

Axioms

The Field Axioms

A *field* is a set F with two operations, called *addition* and *multiplication* which satisfy the following axioms (A1–5), (M1–5) and (D).

(A) Axioms for addition

- (A1) $x, y \in F \implies x + y \in F$
- (A2) $x + y = y + x$ for all $x, y \in F$ (addition is commutative)
- (A3) $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$ (addition is associative)
- (A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
- (A5) For each $x \in F$ there is an element $-x \in F$ such that $x + (-x) = 0$.

(M) Axioms for multiplication

- (M1) $x, y \in F \implies xy \in F$
- (M2) $xy = yx$ for all $x, y \in F$ (multiplication is commutative)
- (M3) $(xy)z = x(yz)$ for all $x, y, z \in F$ (multiplication is associative)
- (M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
- (M5) For each $0 \neq x \in F$ there is an element $\frac{1}{x} \in F$ such that $x(\frac{1}{x}) = 1$.

(D) The distributive law

- (D) $x(y + z) = xy + xz$ for all $x, y, z \in F$

Ordered Field

An *ordered field* is a field F with a relation, denoted $<$, obeying the

(O) Order axioms

- (O1) For each pair $x, y \in F$ precisely one of $x < y$, $x = y$, $x > y$ is true.
- (O2) $x < y, y < z \implies x < z$
- (O3) $y < z \implies x + y < x + z$
- (O4) $x > 0, y > 0 \implies xy > 0$