Completion

**Theorem.** Let \((M, d)\) be a metric space. Then there exists a metric space \((M^*, \Delta)\) and a map \(\varphi: M \rightarrow M^*\) such that

1. \(\varphi\) is one–to–one. That is, \(\varphi(x) = \varphi(y)\) if and only if \(x = y\).
2. \(\Delta(\varphi(x), \varphi(y)) = d(x, y)\) for all \(x, y \in M\).
3. \(M^*\) is complete.
4. \(\varphi(M)\) is dense in \(M^*\). This means that each element of \(M^*\) is a limit of elements of

\[\varphi(M) = \{ \varphi(x) \mid x \in M \}\]

Equivalently, for each \(P \in M^*\) and each \(\varepsilon > 0\), there is a \(p \in M\) with \(\Delta(\varphi(p), P) < \varepsilon\).
5. If \(M\) is complete, then \(\varphi(M) = M^*\).

\[
\begin{array}{ccc}
M, d & \overset{\varphi \ 1-1}{\longrightarrow} & \varphi(M), \Delta \\
\downarrow \scriptstyle{d(p,q) = \Delta(\varphi(p), \varphi(q))} & & \downarrow \scriptstyle{\text{dense complete}} \\
\end{array}
\]

**Remark.**

(a) \(M^*\) is called the completion of \(M\).

(b) \(\varphi\) gives a one-to-one correspondence between elements of \(M\) and elements of \(\varphi(M)\). So we can think of \(\varphi\) as giving the new name \(\varphi(x)\) to each \(x \in M\) and we can think of \(\varphi(M)\) as being the same as \(M\) but with the names of the elements changed. Thus, we can think of \(M\) as being \(\varphi(M) \subset M^*\). Using this point of view, the above theorem says that any metric space can be extended to a complete metric space. I.e. can have its holes filled in.

(c) If \(M = \mathbb{Q}\), then we can take \(M^* = \mathbb{R}\), \(\Delta(x, y) = |x - y|\) and \(\varphi(x) = x\). It is easy to check that (1)–(4) are obeyed.

**Motivation.**

The hard part of the proof is to make a guess as to what \(M^*\) should be. That’s what we’ll do now. A good strategy is to work backwards. Suppose that, somehow, we have found a suitable \(M^*\) with \(M \subset M^*\). If we can find a way to describe each element of \(M^*\) purely in terms of elements of \(M\), then we can turn around and take that as the definition of \(M^*\).

Consider, for example \(M = \mathbb{Q}\) and \(M^* = \mathbb{R}\). Each element of \(\mathbb{R}\) may be written as the limit of a sequence of rational numbers, and each such sequence is Cauchy. For example, \(\sqrt{2}\) is the limit of the Cauchy sequence \(1, 1.4, 1.41, 1.414, 1.4142, \ldots\). Thus specifying a real number is equivalent to specifying a Cauchy sequence of rational numbers. But there is not a one-to-one correspondence between real numbers and Cauchy sequences of rational numbers, because many different Cauchy sequences of rational numbers converge to the same real number. For example

\[
\begin{align*}
1, & 1.4, 1.41, 1.414, 1.4142, \ldots \\
0, & 1.3, 1.40, 1.413, 1.4141, \ldots \\
2, & 1.5, 1.42, 1.415, 1.4143, \ldots
\end{align*}
\]
all converge to \( \sqrt{2} \). (For the second sequence, I just subtracted one from the last nonzero decimal place. For the third sequence, I added one to the last nonzero decimal place.) To get a one-to-one correspondence, we can identify each real number \( P \) with the set of all Cauchy sequences in \( \mathbb{Q} \) that converge to \( P \).

**Outline of Proof.** First define

\[
M' = \left\{ (p_n)_{n \in \mathbb{N}} \mid (p_n)_{n \in \mathbb{N}} \text{ is a Cauchy sequence in } M \right\}
\]

That is, \( M' \) is the set of all Cauchy sequences in \( M \). Next we define two Cauchy sequences \( (p_n)_{n \in \mathbb{N}} \) and \( (q_n)_{n \in \mathbb{N}} \) in \( M \) to be “equivalent”, written \( (p_n)_{n \in \mathbb{N}} \sim (q_n)_{n \in \mathbb{N}} \), if and only if

\[
\lim_{n \to \infty} d(p_n, q_n) = 0
\]

This definition is rigged so that any two convergent sequences have the same limit if and only if they are equivalent. Next, if \( (p_n)_{n \in \mathbb{N}} \), we define the “equivalence class of \( (p_n)_{n \in \mathbb{N}} \)” to be the set

\[
[(p_n)_{n \in \mathbb{N}}] = \left\{ (q_n)_{n \in \mathbb{N}} \in M' \mid (q_n)_{n \in \mathbb{N}} \sim (p_n)_{n \in \mathbb{N}} \right\}
\]

of all Cauchy sequences that are equivalent to \( (p_n)_{n \in \mathbb{N}} \). We shall shortly prove

**Lemma 1** If \( (p_n)_{n \in \mathbb{N}}, (q_n)_{n \in \mathbb{N}} \in M' \) then

either \( [(p_n)_{n \in \mathbb{N}}] = [(q_n)_{n \in \mathbb{N}}] \) or \( [(p_n)_{n \in \mathbb{N}}] \cap [(q_n)_{n \in \mathbb{N}}] = \emptyset \)

If you think of a Cauchy sequence as one person and an equivalence class of Cauchy sequences as a “family” of related people, then the above Lemma says, that the whole world is divided into a collection of nonoverlapping families. Next, we define

\[
M^* = \left\{ [(p_n)_{n \in \mathbb{N}}] \mid (p_n)_{n \in \mathbb{N}} \in M' \right\}
\]

as the set of all “families” and prove

**Lemma 2** If \( (p_n)_{n \in \mathbb{N}}, (q_n)_{n \in \mathbb{N}} \in M' \) then \( \lim_{n \to \infty} d(p_n, q_n) \) exists.

**Lemma 3** If \( (p_n)_{n \in \mathbb{N}} \) and \( (p'_n)_{n \in \mathbb{N}} \) are two representatives of \( P \in M^* \) (this means that \( P = [(p_n)_{n \in \mathbb{N}}] = [(p'_n)_{n \in \mathbb{N}}] \)) and \( (q_n)_{n \in \mathbb{N}} \) and \( (q'_n)_{n \in \mathbb{N}} \) are two representatives of \( Q \in M^* \), then

\[
\lim_{n \to \infty} d(p_n, q_n) = \lim_{n \to \infty} d(p'_n, q'_n)
\]

This allows us to define \( \Delta(P, Q) \). If \( P, Q \in M^* \) and \( (p_n)_{n \in \mathbb{N}} \) is a representative of \( P \) and \( (q_n)_{n \in \mathbb{N}} \) is a representative of \( Q \in M^* \), we define

\[
\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n)
\]

(1)

By Lemma 3, we always get the same value for \( \Delta(P, Q) \) no matter which representatives for \( P \) and \( Q \) are used in (1). Finally, we define \( \varphi : M \to M^* \) by

\[
\varphi(p) = [(p, p, p, \ldots)]
\]

The conclusions of the Theorem are now proven as a series of Lemmata.
Lemma 4 \( \Delta \) is a metric on \( M^* \).

Lemma 5 \( M^* \) is complete

Lemma 6 \( \Delta(\varphi(p), \varphi(q)) = d(p, q) \) for all \( p, q \in M \).

Lemma 7 \( \varphi \) is one-to-one.

Lemma 8 \( \varphi(M) \) is dense in \( M^* \).

Lemma 9 If \( M \) is complete, then \( \varphi(M) = M^* \).

So now we just have to prove all of the Lemmata.

Lemma 1 If \( (p_n)_{n \in \mathbb{N}}, (q_n)_{n \in \mathbb{N}} \in M' \) then

\[
either (p_n)_{n \in \mathbb{N}} = (q_n)_{n \in \mathbb{N}} \quad \text{or} \quad (p_n)_{n \in \mathbb{N}} \cap (q_n)_{n \in \mathbb{N}} = \emptyset
\]

Proof: Suppose that the conclusion is false. Then \( (p_n)_{n \in \mathbb{N}} \cap (q_n)_{n \in \mathbb{N}} \) must be nonempty so that there is a Cauchy sequence \( (a_n)_{n \in \mathbb{N}} \) that is in both \( (p_n)_{n \in \mathbb{N}} \) and \( (q_n)_{n \in \mathbb{N}} \). Consequently \( (a_n) \) is equivalent to both \( (p_n) \) and \( (q_n) \) so that

\[
\lim_{n \to \infty} d(a_n, p_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(a_n, q_n) = 0
\]

On the other hand \( (p_n)_{n \in \mathbb{N}} \) and \( (q_n)_{n \in \mathbb{N}} \) must be different, so that there is a Cauchy sequence \( (b_n) \) that is in one of them (say \( (p_n)_{n \in \mathbb{N}} \)) and not in the other (say \( (q_n)_{n \in \mathbb{N}} \)). Since \( (b_n) \) is in \( (p_n)_{n \in \mathbb{N}} \),

\[
\lim_{n \to \infty} d(b_n, p_n) = 0
\]

By the triangle inequality

\[
d(b_n, q_n) \leq d(b_n, p_n) + d(p_n, a_n) + d(a_n, q_n)
\]

But this forces \( \lim_{n \to \infty} d(b_n, q_n) = 0 \), which contradicts the assumption that \( (b_n) \) is not in \( (q_n)_{n \in \mathbb{N}} \).

Lemma 2 If \( (p_n)_{n \in \mathbb{N}}, (q_n)_{n \in \mathbb{N}} \in M' \) then \( \lim_{n \to \infty} d(p_n, q_n) \) exists.
Proof: We first prove
\[ |d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) \]  \hspace{1cm} (2)
by combining the two right hand inequalities of
\[
\begin{align*}
d(p_n, q_n) &\leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \implies d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_m, q_n) \\
d(p_m, q_m) &\leq d(p_m, p_n) + d(p_n, q_n) + d(q_n, q_m) \implies d(p_m, q_m) - d(p_n, q_n) \leq d(p_n, p_m) + d(q_m, q_n)
\end{align*}
\]
Since \((p_n)_{n \in \mathbb{N}}\) and \((q_n)_{n \in \mathbb{N}}\) are both Cauchy, both \(d(p_n, p_m)\) and \(d(q_m, q_n)\) converge to zero as \(m, n \to \infty\). So the same is true for \(|d(p_n, q_n) - d(p_m, q_m)|\). So \((d(p_n, q_n))_{n \in \mathbb{N}}\) is a Cauchy sequence of real numbers. Since \(\mathbb{R}\) is complete, this Cauchy sequence converges. \(\blacksquare\)

Lemma 3 If \((p_n)_{n \in \mathbb{N}}\) and \((p'_n)_{n \in \mathbb{N}}\) are two representatives of \(P \in \mathcal{M}^*\) (this means that \(P = [(p_n)_{n \in \mathbb{N}}] = [(p'_n)_{n \in \mathbb{N}}]\)) and \((q_n)_{n \in \mathbb{N}}\) and \((q'_n)_{n \in \mathbb{N}}\) are two representatives of \(Q \in \mathcal{M}^*\), then
\[
\lim_{n \to \infty} d(p_n, q_n) = \lim_{n \to \infty} d(p'_n, q'_n)
\]

Proof: Replace, in (2), \(p_m\) and \(q_m\) by \(p'_m\) and \(q'_m\) respectively. This gives
\[
|d(p_n, q_n) - d(p'_n, q'_n)| \leq d(p_n, p'_n) + d(q'_n, q_n)
\]
Since \((p'_n) \in [(p_n)_{n \in \mathbb{N}}]\), we have that \(\lim_{n \to \infty} d(p_n, p'_n) = 0\). Since \((q'_n) \in [(q_n)_{n \in \mathbb{N}}]\), we have that \(\lim_{n \to \infty} d(q_n, q'_n) = 0\). Consequently, \(\lim_{n \to \infty} |d(p_n, q_n) - d(p'_n, q'_n)| = 0\). \(\blacksquare\)

Lemma 4 \(\Delta\) is a metric on \(\mathcal{M}^*\).

Proof: Let
\[
P = [(p_n)_{n \in \mathbb{N}}] \in \mathcal{M}^* \quad Q = [(q_n)_{n \in \mathbb{N}}] \in \mathcal{M}^* \quad R = [(r_n)_{n \in \mathbb{N}}] \in \mathcal{M}^*
\]
(i) Since every \(d(p_n, q_n) \geq 0\), we have \(\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n) \geq 0\). On the other hand, if \(\Delta(P, Q) = 0\), then \(\lim_{n \to \infty} d(p_n, q_n) = 0\) so that \((p_n)_{n \in \mathbb{N}} \sim (q_n)_{n \in \mathbb{N}}\) and \(P = Q\).
(ii) \(\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n) = \lim_{n \to \infty} d(q_n, p_n) = \Delta(Q, P)\).
(iii) \(\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n) \leq \lim_{n \to \infty} \left( d(p_n, r_n) + d(r_n, q_n) \right) = \Delta(P, R) + \Delta(R, Q)\). \(\blacksquare\)
Lemma 5 $\mathcal{M}^*$ is complete

Proof: Let $(Q^{(n)} \in \mathcal{M}^*)_{n \in \mathbb{N}}$ be a Cauchy sequence. We must prove that it has a limit, $Q \in \mathcal{M}^*$, as $n \to \infty$. Each $Q^{(n)}$ is an equivalence class of Cauchy sequences in $\mathcal{M}$. Say $Q^{(n)} = [(q^{(n)}_\ell)_{\ell \in \mathbb{N}}]$. We shall guess $Q = [(q_n)_{n \in \mathbb{N}}]$ by choosing each $q_n$ to be a $q^{(n)}_\ell$ with $\ell$ chosen larger and larger as $n$ increases. Here we go:

$$n, m \geq 1$$

Let

$\begin{align*}
&(q^{(1)}_\ell)_{\ell \in \mathbb{N}} \text{ Cauchy } \Rightarrow \exists \ell_1 \in \mathbb{N} \text{ s.t. } \ell \geq \ell_1 \Rightarrow d(q^{(1)}_\ell, q^{(1)}_{\ell_1}) < 1. \text{ Pick } q_1 = q^{(1)}_{\ell_1}. \\
&(q^{(2)}_\ell)_{\ell \in \mathbb{N}} \text{ Cauchy } \Rightarrow \exists \ell_2 > \ell_1 \text{ s.t. } \ell \geq \ell_2 \Rightarrow d(q^{(2)}_\ell, q^{(2)}_{\ell_2}) < \frac{1}{2}. \text{ Pick } q_2 = q^{(2)}_{\ell_2}. \\
&(q^{(3)}_\ell)_{\ell \in \mathbb{N}} \text{ Cauchy } \Rightarrow \exists \ell_3 > \ell_2 \text{ s.t. } \ell \geq \ell_3 \Rightarrow d(q^{(3)}_\ell, q^{(3)}_{\ell_3}) < \frac{1}{3}. \text{ Pick } q_3 = q^{(3)}_{\ell_3}. \\
&\vdots \\
&(q^{(n)}_\ell)_{\ell \in \mathbb{N}} \text{ Cauchy } \Rightarrow \exists \ell_n > \ell_{n-1} \text{ s.t. } \ell \geq \ell_n \Rightarrow d(q^{(n)}_\ell, q^{(n)}_{\ell_n}) < \frac{1}{n}. \text{ Pick } q_n = q^{(n)}_{\ell_n}. \\
&\vdots
\end{align*}$$

In the example sketched below, the $q^{(n)}_{\ell_n}$'s are circled.

\[
\begin{array}{cccc}
Q^{(1)} & Q^{(2)} & Q^{(3)} & \ldots \\
q^{(1)}_4 & q^{(2)}_4 & \bullet & \circ \\
q^{(1)}_3 & q^{(2)}_3 & \circ & \bullet \\
q^{(1)}_2 & q^{(2)}_2 & \bullet & \bullet \\
q^{(1)}_1 & q^{(2)}_1 & \circ & \bullet \circ \\
\end{array}
\]

Proof that $(q_n)_{n \in \mathbb{N}}$ is Cauchy: Let $\varepsilon > 0$. By the triangle inequality

$$d(q_n, q_m) \leq d(q^{(n)}_{\ell_n}, q^{(n)}_{\ell_n}) + d(q^{(n)}_{\ell_n}, q^{(m)}_{\ell_m}) + d(q^{(m)}_{\ell_m}, q^{(m)}_{\ell_m})$$

$$= d(q^{(n)}_{\ell_n}, q^{(n)}_{\ell_n}) + [d(q^{(n)}_{\ell_n}, q^{(m)}_{\ell_m}) - \Delta(Q^{(n)}, Q^{(m)})] + \Delta(Q^{(n)}, Q^{(m)}) + d(q^{(m)}_{\ell_m}, q^{(m)}_{\ell_m})$$

for any $\ell \in \mathbb{N}$.

- If $\ell \geq \ell_n$, the first term is smaller than $\frac{1}{n}$. 
- By definition, $\Delta(Q^{(n)}, Q^{(m)}) = \lim_{\ell \to \infty} d(q^{(n)}_{\ell_n}, q^{(m)}_{\ell_m})$. So there is a natural number $N_{n,m}$ such that the second term is smaller than $\frac{1}{4}$ whenever $\ell \geq N_{n,m}$.
- By hypothesis, the sequence $(Q^{(n)})$ is Cauchy. So there is a natural number $\tilde{N}$ such that the third term is smaller than $\frac{1}{4}$ whenever $n, m \geq \tilde{N}$.
- Finally, if $\ell \geq \ell_m$, the last term is smaller than $\frac{1}{m}$.

Choose any natural number $N \geq \max \{\tilde{N}, \frac{1}{\varepsilon} \}$. I claim that $d(q_n, q_m) < \varepsilon$ whenever $n, m \geq N$. So let $n, m \geq N$. Now choose $\ell$ to be any natural number bigger than $\max \{N_{n,m}, \ell_n, \ell_m\}$. Then the four terms in (3) are each smaller than $\frac{1}{4}$. 

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Proof that $Q = \lim_{n \to \infty} Q^{(n)}$: Let $\varepsilon > 0$. By definition

$$\Delta(Q, Q^{(n)}) = \lim_{m \to \infty} d(q_m, q_m^{(n)}) = \lim_{m \to \infty} d(q_m^{(n)}, q_m^{(n)})$$

By the triangle inequality

$$d(q_m^{(n)}, q_m^{(n)}) \leq d(q_m^{(n)}, q_m^{(n)}) + d(q_m^{(n)}, q_m^{(n)}) \quad (3)$$

○ Since the sequence $(q_n = q_{\ell_n}^{(n)})$ is Cauchy, there is a natural number $N'$ such that the first term is smaller than $\frac{\varepsilon}{2}$ whenever $n, m \geq N'$.

○ By the construction of $\ell_n$, the second term is smaller than $\frac{1}{n}$ whenever $m \geq \ell_n$.

Choose any natural number $N \geq \max \{N', \frac{2}{\varepsilon} \}$. I claim that $\Delta(Q, Q^{(n)}) < \varepsilon$ whenever $n \geq N$. So let $n \geq N$. Now choose $m$ to be any natural number bigger than $\max \{N', \ell_n \}$. Then the two terms in (4) are each smaller than $\varepsilon$.

Lemma 6 $\Delta(\varphi(p), \varphi(q)) = d(p, q)$ for all $p, q \in \mathcal{M}$.

Proof: $\Delta(\varphi(p), \varphi(q)) = \lim_{n \to \infty} d(\varphi(p)_n, \varphi(q)_n) = \lim_{n \to \infty} d(p, q) = d(p, q)$

Lemma 7 $\varphi$ is one–to–one.

Proof: $\varphi(p) = \varphi(q) \implies \Delta(\varphi(p), \varphi(q)) = 0 \implies d(p, q) = 0 \implies p = q$.

Lemma 8 $\varphi(\mathcal{M})$ is dense in $\mathcal{M}^*$.

Proof: Let $P = \{(p_n)_{n \in \mathbb{N}}\} \in \mathcal{M}^*$. Then I claim that the sequence $(\varphi(p_m))_{m \in \mathbb{N}}$ converges in $\mathcal{M}^*$ to $P$. To check this, it suffices to observe that

$$\Delta(P, \varphi(p_m)) = \lim_{n \to \infty} d(p_n, \varphi(p_m)_n) = \lim_{n \to \infty} d(p_n, p_m)$$

Since $(p_n)_{n \in \mathbb{N}}$ is Cauchy, this converges to zero as $m \to \infty$.

Lemma 9 If $\mathcal{M}$ is complete, then $\varphi(\mathcal{M}) = \mathcal{M}^*$.

Proof: Let $Q \in \mathcal{M}^*$. We must find $q \in \mathcal{M}$ with $\varphi(q) = Q$. By Lemma 8

$$\exists \ (q_n \in \mathcal{M})_{n \in \mathbb{N}} \text{ s.t. } Q = \lim_{n \to \infty} \varphi(q_n)$$

$$\implies (\varphi(q_n))_{n \in \mathbb{N}} \text{ is Cauchy in } \mathcal{M}^*$$

$$\implies (q_n)_{n \in \mathbb{N}} \text{ is Cauchy in } \mathcal{M}, \text{ by Lemma 6}$$

$$\implies q = \lim_{n \to \infty} q_n \text{ exists, since } \mathcal{M} \text{ is complete}$$

$$\implies \varphi(q) = \lim_{n \to \infty} \varphi(q_n) \text{ by Lemma 6}$$

$$\implies \varphi(q) = Q$$