_n) = 0$, which contradicts the assumption that (b_n) is not in $\big[\big(q_n\big)_{n\in\mathbb{N}}\big]$.

■

Lemma 2 *If $\big(p_n\big)_{n\in\mathbb{N}},\ \big(q_n\big)_{n\in\mathbb{N}} \in \mathcal{M}'$ then $\lim\limits_{n\to\infty} d(p_n, q_n)$ exists.*

Proof: We first prove

$$\big|d(p_n,q_n)-d(p_m,q_m)\big| \le d(p_n,p_m)+d(q_m,q_n) \tag{2}$$



by combining the two right hand inequalities of

$$\begin{aligned} d(p_n,q_n) \le d(p_n,p_m)+d(p_m,q_m)+d(q_m,q_n) \;&\Rightarrow\; d(p_n,q_n)-d(p_m,q_m) \le d(p_n,p_m)+d(q_m,q_n) \\ d(p_m,q_m) \le d(p_m,p_n)+d(p_n,q_n)+d(q_n,q_m) \;&\Rightarrow\; d(p_m,q_m)-d(p_n,q_n) \le d(p_n,p_m)+d(q_n,q_m) \end{aligned}$$

Since $\big(p_n\big)_{n\in\mathbb{N}}$ and $\big(q_n\big)_{n\in\mathbb{N}}$ are both Cauchy, both $d(p_n,p_m)$ and $d(q_m,q_n)$ converge to zero as $m,n\to\infty$. So the same is true for $\big|d(p_n,q_n)-d(p_m,q_m)\big|$. So $\big(d(p_n,q_n)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, this Cauchy sequence converges. ■

Lemma 3 *If $\big(p_n\big)_{n\in\mathbb{N}}$ and $\big(p'_n\big)_{n\in\mathbb{N}}$ are two representatives of $P\in\mathcal{M}^*$ (this means that $P=\big[\big(p_n\big)_{n\in\mathbb{N}}\big]=\big[\big(p'_n\big)_{n\in\mathbb{N}}\big]$) and $\big(q_n\big)_{n\in\mathbb{N}}$ and $\big(q'_n\big)_{n\in\mathbb{N}}$ are two representatives of $Q\in\mathcal{M}^*$, then*

$$\lim_{n\to\infty} d(p_n,q_n) = \lim_{n\to\infty} d(p'_n,q'_n)$$

Proof: Replace, in (2), p_m and q_m by p'_n and q'_n respectively. This gives

$$\big|d(p_n,q_n)-d(p'_n,q'_n)\big| \le d(p_n,p'_n)+d(q'_n,q_n)$$

Since $(p'_n)\in\big[\big(p_n\big)_{n\in\mathbb{N}}\big]$, we have that $\lim\limits_{n\to\infty} d(p_n,p'_n)=0$. Since $(q'_n)\in\big[\big(q_n\big)_{n\in\mathbb{N}}\big]$, we have that $\lim\limits_{n\to\infty} d(q_n,q'_n)=0$. Consequently, $\lim\limits_{n\to\infty}\big|d(p_n,q_n)-d(p'_n,q'_n)\big|=0$. ■

Lemma 4 *Δ is a metric on $\mathcal{M}^*$.*

Proof: Let

$$P=\big[\big(p_n\big)_{n\in\mathbb{N}}\big]\in\mathcal{M}^* \qquad Q=\big[\big(q_n\big)_{n\in\mathbb{N}}\big]\in\mathcal{M}^* \qquad R=\big[\big(r_n\big)_{n\in\mathbb{N}}\big]\in\mathcal{M}^*$$

(i) Since every $d(p_n,q_n)\ge 0$, we have $\Delta(P,Q)=\lim\limits_{n\to\infty} d(p_n,q_n)\ge 0$. On the other hand, if $\Delta(P,Q)=0$, then $\lim\limits_{n\to\infty} d(p_n,q_n)=0$ so that $\big(p_n\big)_{n\in\mathbb{N}}\sim\big(q_n\big)_{n\in\mathbb{N}}$ and $P=Q$.

(ii) $\Delta(P,Q)=\lim\limits_{n\to\infty} d(p_n,q_n)=\lim\limits_{n\to\infty} d(q_n,p_n)=\Delta(Q,P)$.

(iii) $\Delta(P,Q)=\lim\limits_{n\to\infty} d(p_n,q_n)\le\lim\limits_{n\to\infty}\Big(d(p_n,r_n)+d(r_n,q_n)\Big)=\Delta(P,R)+\Delta(R,Q)$.

■

Lemma 5 *$\mathcal{M}^*$ is complete*

Proof: Let $\big(Q^{(n)} \in \mathcal{M}^*\big)_{n\in\mathbb{N}}$ be a Cauchy sequence. We must prove that it has a limit, $\mathrm{Q} \in \mathcal{M}^*$, as $n \to \infty$. Each $Q^{(n)}$ is an equivalence class of Cauchy sequences in $\mathcal{M}$. Say $Q^{(n)} = \big[\big(q_\ell^{(n)}\big)_{\ell\in\mathbb{N}}\big]$. We shall guess $\mathrm{Q} = \big[\big(\mathrm{q}_n\big)_{n\in\mathbb{N}}\big]$ by choosing each q_n to be a $q_\ell^{(n)}$ with ℓ chosen larger and larger as n increases. Here we go:

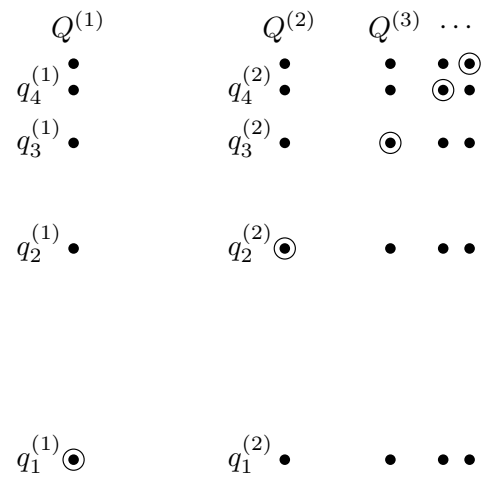
$\big(q_\ell^{(1)}\big)_{\ell\in\mathbb{N}}$ Cauchy $\Rightarrow \exists\, \ell_1 \in \mathbb{N}$ s.t. $\ell \ge \ell_1 \Rightarrow d(q_\ell^{(1)}, q_{\ell_1}^{(1)}) < 1$. Pick $\mathrm{q}_1 = q_{\ell_1}^{(1)}$.
$\big(q_\ell^{(2)}\big)_{\ell\in\mathbb{N}}$ Cauchy $\Rightarrow \exists\, \ell_2 > \ell_1$ s.t. $\ell \ge \ell_2 \Rightarrow d(q_\ell^{(2)}, q_{\ell_2}^{(2)}) < \frac{1}{2}$. Pick $\mathrm{q}_2 = q_{\ell_2}^{(2)}$.
$\big(q_\ell^{(3)}\big)_{\ell\in\mathbb{N}}$ Cauchy $\Rightarrow \exists\, \ell_3 > \ell_2$ s.t. $\ell \ge \ell_3 \Rightarrow d(q_\ell^{(3)}, q_{\ell_3}^{(3)}) < \frac{1}{3}$. Pick $\mathrm{q}_3 = q_{\ell_3}^{(3)}$.

$\vdots$

$\big(q_\ell^{(n)}\big)_{\ell\in\mathbb{N}}$ Cauchy $\Rightarrow \exists\, \ell_n > \ell_{n-1}$ s.t. $\ell \ge \ell_n \Rightarrow d(q_\ell^{(n)}, q_{\ell_n}^{(n)}) < \frac{1}{n}$. Pick $\mathrm{q}_n = q_{\ell_n}^{(n)}$.

$\vdots$

In the example sketched below, the $q_{\ell_n}^{(n)}$'s are circled.



Proof that $\big(\mathrm{q}_n\big)_{n\in\mathbb{N}}$ is Cauchy: Let $\varepsilon > 0$. By the triangle inequality

$$\begin{aligned} d(\mathrm{q}_n, \mathrm{q}_m) &\le d\big(q_{\ell_n}^{(n)}, q_\ell^{(n)}\big) + d\big(q_\ell^{(n)}, q_\ell^{(m)}\big) + d\big(q_\ell^{(m)}, q_{\ell_m}^{(m)}\big) \\ &= d\big(q_{\ell_n}^{(n)}, q_\ell^{(n)}\big) + \big[d\big(q_\ell^{(n)}, q_\ell^{(m)}\big) - \Delta(Q^{(n)}, Q^{(m)})\big] + \Delta(Q^{(n)}, Q^{(m)}) + d\big(q_\ell^{(m)}, q_{\ell_m}^{(m)}\big) \qquad (3) \end{aligned}$$

for any $\ell \in \mathbb{N}$.

- If $\ell \ge \ell_n$, the first term is smaller than $\frac{1}{n}$.
- By definition, $\Delta(Q^{(n)}, Q^{(m)}) = \lim\limits_{\ell\to\infty} d\big(q_\ell^{(n)}, q_\ell^{(m)}\big)$. So there is a natural number $N_{n,m}$ such that the second term is smaller than $\frac{\varepsilon}{4}$ whenever $\ell \ge N_{n,m}$.
- By hypothesis, the sequence $(Q^{(n)})$ is Cauchy. So there is a natural number $\tilde{N}$ such that the third term is smaller than $\frac{\varepsilon}{4}$ whenever $n, m \ge \tilde{N}$.
- Finally, if $\ell \ge \ell_m$, the last term is smaller than $\frac{1}{m}$.

Choose any natural number $N \ge \max\big\{\tilde{N}, \frac{4}{\varepsilon}\big\}$. I claim that $d(\mathrm{q}_n, \mathrm{q}_m) < \varepsilon$ whenever $n, m \ge N$. So let $n, m \ge N$. Now choose ℓ to be any natural number bigger than $\max\big\{N_{n,m}, \ell_n, \ell_m\big\}$. Then the four terms in (3) are each smaller than $\frac{\varepsilon}{4}$.

Proof that $\mathrm{Q} = \lim_{n\to\infty} Q^{(n)}$*:* Let $\varepsilon > 0$. By definition

$$\Delta(\mathrm{Q}, Q^{(n)}) = \lim_{m\to\infty} d(\mathrm{q}_m, q_m^{(n)}) = \lim_{m\to\infty} d\big(q_{\ell_m}^{(m)}, q_m^{(n)}\big)$$

By the triangle inequality

$$d\big(q_{\ell_m}^{(m)}, q_m^{(n)}\big) \le d\big(q_{\ell_m}^{(m)}, q_{\ell_n}^{(n)}\big) + d\big(q_{\ell_n}^{(n)}, q_m^{(n)}\big) \tag{3}$$

- Since the sequence $\big(\mathrm{q}_n = q_{\ell_n}^{(n)}\big)$ is Cauchy, there is a natural number N' such that the first term is smaller than $\frac{\varepsilon}{2}$ whenever $n, m \ge N'$.
- By the construction of ℓ_n, the second term is smaller than $\frac{1}{n}$ whenever $m \ge \ell_n$.

Choose any natural number $N \ge \max\big\{N', \frac{2}{\varepsilon}\big\}$. I claim that $\Delta(\mathrm{Q}, Q^{(n)}) < \varepsilon$ whenever $n \ge N$. So let $n \ge N$. Now choose m to be any natural number bigger than $\max\big\{N', \ell_n\big\}$. Then the two terms in (4) are each smaller than $\frac{\varepsilon}{2}$. ■

Lemma 6 $\Delta\big(\varphi(p), \varphi(q)\big) = d(p, q)$ *for all* $p, q \in \mathcal{M}$.

Proof: $\Delta\big(\varphi(p), \varphi(q)\big) = \lim_{n\to\infty} d\big(\varphi(p)_n, \varphi(q)_n\big) = \lim_{n\to\infty} d(p, q) = d(p, q)$ ■

Lemma 7 φ *is one–to–one.*

Proof: $\varphi(p) = \varphi(q) \implies \Delta\big(\varphi(p), \varphi(q)\big) = 0 \implies d(p, q) = 0 \implies p = q.$ ■

Lemma 8 $\varphi(\mathcal{M})$ *is dense in* $\mathcal{M}^*$.

Proof: Let $P = \big[\big(p_n\big)_{n\in\mathbb{N}}\big] \in \mathcal{M}^*$. Then I claim that the sequence $\big(\varphi(p_m)\big)_{m\in\mathbb{N}}$ converges in $\mathcal{M}^*$ to P. To check this, it suffices to observe that

$$\Delta\big(P, \varphi(p_m)\big) = \lim_{n\to\infty} d\big(p_n, \varphi(p_m)_n\big) = \lim_{n\to\infty} d(p_n, p_m)$$

Since $\big(p_n\big)_{n\in\mathbb{N}}$ is Cauchy, this converges to zero as $m \to \infty$. ■

Lemma 9 *If* $\mathcal{M}$ *is complete, then* $\varphi(\mathcal{M}) = \mathcal{M}^*$.

Proof: Let $Q \in \mathcal{M}^*$. We must find $q \in \mathcal{M}$ with $\varphi(q) = Q$. By Lemma 8

$$\begin{aligned}
&\exists\ \big(q_n \in \mathcal{M}\big)_{n\in\mathbb{N}} \text{ s.t. } Q = \lim_{n\to\infty} \varphi(q_n)\\
&\implies \big(\varphi(q_n)\big)_{n\in\mathbb{N}} \text{ is Cauchy in } \mathcal{M}^*\\
&\implies \big(q_n\big)_{n\in\mathbb{N}} \text{ is Cauchy in } \mathcal{M}, \text{ by Lemma 6}\\
&\implies q = \lim_{n\to\infty} q_n \text{ exists, since } \mathcal{M} \text{ is complete}\\
&\implies \varphi(q) = \lim_{n\to\infty} \varphi(q_n) \text{ by Lemma 6}\\
&\implies \varphi(q) = Q
\end{aligned}$$

■