Stokes' Theorem

The statement

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_S \vec{\nabla} \times \vec{F} \cdot \hat{\mathbf{n}}\, dS$$

provided

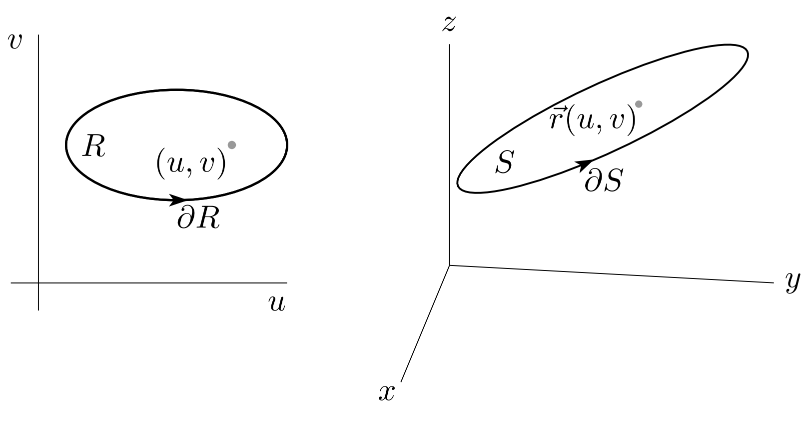
- The curve ∂S is the boundary of the surface S
- The orientations of ∂S and $\hat{\mathbf{n}}$ obey the right hand rule
- The vector field $\vec{F}$ has continuous first partial derivatives at every point of S

The proof Both integrals involve F_1 terms and F_2 terms and F_3 terms. We shall show that the F_1 terms in the two integrals agree. In other words, we shall assume that $\vec{F} = F_1\hat{\imath}$. The proofs that the F_2 and F_3 terms also agree are similar.

Suppose that S is the parametrized surface

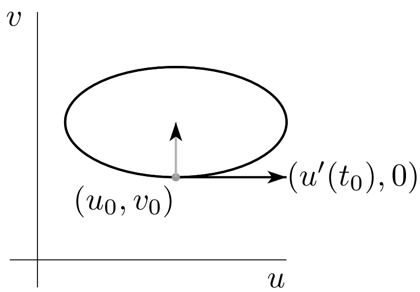
$$S = \left\{ \vec{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big) \;\big|\; (u,v) \text{ in } R \right\}$$

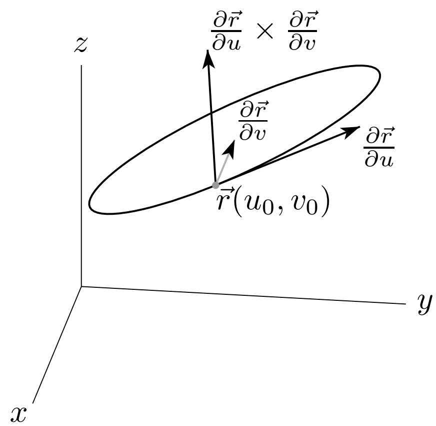
If the curve, ∂R, bounding R is parametrized as $\big(u(t), v(t)\big)$, $a \le t \le b$, then the curve ∂S bounding S is parametrized as $\vec{r}\big(u(t), v(t)\big)$, $a \le t \le b$.



The surface integral For the surface S, $\hat{\mathbf{n}}\, dS = \pm \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\, du\, dv$. I claim that $+\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ gives the correct orientation of $\hat{\mathbf{n}}$. Because $\hat{\mathbf{n}}$ must vary continuously over S, it suffices to check that $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ gives the correct orientation of $\hat{\mathbf{n}}$ at one point of S. Find a point (u_0, v_0) on ∂R where the forward pointing tangent vector is a positive multiple of $\hat{\imath}$. The arrow on ∂R in the left figure above is at such a point. Suppose that $t = t_0$ at this point – in other words, suppose that $(u_0, v_0) = \big(u(t_0), v(t_0)\big)$. Because the forward pointing tangent vector to ∂R at (u_0, v_0), namely $\big(u'(t_0), v'(t_0)\big)$, is a positive multiple of $\hat{\imath}$, we have $u'(t_0) > 0$ and $v'(t_0) = 0$. The forward pointing tangent vector to ∂S at $\vec{r}\big(u_0, v_0\big)$ is $\frac{d}{dt}\vec{r}\big(u(t), v(t)\big)\big|_{t=t_0} = u'(t_0)\frac{\partial \vec{r}}{\partial u}(u_0, v_0) + v'(t_0)\frac{\partial \vec{r}}{\partial v}(u_0, v_0) = u'(t_0)\frac{\partial \vec{r}}{\partial u}(u_0, v_0)$ and so is a positive multiple of $\frac{\partial \vec{r}}{\partial u}(u_0, v_0)$.

If we now walk along a path in the uv–plane which starts at (u_0, v_0), holds u fixed at u_0 and increases v, we move into the interior of R starting at (u_0, v_0). Correspondingly, if we walk along a path, $\vec{r}(u_0, v)$, in $\mathbb{R}^3$ with v starting at v_0 and increasing, we move into the interior of S. The forward tangent to this new path, $\frac{\partial \vec{r}}{\partial v}(u_0, v_0)$, points from $\vec{r}(u_0, v_0)$ into the interior of S.





According to the right hand rule, the desired normal, $\hat{\mathbf{n}}$, to S at $\vec{r}(u_0, v_0)$ is to be constructed as follows. Orient your right hand with its fingers pointing in the forward direction along ∂S at $\vec{r}(u_0, v_0)$ and its palm facing the interior of S. Then your thumb is pointing in the direction of $\hat{\mathbf{n}}$. Your right hand now has its fingers pointing in the direction $\frac{\partial \vec{r}}{\partial u}(u_0, v_0)$ and its palm pointing in the direction $\frac{\partial \vec{r}}{\partial v}(u_0, v_0)$. So your thumb is pointing in the direction of $\frac{\partial \vec{r}}{\partial u}(u_0, v_0) \times \frac{\partial \vec{r}}{\partial v}(u_0, v_0)$.

Hence we have the correct orientation of $\hat{\mathbf{n}}$ and, recalling that $\vec{F} = F_1 \hat{\boldsymbol{\imath}}$,

$$\begin{aligned} \int_S \vec{\nabla} \times \vec{F} \cdot \hat{\mathbf{n}}\, dS &= \int_R \big(0, \tfrac{\partial F_1}{\partial z}, -\tfrac{\partial F_1}{\partial y}\big) \cdot \tfrac{\partial \vec{r}}{\partial u} \times \tfrac{\partial \vec{r}}{\partial v}\; du\, dv \\ &= \int_R \Big\{ \tfrac{\partial F_1}{\partial z} \big(\tfrac{\partial z}{\partial u} \tfrac{\partial x}{\partial v} - \tfrac{\partial x}{\partial u} \tfrac{\partial z}{\partial v} \big) - \tfrac{\partial F_1}{\partial y} \big(\tfrac{\partial x}{\partial u} \tfrac{\partial y}{\partial v} - \tfrac{\partial y}{\partial u} \tfrac{\partial x}{\partial v} \big) \Big\}\, du\, dv \end{aligned}$$

The line integral

$$\begin{aligned} \oint_{\partial S} \vec{F} \cdot d\vec{r} &= \int_a^b \vec{F}\Big(\vec{r}\big(u(t), v(t)\big)\Big) \cdot \tfrac{d}{dt} \vec{r}\big(u(t), v(t)\big)\; dt \\ &= \int_a^b \vec{F}\Big(\vec{r}\big(u(t), v(t)\big)\Big) \cdot \Big[\tfrac{\partial \vec{r}}{\partial u}\big(u(t), v(t)\big) \tfrac{du}{dt}(t) + \tfrac{\partial \vec{r}}{\partial v} \tfrac{dv}{dt} \Big]\; dt \end{aligned}$$

This agrees exactly with the line integral

$$\oint_{\partial R} M(u, v)\; du + N(u, v)\; dv = \int_a^b \Big[M\big(u(t), v(t)\big)\; \tfrac{du}{dt}(t) + N\big(u(t), v(t)\big)\; \tfrac{dv}{dt}(t) \Big]\; dt$$

around ∂R, if we choose

$$\begin{aligned} M(u,v) &= \vec{F}\big(\vec{r}(u,v)\big) \cdot \tfrac{\partial \vec{r}}{\partial u}(u,v) \\ &= F_1\big(x(u,v), y(u,v), z(u,v)\big)\tfrac{\partial x}{\partial u}(u,v) \\ N(u,v) &= \vec{F}\big(\vec{r}(u,v)\big) \cdot \tfrac{\partial \vec{r}}{\partial v}(u,v) \\ &= F_1\big(x(u,v), y(u,v), z(u,v)\big)\tfrac{\partial x}{\partial v}(u,v) \end{aligned}$$

By Green's Theorem, we have

$$\begin{aligned} &\oint_{\partial S} \vec{F} \cdot d\vec{r} = \oint_{\partial R} M(u,v)\ du + N(u,v)\ dv \\ &\quad = \int_R \Big\{ \tfrac{\partial N}{\partial u} - \tfrac{\partial M}{\partial v} \Big\}\ dudv \\ &\quad = \int_R \Big\{ \big(\tfrac{\partial F_1}{\partial x}\tfrac{\partial x}{\partial u} + \tfrac{\partial F_1}{\partial y}\tfrac{\partial y}{\partial u} + \tfrac{\partial F_1}{\partial z}\tfrac{\partial z}{\partial u}\big)\tfrac{\partial x}{\partial v} + F_1\tfrac{\partial^2 x}{\partial u \partial v} \\ &\qquad\quad - \big(\tfrac{\partial F_1}{\partial x}\tfrac{\partial x}{\partial v} + \tfrac{\partial F_1}{\partial y}\tfrac{\partial y}{\partial v} + \tfrac{\partial F_1}{\partial z}\tfrac{\partial z}{\partial v}\big)\tfrac{\partial x}{\partial u} - F_1\tfrac{\partial^2 x}{\partial v \partial u} \Big\}\ dudv \\ &\quad = \int_R \Big\{ \big(\tfrac{\partial F_1}{\partial y}\tfrac{\partial y}{\partial u} + \tfrac{\partial F_1}{\partial z}\tfrac{\partial z}{\partial u}\big)\tfrac{\partial x}{\partial v} - \big(\tfrac{\partial F_1}{\partial y}\tfrac{\partial y}{\partial v} + \tfrac{\partial F_1}{\partial z}\tfrac{\partial z}{\partial v}\big)\tfrac{\partial x}{\partial u} \Big\}\, du\, dv \\ &\quad = \int_S \vec{\boldsymbol{\nabla}} \times \vec{F} \cdot \hat{\mathbf{n}}\, dS \end{aligned}$$

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